

From contact relations to modal operators, and back

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Any Boolean algebra is turned into a *Boolean contact algebra* by expanding it to a structure $\mathfrak{B} = \langle B, \cdot, +, -, \mathbf{0}, \mathbf{1}, C \rangle$ where $C \subseteq B^2$ is a *contact* relation which satisfies the following five axioms:

$$\neg(\mathbf{0} C x), \quad (\text{C0})$$

$$x \leq y \wedge x \neq \mathbf{0} \longrightarrow x C y, \quad (\text{C1})$$

$$x C y \longrightarrow y C x, \quad (\text{C2})$$

$$x \leq y \wedge z C x \longrightarrow z C y, \quad (\text{C3})$$

$$x C y + z \longrightarrow x C y \vee x C z. \quad (\text{C4})$$

Stronger condition

Our intention is to examine a stronger version of this axiom according to which if x is in contact with the supremum of some family S of regions, then there is y in S that is in contact with x . We consider Boolean contact algebras $\mathfrak{B} = \langle B, \cdot, +, -, \mathbf{0}, \mathbf{1}, C \rangle$ which satisfy the following axioms:

$$\neg(\mathbf{0} C x), \quad (\text{C0})$$

$$x \leq y \wedge x \neq \mathbf{0} \longrightarrow x C y, \quad (\text{C1})$$

$$x C y \longrightarrow y C x, \quad (\text{C2})$$

$$x \leq y \wedge z C x \longrightarrow z C y, \quad (\text{C3})$$

$$x C \bigvee_{i \in I} x_i \longrightarrow (\exists i \in I) x C x_i. \quad (\text{C4}^c)$$

Examples and counterexamples

Overlap

If \mathfrak{B} is a BCA in which $C = O$ where

$$x O y :\longleftrightarrow x \cdot y \neq \mathbf{0}, \quad (\text{df } O)$$

then in \mathfrak{B} contact completely distributes over join.

Relational contact algebras

Given a reflexive and symmetric frame $\langle W, R \rangle$, i.e., a non-empty set of worlds with an accessibility relation $R \subseteq W \times W$ that is reflexive and symmetric, we expand the power set algebra $\mathcal{P}(W)$ with the following relation:

$$A C_R B :\longleftrightarrow (\exists x \in A)(\exists y \in B) x R y. \quad (\text{df } C_R)$$

It is routine to verify that C_R satisfies axioms (C0)–(C4), so it is a Boolean contact algebra. Every relational contact algebra belongs to class $\mathbf{C4}^c$.

The algebra of regular closed subsets

Given a topological space X , let $\text{RC}(X)$ be the complete Boolean algebra of all regular closed subsets of X , i.e., these $A \in \mathcal{P}(X)$ for which $A = \text{Cl Int } A$. The operations are given by the following identities:

$$\begin{aligned}A \cdot B &:= \text{Cl Int}(A \cap B), \\A + B &:= A \cup B, \\-A &:= \text{Cl}(X \setminus A), \\\bigvee \mathcal{S} &:= \text{Cl} \bigcup \mathcal{S},\end{aligned}$$

For any topological space X , its complete Boolean algebra of regular closed subsets of X with

$$A C B :\longleftrightarrow A \cap B \neq \emptyset,$$

is a contact algebra.

If X is an Alexandroff space, then $\langle \text{RC}(X), C \rangle$ satisfies ($C4^c$).

Regular-closed sets of \mathbb{R} does not satisfy (C4^c)

To show that (C4) does not entail (C4^c) consider the set of reals, the algebra $\langle \text{RC}(\mathbb{R}), \mathcal{C} \rangle$ and the following family of regular closed subsets of the reals:

$$\mathcal{S} := \left\{ [-x, x] : 0 < x < \sqrt{2} \right\}.$$

Since $\bigvee \mathcal{S} = \text{Cl} \bigcup \mathcal{S} = [-\sqrt{2}, \sqrt{2}]$, we have that:

$$(-\infty, -\sqrt{2}] \subset \bigvee \mathcal{S} \quad \text{but} \quad \text{for all } S \in \mathcal{S}, (-\infty, -\sqrt{2}] \not\subset S. \quad \square$$

A zero-dimensional space that fails to meet the axiom

Take the cantor space 2^ω , i.e., the countable product of the discrete space $\{0, 1\}$ with the standard product topology.

- Let Odd be the set of all sequences that begin with odd number of zeroes followed by 1, and Even be the set of all sequences that begin with even number of zeroes followed by 1.
- Let (0) be the constant zero sequence.

Both $O := \text{Odd} \cup \{(0)\}$ and $E := \text{Even} \cup \{(0)\}$ are regular closed, while both Odd and Even are regular open.

We see that $O \subset E$. Yet the space is zero-dimensional, so there is a family of clopen sets $\{A_i \mid i \in I\}$ such that $E = \bigvee_{i \in I} A_i$, and none of A_i contains (0) .

Thus O is contact with the supremum of all A_i 's, but with none of the A_i 's themselves.

Every contact algebra can be expanded with the standard binary relation of *non-tangential* inclusion:

$$x \ll y :\longleftrightarrow x \not\ll -y. \quad (\text{df } \ll)$$

It is well-know that non-tangential inclusion is a special case of the subordination relation:

$$0 \ll 0 \quad \text{and} \quad \mathbf{1} \ll \mathbf{1}, \quad (\text{S1})$$

$$x \ll y \wedge x \ll z \longrightarrow x \ll y \cdot z, \quad (\text{S2})$$

$$x \ll y \wedge z \ll y \longrightarrow x + z \ll y, \quad (\text{S3})$$

$$x \leq y \ll z \leq u \longrightarrow x \ll u. \quad (\text{S4})$$

Moreover, we have:

$$x \ll y \longrightarrow x \leq y, \quad (\text{S5})$$

$$x \ll y \longrightarrow -y \ll -x, \quad (\text{S6})$$

$$x \ll y \wedge y \cdot z = \mathbf{0} \longrightarrow x \not\ll z. \quad (\text{S7})$$

Let us observe that by means of the non-tangential part of relation we can express (C4^c) in an alternative form:

$$(\forall i \in I) x \ll x_i \longrightarrow x \ll \bigwedge_{i \in I} x_i.$$

We can define a quasi-modal operator $\Delta: B \rightarrow \text{Id}(\mathfrak{B})$

$$\Delta(x) = \{y: y \ll x\}$$

and we can express (C4^c) in an alternative form:

$$\Delta\left(\bigwedge_{i \in I} x_i\right) = \bigcap_{i \in I} \Delta(x_i)$$

The axiom in extensional algebras

Standardly, what is called *the extensionality axiom* for BCAs is

$$(\forall x \in B)(x \mathbf{C} a \longrightarrow x \mathbf{C} b) \longrightarrow a \leq b. \quad (\text{C5})$$

All three following constraints are its equivalents (in the class **BCA**):

$$\begin{aligned} &(\forall x \neq \mathbf{1})(\exists y \neq \mathbf{0}) x \mathbf{C} y, \\ &x \neq \mathbf{0} \longrightarrow (\exists y \in B)(y \neq \mathbf{0} \wedge y \ll x), \\ &x \neq \mathbf{1} \longrightarrow (\exists y \in B)(y \neq \mathbf{1} \wedge x \ll y). \end{aligned}$$

Corollary

If $\mathfrak{B} \in \mathbf{BCA} + (C5) + (C4^c)$, then for every region x , $x \ll x$.

Theorem

If $\mathfrak{B} \in \mathbf{BCA} + (C5)$, then \mathfrak{B} satisfies the complete distributivity of contact over join if and only if every region of \mathfrak{B} is isolated iff contact and overlap relations coincide.

(C4^c) does not entail (C5)

Take any Boolean algebra with at least 4 elements and extend it with the largest contact relation on any algebra:

$$x C_L y :\longleftrightarrow x \neq \mathbf{0} \wedge y \neq \mathbf{0}.$$

(C5) fails, since the algebra has at least four elements, and thus there is a region $x \neq \mathbf{0}$ such that $-x \neq \mathbf{0}$, and thus every non-zero tangential part of x is in contact with $-x$. If the algebra is either finite or infinite, it is equally easy to see that if x is in contact with $\bigvee_{i \in I} x_i$, then there must be non-zero x_j , for some $j \in I$. And thus $x C x_j$.

For any region x let $C(x)$ be the set of all regions that are in contact with x :

$$C(x) := \{y \in B \mid y C x\}.$$

It is easy to see that the axioms (C0), (C3) and (C4) say that:

$C(x)$ is a grill.

Further, it can be observed that (C4^c) says that every $C(x)$ is a completely prime grill:

$$\bigvee_{i \in I} x_i \in C(x) \longrightarrow (\exists i \in I) x_i \in C(x).$$

From the contact relation to a modal operator

If \mathfrak{B} is a complete BA, then \mathfrak{B} satisfies ($\mathbf{C4}^c$) iff for every region x there exists a unique region y such that

$$C(x) = O(y) = \{z : z \cdot y \neq 0\}.$$

In particular, every finite Boolean contact algebra satisfies the latter property.

This offers a different axiomatization of the class $\mathbf{C4}^c$ via axioms for complete BAs plus (C0), (C1), (C2) and:

$$(\forall x \in B)(\exists y \in B) C(x) = O(y),$$

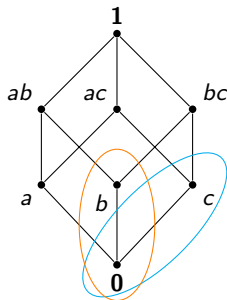
This entails existence of an operation $m: B \longrightarrow B$ such that:

$$m(x) := (\iota y) C(x) = O(y). \quad (\text{df } m)$$

Corollary

m is a modal possibility operator.

m does not have to be a closure operator



Let $\mathcal{G}_1 := (\downarrow b)^{\complement}$ and $\mathcal{G}_2 := (\downarrow c)^{\complement}$. Take $C_{\mathbf{G}}$ for $\mathbf{G} := \{\mathcal{G}_1, \mathcal{G}_2\}$ and consider

$$x C_{\mathbf{G}} y :\longleftrightarrow x O y \vee (\exists \mathcal{G} \in \mathbf{G}) \{x, y\} \subseteq \mathcal{G}.$$

We see that:

$$\begin{aligned} C_{\mathbf{G}}(a) &= C_{\mathbf{G}}(ab) = C_{\mathbf{G}}(ac) = C_{\mathbf{G}}(bc) = O(\mathbf{1}) \\ C_{\mathbf{G}}(b) &= O(ab) \quad \text{and} \quad C_{\mathbf{G}}(c) = O(bc). \end{aligned}$$

In consequence $m(b) = ab$ and $m(m(b)) = m(ab) = \mathbf{1}$.

Properties of the modal operator

More specifically, we have that:

$$m(x) = \bigwedge \uparrow x.$$

Obviously, we have that:

$$x \mathsf{C} y \longleftrightarrow m(x) \cdot y \neq \mathbf{0},$$

and so:

$$x \ll y \longleftrightarrow m(x) \leq y.$$

Lemma

If $\mathfrak{B} \in \mathbf{C4}^c$, then $m: B \longrightarrow B$ is a completely additive modal possibility operator such that:

- ① $x \ll m(x)$, and so $x \leq m(x)$, i.e., $x + m(x) = m(x)$,
- ② $m(x) \leq -y \longleftrightarrow m(y) \leq -x$, which is equivalent to $m(-m(-x)) \leq x$.

From a modal operator to the contact relation

Definition

Any modal algebra $\mathfrak{B} := \langle B, \Diamond \rangle$ whose possibility operator satisfies the following two conditions:

$$x \leq \Diamond x, \quad (\text{T}_\Diamond)$$

$$\Diamond \Box x \leq x, \quad (\text{B}_\Diamond)$$

where $\Box := -\Diamond-$, is called a *KTB-algebra*.

Lemma

If \mathfrak{B} is a complete KTB-algebra, then:

$$C_\Diamond := \{ \langle x, y \rangle \mid x \cdot \Diamond y \neq \mathbf{0} \} \quad (\text{df } C_\Diamond)$$

is a contact relation that satisfies (C4^c). Moreover, $\Diamond = m$, where m is the modal operator for C_\Diamond introduced by (df m).

An isomorphism of categories

Definition

Given two algebras $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathbf{C4}^c$, a mapping $h: B_1 \rightarrow B_2$ is a *p-morphism*^a iff it is a homomorphism such that:

$$h(x) C_2 h(y) \longrightarrow x C_1 y, \quad (\text{P1})$$

$$h(z) \ll_2 y \longrightarrow (\exists x \in B_1)(z \ll_1 x \wedge h(x) \leq_2 y). \quad (\text{P2})$$

Of course, (P1) is equivalent to:

$$x \ll_1 y \longrightarrow h(x) \ll_2 h(y).$$

^aThe idea of this comes from (Celani, 2001), where similar morphisms are called *q-morphism*.

An isomorphism of categories

The class $\mathbf{C4}^c$ together with p -morphisms form a category with the identity functions serving as the identity morphisms.

It is obvious that the class \mathbf{KTB}^c of the modal KTB complete algebras together as with the standard homomorphism form a category.

Theorem

The categories $\mathbf{C4}^c$ and \mathbf{KTB}^c are isomorphic.

Definition

A *partition* of a Boolean algebra \mathfrak{B} is any non-empty set P of non-zero and disjoint regions of B that add up to the unity: $\bigvee P = \mathbf{1}$.

Let $\mathfrak{B} \in \mathbf{BCA}^c$ be given, let $P := \{p_i \mid i \in I\}$ be its partition. Define:

$$x C_P y :\longleftrightarrow (\exists i \in I) (x O p_i \wedge y O p_i). \quad (\text{df } C_P)$$

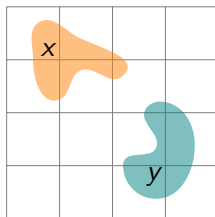


Figura: Regions x and y that are in contact with respect to a partition.

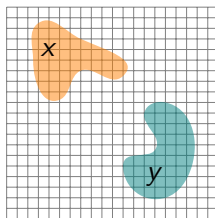


Figura: Regions x and y are no longer in contact if we take a finer partition as the frame of reference.

Further, for any region x define

$$O_P(x) := \{p_i \in P \mid x O p_i\}$$

We have that $m(x) = \bigvee O_P(x)$, and so if $x \leq p_i$, for some $i \in I$, then $m(x) = p_i$.

If x and y are regions of a pixel algebra \mathfrak{B} with a partition P ,

then $x C_P y$ iff $m(x) \cdot m(y) \neq \mathbf{0}$.

Definition

Given two partitions, P and P' , P is *finer* than P' (in symbols: $P \trianglelefteq P'$) iff for every $p \in P$ there is a p' in P' such that $p \leq p'$.

Given a sequence of partitions $\langle P_\alpha \mid \alpha < \kappa \rangle$ (with κ being a limit ordinal) such that $P_{\alpha+1} \trianglelefteq P_\alpha$, with each of them there is associated the contact relation C_α determined by P_α . We have that:

$$\beta < \alpha \longrightarrow C_\alpha \subseteq C_\beta.$$

Theorem

If $\mathfrak{B} \in \mathbf{RCA}^c$ has a finite resolution $P = \{p_i \mid i \leq n\}$ for some $n \in \mathbb{N}$, then the Kripke relation on the set $\text{Ult } B$ is an equivalence relation and there is a one-to-one correspondence $f: P \rightarrow \text{Ult } B /_R$ between cells and equivalence classes of ultrafilters.

Theorem

Given an S5 modal algebra $\mathfrak{B} = \langle B, \Diamond \rangle$, its expansion $\mathfrak{B}^ = \langle B, \Diamond, C_\Diamond \rangle$ can be embedded into a modal expansion of a resolution algebra.*

Thank you!

Celani, S. A. (2001). Quasi-modal algebras. *Mathematica Bohemica*, 126(4):721–736.