LATD 2022 and MOSAIC KICK OFF MEETING
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Free algebras in all subvarieties of the variety generated by the MG t-norm

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UNLP - INMABB - CONICET

Definition

A **t-norm** *is a binary operation* $*: [0,1]^2 \to [0,1]$ *which satisfies the following conditions:*

1. $*$ *is commutative and associative.*
2. $*$ *is non decreasing in both arguments, i.e., for every* $x, y, z \in [0,1]$
$$x \leq y \text{ implies } x * z \leq y * z \text{ and } z * x \leq z * y,$$
3. $1 * x = x$ *and* $0 * x = 0$ *for every* $x \in [0,1]$.

A **continuous t-norm** *is a t-norm which is continuous as a map from* $[0,1]^2$ *into* $[0,1]$. *For every continuous t-norm a residuum can be defined by:*

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The algebra $([0,1], *, \to, max, min, 0, 1)$ is the standard algebra associated with the continuous t-norm $*$.

Examples

1. *Łukasiewicz t-norm:* $x *_L y = max(0, x + y - 1)$
   Łukasiewicz implication: $x \rightarrow_L y = min(1, 1 - x + y)$,
2. *Gödel t-norm:* $x *_G y = min(x, y)$,
   Gödel implication:
$$x \rightarrow_G y = \begin{cases} y & \text{if} \quad x > y; \\ 1 & \text{if} \quad x \leq y. \end{cases}$$
3. *Product t-norm:* $x *_P y = x \cdot y$,
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$$x \rightarrow_P y = \begin{cases} y/x & \text{if} \quad x > y; \\ 1 & \text{if} \quad x \leq y. \end{cases}$$

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The algebras $([0,1], *_L, \rightarrow_L, \vee, \wedge, 0, 1)$, $([0,1], *_G, \rightarrow_G, \vee, \wedge, 0, 1)$ and $([0,1], *_P, \rightarrow_P, \vee, \wedge, 0, 1)$ are the Łukasiewicz, Gödel and Product standard algebras, respectively.

If $(a_i, b_i)_{i\in I}$ is a family of disjoint intervals, with $0 \leq a_i < b_i \leq 1$ such that $*^i$ is a continuous t-norm on (a_i, b_i), we define for every $x, y \in [0,1]$ a continuous t-norm called **ordinal sum of t-norms** by:

$$x * y = \begin{cases} x *^i_{[a_i,b_i]} y & \text{if } x, y \in (a_i, b_i); \\ min\{x, y\} & \text{otherwise.} \end{cases}$$

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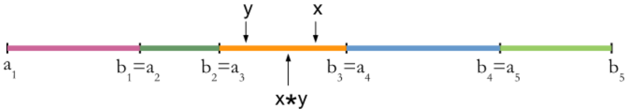

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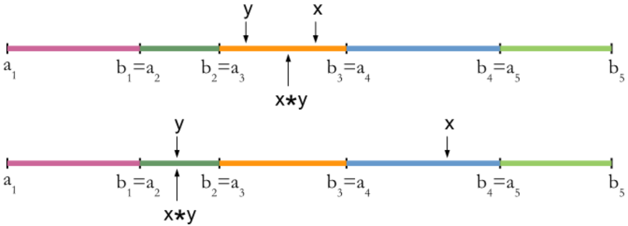

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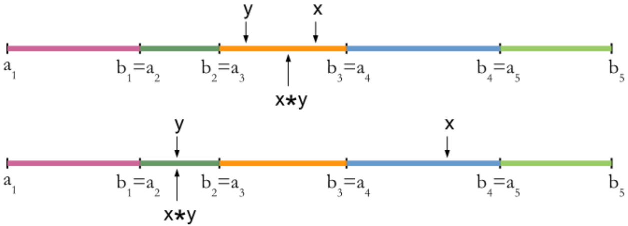

Theorem (Mostert-Schields)

Every continuous t-norm is the ordinal sum of a family of Łukasiewicz, Gödel and product t-norms.

Hoops and BL-algebras

Definitions

A **hoop** *is an algebra* $\mathbf{A} = (A, *, \to, \top)$ *of type* $(2, 2, 0)$*, where* $(A, *, \top)$ *is a commutative monoid such that for every* $x, y, z \in A$:

1. $x \to x = \top$,
2. $x * (x \to y) = y * (y \to x)$,
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A **BL-algebra** *is a bounded basic hoop, i.e., an algebra* $\mathbf{A} = (A, *, \to, \bot, \top)$ *of type* $(2,2,0,0)$ *such that* $(A, *, \to, \top)$ *is a basic hoop and* $\bot$ *is the minimum of* A.

In every basic hoop **A** we can define the operations

$$x \wedge y := x \cdot (x \to y) = y \cdot (y \to x),$$

$$x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x)$$

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Theorem (Cignoli, Esteva, Godo, Torrens)

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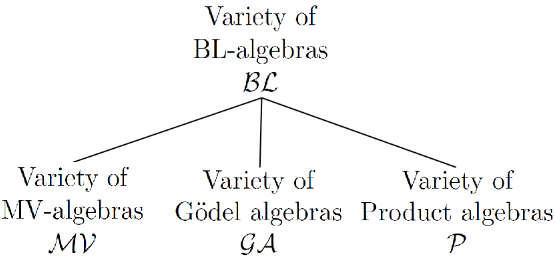

Ordinal sums

Definition

Let $\mathbf{A} = \langle A, \cdot_A, \rightarrow_A, \top\rangle$ *and* $\mathbf{B} = \langle B, \cdot_B, \rightarrow_B, \top\rangle$ *be two hoops such that* $A \cap B = \{\top\}$*. We can define the* **ordinal sum** *of* $\mathbf{A}$ *and* $\mathbf{B}$ *as the hoop* $\mathbf{A} \oplus \mathbf{B} = \langle A \cup B, \cdot, \rightarrow, \top, \rangle$*, where the operations* $\cdot$ *and* $\rightarrow$ *are given by:*

$$x \cdot y = \begin{cases} x \cdot_A y & \text{if} \quad x, y \in A; \\ x \cdot_B y & \text{if} \quad x, y \in B; \\ x & \text{if} \quad x \in A \setminus \{\top\},\ y \in B; \\ y & \text{if} \quad y \in B \setminus \{\top\},\ x \in A. \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow_A y & \text{if} \quad x, y \in A; \\ x \rightarrow_B y & \text{if} \quad x, y \in B; \\ \top & \text{if} \quad x \in A \setminus \{\top\},\ y \in B; \\ y & \text{if} \quad y \in A,\ x \in B. \end{cases}$$

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This variety is generated by the t-norm which we called MG t-norm $t_{MG} : [0,1]^2 \to [0,1]$ defined by

$$t_{MG}(x,y) = \begin{cases} max(0, x+y-\frac{1}{2}) & \text{if } x, y \in [0, \frac{1}{2}); \\ min(x,y) & \text{otherwise.} \end{cases}$$

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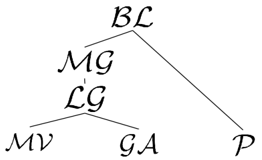

The lattice of subvarieties $\mathbf{\Lambda}(\mathcal{MV})$

For $n, k \geq 1$ we define

$$\begin{aligned} \text{Ł}_n &= \Gamma(\mathbb{Z}, n), \\ \text{Ł}_n^\infty &= \Gamma(\mathbb{Z} \times \mathbb{Z}, (n, 0)), \end{aligned}$$

where $\mathbb{Z}$ has the natural order and $\mathbb{Z} \circ \mathbb{Z}$ is the product of two copies of $\mathbb{Z}$ ordered lexicographically.

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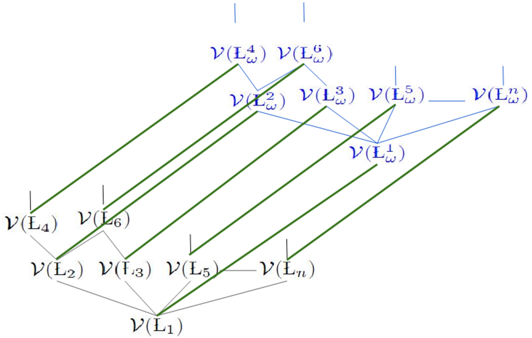

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Hetch and Katriñak proved that the subvarieties of the variety of Gödel algebras form a chain, but since the Gödel hoops are the subreducts of these algebras, the results can be naturally extended for our case.

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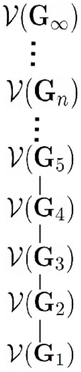

Theorem

If $\mathcal{A}$ is a join-irreducible element in the lattice of subvarieties of $\mathcal{MG}$ then $\mathcal{A}$ is the variety generated by $\mathbf{A} \oplus \mathbf{B}$ with $\mathbf{A}$ is a chain in $\mathcal{MV}$ and $\mathbf{B}$ is a chain in $\mathcal{G}$ such that

$$\mathbf{A} \in \{\mathbf{Ł}_n : n \in \mathbb{N}\} \cup \{\mathbf{Ł}_n^{\infty} : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$$

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Corollary

The join-irreducible elements in the lattice of subvarieties of $\mathcal{MG}$ form an ordered lattice.

$$(2 \times D(\mathbb{N})) \times (\omega + 1)$$

Theorem

Any subvariety $\mathcal{U}$ of $\mathcal{MG}$ is generated by a finite number of chains $\mathbf{A} \oplus \mathbf{B}$ *with* $\mathbf{A} \in \{\mathbf{Ł}_n : n \in \mathbb{N}\} \cup \{\mathbf{Ł}_n^\infty : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$ *and* $\mathbf{B} \in \{\mathbf{G}_n : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{G}}\}$.

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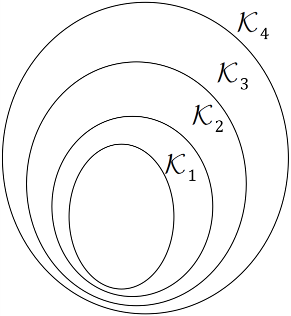
Idea of the proof:

$$\mathcal{K}_n = \mathcal{V}([0,1]_{\mathbf{MV}} \oplus \mathbf{G}_n)$$



Corollary

Any variety in $\Lambda(\mathcal{MG})$ is a join of finitely many varieties generated by a single chain $\mathbf{A} \oplus \mathbf{B}$ where $\mathbf{A} \in \{\text{Ł}_n : n \in \mathbb{N}\} \cup \{\text{Ł}_n^{\infty} : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$ and $\mathbf{B} \in \{\mathbf{G}_n : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{G}}\}$.

Equational bases

Lemma (Di Nola - Lettieri)

For $n \geq 2$, the subvariety $\mathcal{V}(\text{Ł}_n^\infty)$ of $\mathcal{MV}$ is characterized by the identity:

$$(((n+1)x^n)^2 \leftrightarrow 2x^{n+1}) \wedge ((p.x^{p-1})^{n+1} \leftrightarrow (n+1)x^p) \approx 1 \quad (\alpha_\infty^n)$$

for every positive integer $1 < p < n$ such that p is not a divisor of n.

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for every positive integer $1 < p < n$ such that p is not a divisor of n and every integer q such that $1 < q < n$ and q divides n.

Equational bases

Lemma (Hecht - Katriñak)

For $n \geq 2$, the subvariety $\mathcal{V}(G_n)$ of $\mathcal{G}$ is characterized by the following identity:

$$\bigvee_{i=1}^{n+1} (x_i \leftrightarrow x_{i+1}) \approx 1 \;(\beta_n)$$

Equational bases

Theorem

If **A** *is a subvariety of* $\mathcal{MV}$ *characterized by the identity* $\alpha \approx 1$ *and* **B** *is a subvariety of* $\mathcal{G}$ *characterized by the identity* $\beta \approx 1$*, then* **A** $\oplus$ **B** *is a subvariety of* $\mathcal{MG}$ *characterized by the identity*

$$\alpha^{'} \cdot \beta^{'} \approx 1,$$

where $\alpha^{'}$ *is the term given by substituying* $\neg\neg x$ *for every variable* x *in* α *and* $\beta^{'}$ *is the term given by substituying* $\neg\neg y \to y$ *for every variable* y *in* β*.*

Equational bases

Theorem

If $\mathcal{A}$ is a subvariety in the lattice of subvarieties of $\mathcal{MG}$ given by $\mathcal{A} = \bigvee_{i=1}^{n} \mathcal{A}_i$ for n subvarieties $\mathcal{A}_1, \ldots, \mathcal{A}_n$, where every variety $\mathcal{A}_i$ is generated by a chain $\mathbf{A}_i \oplus \mathbf{B}_i$ where $\mathbf{A} \in \{\mathbf{Ł}_n : n \in \mathbb{N}\} \cup \{\mathbf{Ł}_n^\infty : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$, and there are identities $\gamma_i(x_1^i, \ldots, x_{k_i}^i) \approx 1$ associated with each variety $\mathcal{A}_i$, then, the variety $\mathcal{A}$ as a subvariety of $\mathcal{MG}$ is given by the identity

$$\gamma_{\mathcal{A}}(x_1^1, \ldots, x_{n_1}^1, \ldots, x_n^1, \ldots, x_{k_n}^n) \approx 1$$

where

$$\gamma_{\mathcal{A}}(x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^n, \ldots, x_{k_n}^n) \approx \bigvee_{i=1}^{n} \gamma_i(x_1^i, \ldots, x_{k_i}^i).$$

Example

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$$\underbrace{\bigvee_{i=1}^{2} (\neg\neg x_i^3 \to x_i^3) \leftrightarrow (\neg\neg x_{i+1}^3 \to x_{i+1}^3)}_{\beta_1(x_1^3, x_2^3, x_3^3)} \approx 1$$

$$\underbrace{\bigvee_{i=1}^{4} (\neg\neg x_i^4 \to x_i^4) \leftrightarrow (\neg\neg x_{i+1}^4 \to x_{i+1}^4)}_{\beta_4(x_1^4, x_2^4, x_3^4, x_4^4, x_5^4)} \approx 1$$

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Hence, $\mathcal{A}$ is characterized as a subvariety of $\mathcal{MG}$ by the identity

$$\underbrace{(\alpha_2^{\infty}(x^1) \cdot \beta_1(x_1^3, x_2^3, x_3^3)) \vee (\alpha_2(x^2) \cdot \beta_4(x_1^4, x_2^4, x_3^4, x_4^4, x_5^4))}_{\gamma(x^1, x^2, x_1^3, x_2^3, x_3^3, x_1^4, x_2^4, x_3^4, x_4^4, x_5^4)} \approx 1.$$

Free algebras: the case of MV-algebras

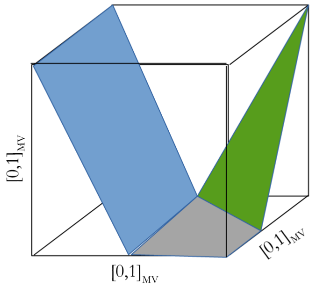
Theorem (McNaughton)

The free n-generated MV-algebra is the subalgebra of $\mathcal{M}_n$ of all continuous piecewise linear functions $f : [0,1]^n \to [0,1]$ where each one of the finitely many linear pieces has integer coefficients.

Free algebras: the case of MV-algebras

Theorem (McNaughton)

The free n-generated MV-algebra is the subalgebra of $\mathcal{M}_n$ of all continuous piecewise linear functions $f : [0,1]^n \to [0,1]$ where each one of the finitely many linear pieces has integer coefficients.



Let $\bar{v}$ be a point in $[0,1]^n$, we denote by

$$Free_{\mathcal{MV}}(n) \restriction \bar{v} := \{[f] : g \in [f] \text{ if } f(\bar{v}) = g(\bar{v}), \text{for } f, g \in Free_{\mathcal{MV}}(n)\}$$

$$Free_{\mathcal{MV}}(n) \restriction (\bar{v}) := \{[(f,U)] : (g,V) \in [(f,U)] \text{ if } f(\bar{x}) = g(\bar{x}) \text{ for every } \bar{x} \in U \cap V$$
$$\text{where } f, g \in Free_{\mathcal{MV}}(n) \text{ and } U, V \text{ are open sets such that } \bar{v} \in U \cap V\}.$$

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THEOREM (PANTI)

Fix $n > 0$ be a natural number and let

$$\mathcal{V} = \mathcal{V}(\{\text{Ł}_{i_1}, \ldots \text{Ł}_{i_k}\} \cup \{\text{Ł}_{j_1}^{\infty}, \ldots, \text{Ł}_{j_l}^{\infty}\})$$

be a proper subvariety of $\mathcal{MV}$. Let X be the set of rational points of the n-cube whose denominator divides at least one $i \in \{i_1, \ldots, i_k\}$, and let Y be the set ot rational points of the n-cube whose denominator divides at least one $j \in \{j_1, \ldots, j_l\}$. Then the free algebra over n generators in $\mathcal{V}$ is isomorphic to the finite product

$$Free_{\mathcal{V}}(n) \cong \prod_{u \in X \setminus Y} Free_{\mathcal{MV}}(n) \restriction u \times \prod_{v \in Y} Free_{\mathcal{MV}}(n) \restriction (v).$$

Free algebras: the case of Gödel hoops

We define the **Gödel chain** $\mathbf{X} = \langle X^1, \ldots, X^r \rangle$ if $X^1, \ldots, X^r$ are subsets of $\{x_1, \ldots, x_n\}$ such that $X^i \cap X^j = \emptyset$ if $i \neq j$ and $X^i \neq \emptyset$, $\forall i = 1, \ldots, r$.

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$$R_{\mathbf{X}} = \left\{ \bar{x} \in [0,1]^n : \begin{array}{lll} x_i = x_j & \text{if} & x_i, x_j \in X^{\sigma(k)}, \text{ for some } k \in \{1, \ldots, r\} \\ x_i < x_j & \text{if} & x_i \in X^{\sigma(k)}, x_j \in X^{\sigma(l)} \text{ for } k < l \\ x_i < x_j & \text{if} & x_i \in X^{\sigma(r)}, x_j \notin \bigcup\limits_{k=1}^{r} X^{\sigma(k)} \end{array} \right\}$$

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Given two Gödel chains $\mathbf{X}_1 = \langle X_1^1, \dots, X_1^r \rangle$ and $\mathbf{X}_2 = \langle X_2^1, \dots, X_2^q \rangle$, we say that $\mathbf{X}_1$ is a subchain of $\mathbf{X}_2$ if $r \leq q$ and $X_1^i = X_2^i$ for $1 \leq i \leq r$.

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We say that a set of Gödel chains defines a Gödel forest if no chain in the set is subchain of other chain.

Free algebras: the case of Gödel hoops

To each Gödel chain $\mathbf{X} = \langle X^1, \ldots, X^r \rangle$ we can associate a function $f_{\mathbf{X}}$:

$$f_{\mathbf{X}} = \begin{cases} x_j & \text{if } \bar{x} \in R_{\mathbf{X}}, \text{ and } x_j \in X^r \\ 1 & \text{otherwise.} \end{cases}$$

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Theorem (Aguzzoli, Bova, Gerla)

A function $f : [0,1]_{\mathbf{G}}^n \to [0,1]_{\mathbf{G}}$ is in $Free_{\mathcal{G}}(n)$ if and only if there is a Gödel forest $\bar{\mathbf{X}}$ containing the Gödel chains $\mathbf{X}_1, \ldots, \mathbf{X}_m$ such that

$$f = \bigwedge_{j=1}^{m} f_{\mathbf{X}_j}.$$

$Free_{\mathcal{G}_l}(n)$

Theorem (Aguzzoli, Bova, Gerla)

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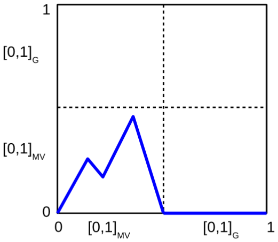
Theorem

The algebra $Free_{\mathcal{G}_l}(n)$ is isomorphic to the quotient of the algebra $Free_{\mathcal{G}}(n)$ over the principal filter generated by the forest where every maximal chain has height $l+1$.

$Free_{\mathcal{MG}}(1)$



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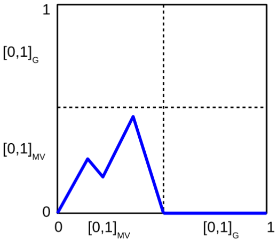

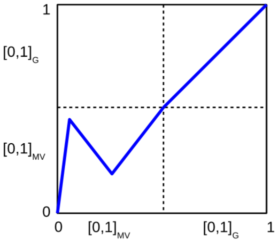

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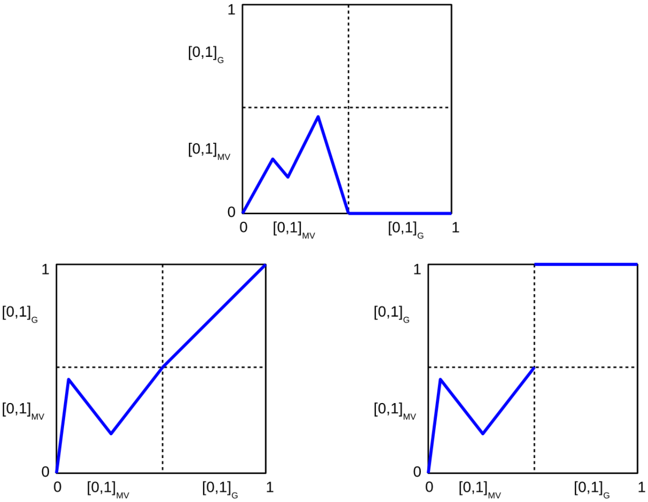
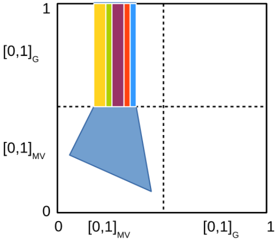
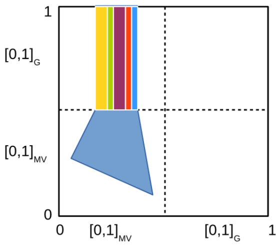
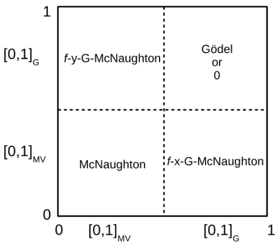


$Free_{\mathcal{MG}}(2)$



$Free_{\mathcal{MG}}(2)$





$Free_{\mathcal{MG}}(n)$

$$\eth[0,1]^n_{\mathbf{MV}} := \{\bar{x} \in [0,1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \leq i \leq n\}.$$

$Free_{\mathcal{MG}}(n)$

$$\eth[0,1]^n_{\mathbf{MV}} := \{\bar{x} \in [0,1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \leq i \leq n\}.$$

For every $\bar{z} = (z_1, \ldots, z_n) \in ([0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}})^n$ we define the projections:

$$\pi_{\mathbf{G}}(\bar{z}) := (z_{j_1}, \ldots, z_{j_m}) \in [0,1]^m_{\mathbf{G}}$$

and

$$\pi_{\mathbf{MV}}(\bar{z}) := (z_{k_1}, \ldots, z_{k_{n-m}}) \in [0,1]^m_{\mathbf{MV}}.$$

$Free_{\mathcal{MG}}(n)$

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If $\bar{x} = (x_1, \ldots, x_n) \in [0,1]^n_{\mathbf{MV}}$ we define:

$$1_{\bar{x}} := \{i \in \{1, \ldots, n\} : x_i = 1\}$$

$$\tilde{x} := \{\bar{z} \in \mathfrak{A}^n \setminus [0,1]^n_{\mathbf{MV}} : \pi_{\mathbf{MV}}(\bar{z}) = \pi_{\mathbf{MV}}(\bar{x})\}$$

and we say that $\tilde{x}$ is the cyllindrification of $\bar{x}$.

$Free_{\mathcal{MG}}(n)$

Theorem

A function $\mathscr{F} : ([0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}})^n \to \mathbf{A}$ is in $Free_{\mathcal{MG}}(n)$ if and only if the following conditions hold:

- *For every $\bar{x} \in ([0,1]_{\mathbf{MV}})^n$, $\mathscr{F}(\bar{x}) = f(\bar{x})$ for some $f \in Free_{\mathcal{MV}}(n)$.*
- *For every $\bar{x} \in \eth[0,1]^n_{\mathbf{MV}}$ such that $\mathscr{F}(\bar{x}) < 1$, $\mathscr{F}(\bar{y}) = \mathscr{F}(\bar{x})$, for every $\bar{y} \in \tilde{x}$.*
- *There is a unimodular triangulation Δ of the rational polyhedra $U = \{\bar{x} \in \eth[0,1]^n_{\mathbf{MV}} : \mathscr{F}(\bar{x}) = 1\}$ such that for every $S \in \Delta$:*
  - *$1_{\bar{y}} = 1_{\bar{z}}$ for every $\bar{y}, \bar{z} \in S \in \Delta$.*
  - *there is a function $g \in Free_{\mathcal{G}}(|1_{\bar{y}}|)$ (for any $\bar{y} \in S$) such that*

    $$\mathscr{F}(\bar{x}) = g(\pi_{[0,1]_{\mathbf{G}}}(\bar{x}))$$

    for every $\bar{x} \in \tilde{y} \in S^\circ$, where

    $$S^\circ := \{\bar{z} \in S : \pi_{\mathbf{MV}}(\bar{z}) \text{ is in the interior of } \pi_{\mathbf{MV}}(S)\}.$$

An **implicative filter** (simply **filter** from now on) in a BL-algebra (or basic hoop) $\mathbf{A}$ is a subset $F \subseteq A$ satisfying that $1 \in F$ and if $x \in F$ and $x \to y \in F$ then $y \in F$.

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For every $\bar{x} \in [0,1]^n_{\mathbf{MV}}$, let $F_{\bar{x}}$ be the filter in $Free_{\mathcal{MG}}(n)$ generated by a function $\mathfrak{F}_{\bar{x}} \in Free_{\mathcal{MG}}(n)$ such that:

- $\mathfrak{F}_{\bar{x}}(\bar{x}) = 1$,
- $\mathfrak{F}_{\bar{x}}(\bar{y}) = 1$ for every $\bar{y} \in \tilde{x}$,
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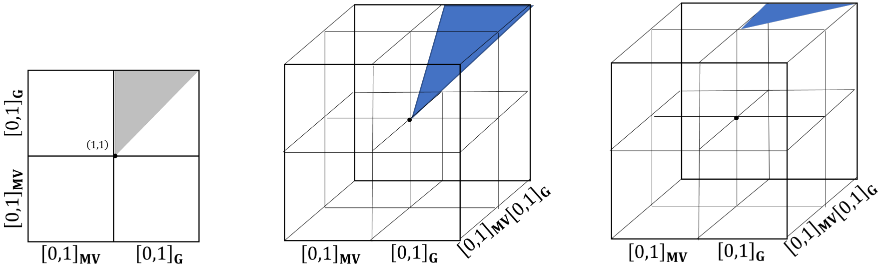
Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and $F_{\bar{x}}$ is the filter in $Free_{\mathcal{MG}}(n)$ generated by a function $\mathfrak{F}_{\bar{x}} \in Free_{\mathcal{MG}}(n)$, then the algebra $Free_{\mathcal{MG}}(n)/F_{\bar{x}}$ is isomorphic to $\text{Ł}_m \oplus Free_{\mathcal{G}}(d)$.

Example

Let $F_{(1,1)} \subseteq Free_{\mathcal{MG}}(2)$ be the filter localized in $(1,1)$. We know that a function $\mathfrak{F} \in Free_{\mathcal{MG}}(2)$ is contained in $F_{(1,1)}$ if and only if $\mathfrak{F}(1,1) = 1$ and for every $(x,y) \in [0,1]^2_{\mathbf{G}}$ such that $x < y$, then either $\mathfrak{F}(x,y) = y$ or $\mathfrak{F}(x,y) = 1$.
If we consider the classes in $Free_{\mathcal{MG}}(2)/F_{(1,1)}$, we have that it is isomorphic to $\mathbf{Ł}_2 \oplus Free_{\mathcal{G}}(2)$, since two functions $\mathfrak{F}_1, \mathfrak{F}_2 \in Free_{\mathcal{MG}}(2)$.



For every $\bar{x} \in [0,1]^n_{\mathbf{MV}}$ and $\bar{y} \in ([0,1]_{\mathbf{MV}})^n \setminus \eth([0,1]_{\mathbf{MV}})^n$ let $F_{[\bar{x},\bar{y})}$ be the principal filter in $Free_{\mathcal{MG}}(n)$ generated by a function $\mathfrak{F}_{[\bar{x},\bar{y})} \in Free_{\mathcal{MG}}(n)$ such that:

- $\mathfrak{F}_{[\bar{x},\bar{y})}(\bar{z}) = 1$ for every $\bar{z} = \bar{x} + \epsilon\bar{y}$, for some $\epsilon \in [0,1)$,
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Definition

Given an MV algebra A, the radical of A, written $Rad(A)$ is the intersection of all maximal filters of A.
A subalgebra S of $\mathbf{Ł}^\infty_m$ is full if it has infinite elements and $S/Rad(S) \cong \mathbf{Ł}_m$.

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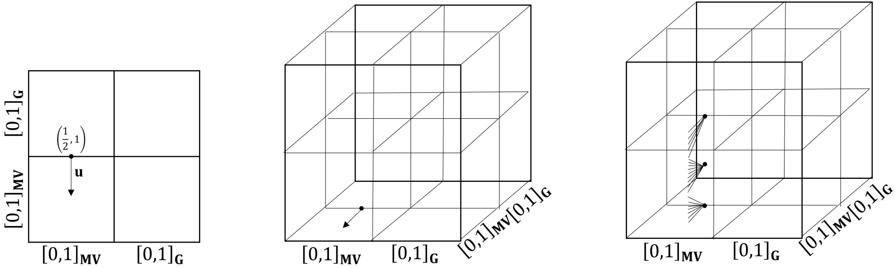
Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and $\bar{y}$ is a rational point contained in $([0,1]_{\mathbf{MV}})^n \setminus \eth([0,1]_{\mathbf{MV}})^n$ then the algebra $Free_{\mathcal{MG}}(n)/F_{[\bar{x},\bar{y})}$ is isomorphic to $\mathbf{A}_i \oplus Free_{\mathcal{G}}(d)$, where $\mathbf{A}_i$ is a full subalgebra of $\mathbf{Ł}^{\infty}_m$, for some $i \in \{0,\ldots,m-1\}$.

Example

Let $F_1 \subseteq Free_{\mathcal{MG}}(2)$ be the prime filter localized in $(\frac{1}{2}, 1)$ and associated with the index $\mathbf{u} = \{(0, -1)\}$. We know that a function $\mathfrak{F} \in Free_{\mathcal{MG}}(2)$ is contained in F_1 if and only if $\mathfrak{F}(\frac{1}{2}, 1 - a) = 1$ for every $a \in [0, \epsilon)$.
We are now in the case when $F_{1MV} \neq MV_{(\frac{1}{2},1)}$.
If we consider the classes in $Free_{\mathcal{MG}}(2)/F_1$, we have that it is isomorphic to a full subalgebra of Ł_2^{∞}, since two functions $\mathfrak{F}_1, \mathfrak{F}_2 \in Free_{\mathcal{MG}}(2)$ are in the same class in the quotient whenever for some $\epsilon > 0$, $\mathfrak{F}_1(\frac{1}{2}, 1 - a) = \mathfrak{F}_2(\frac{1}{2}, 1 - a)$ for every $a \in [0, \epsilon)$.



Generalizations

Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and l is a natural number, with $l \leq n$, then the algebra $Free_{\mathcal{MG}}(n)/F_{\bar{x},l}$ is isomorphic to $\text{Ł}_m \oplus Free_{\mathcal{G}_l}(d)$.

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Let $Free_{\mathcal{MG}}(n) \restriction_{(\bar{x})}$ be the algebra of equivalence classes of pairs $(\mathfrak{F}, U)$, with $\mathfrak{F} \in Free_{\mathcal{MG}}(n)$ and U an open set in $[0,1]^n_{\mathbf{MV}}$ which contains $\bar{x}$. Two such pairs $(\mathfrak{F}_1, U_1)$ and $(\mathfrak{F}_2, U_2)$ are equivalent if $\mathfrak{F}_1 = \mathfrak{F}_2$ on $U_1 \cap U_2$, and the operations are inherited from $Free_{\mathcal{MG}}(n)$.

$Free_{\mathcal{V}}(n)$

THEOREM

Fix $n \in \mathbb{N}$ and let $\mathcal{V} = \mathcal{V}(\text{Ł}_{m_1} \oplus \mathbf{G}_{i_1}, \ldots, \text{Ł}_{m_r} \oplus \mathbf{G}_{i_r}, \text{Ł}_{t_1}^{\infty} \oplus \mathbf{G}_{j_1}, \ldots, \text{Ł}_{t_s}^{\infty} \oplus \mathbf{G}_{j_s})$ be a proper subvariety of $\mathcal{MG}$. Let X be the set of rational points of the cube $[0,1]^n_{\mathbf{MV}}$ whose denominator divides at least one of $m_1, \ldots, m_r$ and let Y be the set of rational points of the cube $[0,1]^n_{\mathbf{MV}}$ whose denominator divides at least one of $t_1, \ldots, t_s$. If A is the algebra in $\mathcal{MG}$ defined by the finite product

$$A = \prod_{\bar{x} \in X} Free_{\mathcal{MG}}(n) \restriction_{\bar{x}, l_x} \times \prod_{\bar{y} \in Y} Free_{\mathcal{MG}}(n) \restriction_{(\bar{y}), l_y}$$

where $l_x = min\{n, \mathcal{O}_{\mathcal{G}}(\bar{x})\}$, $l_y = min\{n, \mathcal{O}_{\mathcal{G}}(\bar{y})\}$ and $\pi_i(\bar{x})$ is the image in A of the i-th projection $\bar{x}_i \in Free_{\mathcal{MG}}(n)$, then the subalgebra $Free_{\mathcal{V}}(n)$ of A generated by $\{\pi_i(\bar{x}) : i < n\}$ is the free algebra over n generators in $\mathcal{V}$, where the elements $\pi_i(\bar{x})$ are the free generators and

$$\mathcal{O}_{\mathcal{G}}(\bar{x}) := max\{i_j : \text{Ł}_m \oplus \mathbf{G}_{i_j} \in \mathcal{V} \text{ and } den(\bar{x}) \text{ divides } m\}.$$

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Aglianò, P., Ferreirim, I. M. A., Montagna, F., *Basic Hoops: an Algebraic Study of Continuous t-norms* Studia Logica, 2007 vol. 87, n. 1, pag. 73-98.

Aglianò, P., Montagna, F., *Varieties of BL-algebras I: general properties* Journal of Pure and Applied Algebra, 2003, vol. 181, n. 2-3, pag. 105-129.

Aglianò, P., Montagna, F., *Varieties of BL-Algebras II* Studia Logica, 2018, vol. 106, n. 4, pag. 721–737.

Aglianò, P., *Varieties of BL-Algebras III* Studia Logica, 2018.

Burris, S., Sankappanavar, H.P., *A Course in Universal Algebra* Springer, New York (1981).

Busaniche, M., Castiglioni, J.L., Lubomirsky, N., *Functional representation of finitely generated free algebras in subvarieties of BL-algebras* Annals of Pure and Applied Logic, 2020, vol. 171, n.2.

Di Nola, A., Lettieri, A., *Equational Characterization of All Varieties of MV-Algebras* Journal of Algebra, 1999, vol. 221, n. 2, pag. 463 - 474.

Esteva, F., Godo, L., Montagna,F., *Equational characterization of the subverieties of BL generated by t-norm algebras* Studia Logica, 2004, vol. 76, n. 2, pag. 161-200.

Hecht, T., Kartiñak, T., *Equational classes of relative Stone algebras* Notre Dame J. Formal Logic 13 (1972), no. 2, 248–254.

] Komori, Y., *Super-Łukasiewicz propositional logic*, Nagoya Math. J. 84 (1981)

] Lubomirsky, N., *Técnicas geométricas y combinatorias en el estudio de subvariedades de BL-álgebras* Tesis Doctoral, Universidad Nacional de La Plata, 2017.

Thank you for your attention!