LATD 2022 and MOSAIC KICK OFF MEETING
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Free algebras in all subvarieties of the variety generated by the MG t-norm

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UNLP - INMABB - CONICET

Definition

A **t-norm** *is a binary operation* $\odot : [0,1]^2 \to [0,1]$ *which satisfies the following conditions:*

1. $\odot$ *is commutative and associative.*
2. $\odot$ *is non decreasing in both arguments, i.e., for every* $x, y, z \in [0,1]$

$$x \leq y \text{ implies } x \odot z \leq y \odot z \text{ and } z \odot x \leq z \odot y,$$

3. $1 \odot x = x$ *and* $0 \odot x = 0$ *for every* $x \in [0,1]$.

A **continuous t-norm** *is a t-norm which is continuous as a map from* $[0,1]^2$ *into* $[0,1]$. *For every continuous t-norm a residuum can be defined by:*

$$x \odot z \leq y \text{ if and only if } x \leq z \to y.$$

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The algebra $([0,1]; \ ; \to; max; min; 0; 1)$ is the standard algebra associated with the continuous t-norm .

Examples

1. Lukasiewicz t-norm: $x *_L y = max(0, x + y - 1)$
   Lukasiewicz implication: $x \to_L y = min(1, 1 - x + y)$,
2. Gödel t-norm: $x *_G y = min(x, y)$,
   Gödel implication:
   $$x \to_G y = \begin{cases} y & \text{if} \quad x > y, \\ 1 & \text{if} \quad x \leq y. \end{cases}$$
3. Product t-norm: $x *_P y = x \cdot y$,
   Goguen implication:
   $$x \to_P y = \begin{cases} y/x & \text{if} \quad x > y, \\ 1 & \text{if} \quad x \leq y. \end{cases}$$

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The algebras $([0,1], *_L, \to_L, \vee, \wedge, 0, 1)$, $([0,1], *_G, \to_G, \vee, \wedge, 0, 1)$ and $([0,1], *_P, \to_P, \vee, \wedge, 0, 1)$ are the Łukasiewicz, Gödel and Product standard algebras, respectively.

If $(a_i,b_i)_{i\in I}$ is a family of disjoint intervals, with $0 \le a_i < b_i \le 1$ such that $*^i$ is a continuous t-norm on (a_i,b_i), we define for every $x,y \in [0,1]$ a continuous t-norm called **ordinal sum of t-norms** by:

$$x * y = \begin{cases} x *^i_{[a_i,b_i]} y & \text{if } x,y \in (a_i,b_i); \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

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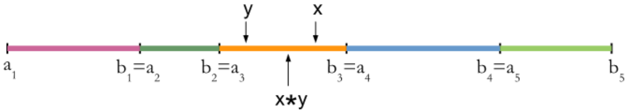

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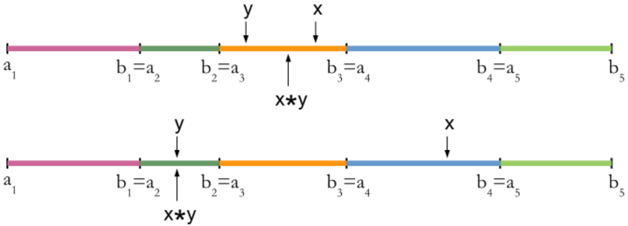
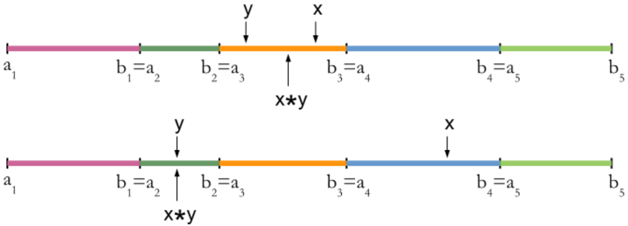


If $(a_i; b_i)_{i2I}$ is a family of disjoint intervals, with 0 $a_i < b_i$ 1 such that ^{i} is a continuous t-norm on $(a_i; b_i)$, we define for every $x; y$ 2 $[0; 1]$ a continuous t-norm called **ordinal sum of t-norms** by:

$$x \quad y = \begin{array}{ll} x \ {}^{i}_{[a_i;b_i]} \ y & \text{if } x; y \ 2 \ (a_i; b_i); \\ min\, fx; yg & \text{otherwise:} \end{array}$$



Theorem (Mostert-Schields)

Every continuous t-norm is the ordinal sum of a family of Lukasiewicz, Gödel and product t-norms.

Hoops and BL-algebras

Definitions

A **hoop** *is an algebra* $\mathbf{A} = (A; \odot; \to; \top)$ *of type* $(2; 2; 0)$, *where* $(A; \odot; \top)$ *is a commutative monoid such that for every* $x; y; z \in A$:

1. $x \to x = \top$,
2. $x \odot (x \to y) = y \odot (y \to x)$,
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A **basic hoop** *es is a hoop which satisfies the equation*

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A **BL-algebra** *is a bounded basic hoop, i.e., an algebra* $\mathbf{A} = (A; \odot; \to; \bot; \top)$ *of type* $(2; 2; 0; 0)$ *such that* $(A; \odot; \to; \top)$ *is a basic hoop and* $\bot$ *is the minimum of* A.

In every basic hoop $\mathbf{A}$ we can define the operations

$$x \wedge y := x \cdot (x \to y) = y \cdot (y \to x),$$

$$x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x)$$

and then $(A, \wedge, \vee, \top)$ is a distributive lattice.

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Theorem (Cignoli, Esteva, Godo, Torrens)

The class of BL-algebras is the variety generated for all the algebras given by continuous t-norms.

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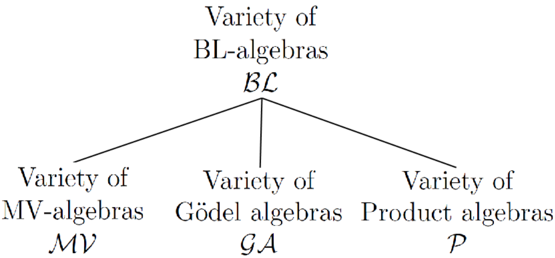

Ordinal sums

Definition

Let $\mathbf{A} = \langle A, \cdot_A, \to_A, \top \rangle$ *and* $\mathbf{B} = \langle B, \cdot_B, \to_B, \top \rangle$ *be two hoops such that* $A \cap B = \{\top\}$. *We can define the* **ordinal sum** *of* $\mathbf{A}$ *and* $\mathbf{B}$ *as the hoop* $\mathbf{A} \oplus \mathbf{B} = \langle A \cup B, \cdot, \to, \top \rangle$, *where the operations* $\cdot$ *and* $\to$ *are given by:*

$$x \cdot y = \begin{cases} x \cdot_A y & \text{if} \quad x, y \in A; \\ x \cdot_B y & \text{if} \quad x, y \in B; \\ x & \text{if} \quad x \in A \setminus \{\top\},\ y \in B; \\ y & \text{if} \quad y \in B \setminus \{\top\},\ x \in A. \end{cases}$$

$$x \to y = \begin{cases} x \to_A y & \text{if} \quad x, y \in A; \\ x \to_B y & \text{if} \quad x, y \in B; \\ \top & \text{if} \quad x \in A \setminus \{\top\},\ y \in B; \\ y & \text{if} \quad y \in A,\ x \in B. \end{cases}$$

The subvariety MG

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This variety is generated by the t-norm which we called MG t-norm $t_{MG} : [0;1]^2 \;\; ! \;\; [0;1]$ defined by

$$t_{MG}(x;y) = \begin{array}{ll} max(0; x + y \quad \frac{1}{2}) & \text{if } x;y \;\, 2 \; [0; \frac{1}{2}); \\ min(x;y) & \text{otherwise:} \end{array}$$

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Moreover, it is the subvariety of BL given by the identity

$$(: :x \; ! \; x)^2 \quad : :x \; ! \; x:$$

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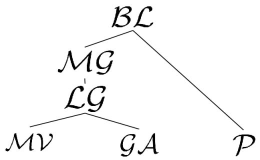
Moreover, it is the subvariety of BL given by the identity

$$(\neg\neg x \to x)^2 \quad \neg\neg x \to x.$$



THE LATTICE OF SUBVARIETIES (MV)

For $n; k$ 1 we define

$$\mathbf{L}_n = \Gamma(Z; n),$$
$$\mathbf{L}_n^{1} = \Gamma(Z \quad Z; (n; 0)),$$

where Z has the natural order and Z   Z is the product of two copies of Z ordered lexicographically.

The lattice of subvarieties (M V)

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The lattice of subvarieties (G)

Hetch and Katriñak proved that the subvarieties of the variety of Gödel algebras form a chain, but since the Gödel hoops are the subreducts of these algebras, the results can be naturally extended for our case.

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Theorem

If A is a join-irreducible element in the lattice of subvarieties of $M\,G$ then A is the variety generated by **A** **B** *with* **A** *is a chain in $M\,V$ and* **B** *is a chain in G such that*

$$\mathbf{A}\ 2\ f\mathbf{L}_n : n\ 2\ \mathrm{N}g\ [\ f\mathbf{L}_n^{1}\ : n\ 2\ \mathrm{N}g\ [\ f[0;1]_{\mathbf{MV}}g$$

$$\mathbf{B}\ 2\ f\mathbf{G}_n : n\ 2\ \mathrm{N}g\ [\ f[0;1]_{\mathbf{G}}g.$$

Theorem

If A is a join-irreducible element in the lattice of subvarieties of MG then A is the variety generated by **A** **B** *with* **A** *is a chain in MV and* **B** *is a chain in G such that*

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$$\mathbf{B}\ 2\ f\mathbf{G}_n : n\ 2\ \mathrm{N}g\ [\ f[0;1]_{\mathbf{G}}g.$$

Corollary

The join-irreducible elements in the lattice of subvarieties of MG form an ordered lattice.

$$(2\quad D(\mathrm{N}))\quad (!+1)$$

Theorem

Any subvariety U of MG is generated by a finite number of chains $\mathbf{A} \times \mathbf{B}$ *with* $\mathbf{A} \in \{\mathbf{L}_n : n \in \mathbb{N}\} \cup \{\mathbf{L}_n^{1} : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$ *and* $\mathbf{B} \in \{\mathbf{G}_n : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{G}}\}$.

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Idea of the proof:

Corollary

Any variety in $\Lambda(\mathcal{MG})$ is a join of finitely many varieties generated by a single chain $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} \in \{\mathbf{L}_n : n \in \mathbb{N}\} \cup \{\mathbf{L}_n^{\infty} : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$ and $\mathbf{B} \in \{\mathbf{G}_n : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{G}}\}$.

Equational bases

Lemma (Di Nola - Lettieri)

For $n \quad 2$, the subvariety $V(\mathbf{L}_n^1)$ of MV is characterized by the identity:

$$(((n+1)x^n)^2 \ \$ \ 2x^{n+1}) \wedge ((p{:}x^{p \quad 1})^{n+1} \ \$ \ (n+1)x^p) \quad 1 \ (\ {}^{n}_{1})$$

for every positive integer $1 < p < n$ such that p is not a divisor of n.

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for every positive integer $1 < p < n$ such that p is not a divisor of n and every integer q such that $1 < q < n$ and q divides n.

Equational bases

Lemma (Hecht - Katriñak)

For n 2, the subvariety $V(G_n)$ of G is characterized by the following identity:

$$\overline{}_{i=1}^{n+1} (x_i \; \$ \; x_{i+1}) \quad 1 \; (\;_n)$$

Equational bases

Theorem

If **A** *is a subvariety of* MV *characterized by the identity* 1 *and* **B** *is a subvariety of* G *characterized by the identity* 1*, then* **A** **B** *is a subvariety of* MG *characterized by the identity*

$$^{0} \quad ^{0} \quad 1;$$

where ^{0} *is the term given by substituying* $: : x$ *for every variable* x *in* *and* ^{0} *is the term given by substituying* $: : y \; ! \; y$ *for every variable* y *in* .

Equational bases

Theorem

If $\mathcal{A}$ is a subvariety in the lattice of subvarieties of $\mathcal{MG}$ given by $\mathcal{A} = \bigvee_{i=1}^{n} \mathcal{A}_i$ for n subvarieties $\mathcal{A}_1, \ldots, \mathcal{A}_n$, where every variety $\mathcal{A}_i$ is generated by a chain $\mathbf{A}_i \times \mathbf{B}_i$ where $\mathbf{A} \in \{\mathbf{L}_n : n \in \mathbb{N}\} \cup \{\mathbf{L}_n^1 : n \in \mathbb{N}\} \cup \{[0,1]_{\mathbf{MV}}\}$, and there are identities $\varphi_i(x_1^i, \ldots, x_{k_i}^i) \approx 1$ associated with each variety $\mathcal{A}_i$, then, the variety $\mathcal{A}$ as a subvariety of $\mathcal{MG}$ is given by the identity

$$\varphi_{\mathcal{A}}(x_1^1, \ldots, x_{n_1}^1, \ldots, x_n^1, \ldots, x_{k_n}^n) \approx 1$$

where

$$\varphi_{\mathcal{A}}(x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^n, \ldots, x_{k_n}^n) \approx \bigvee_{i=1}^{n} \varphi_i(x_1^i, \ldots, x_{k_i}^i).$$

Example

Suppose that we have the variety $A = V(\mathbf{L}_2^{1} \quad \mathbf{G}_1; \mathbf{L}_2 \quad \mathbf{G}_3)$.

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$$\underbrace{(3(: : x^1)^2)^2 \ \$ \ 2(: : x^1)^3}_{{}_2^{1}(x^1)} \quad 1$$

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$$\underbrace{\overset{2}{\underset{i=1}{-}} (: : x_i^3 \ ! \ x_i^3) \ \$ \ (: : x_{i+1}^3 \ ! \ x_{i+1}^3)}_{{}_1(x_1^3; x_2^3; x_3^3)} \quad 1$$

$$\underbrace{\overset{4}{\underset{i=1}{-}} (: : x_i^4 \ ! \ x_i^4) \ \$ \ (: : x_{i+1}^4 \ ! \ x_{i+1}^4)}_{{}_4(x_1^4; x_2^4; x_3^4; x_4^4; x_5^4)} \quad 1$$

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Hence, A is characterized as a subvariety of MG by the identity

$$\underbrace{({}_2^{\mathit{1}}\,(x^1) \quad {}_1(x_1^3;x_2^3;x_3^3)) \_ ({}_2(x^2) \quad {}_4(x_1^4;x_2^4;x_3^4;x_4^4;x_5^4))}_{(x^1;x^2;x_1^3;x_2^3;x_3^3;x_1^4;x_2^4;x_3^4;x_4^4;x_5^4)} \quad 1.$$

Free algebras: the case of MV-algebras

Theorem (McNaughton)

The free n-generated MV-algebra is the subalgebra of M_n *of all continuous piecewise linear functions* $f : [0;1]^n \; ! \; [0;1]$ *where each one of the nitely many linear pieces has integer coe cients.*

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Let $\bar v$ be a point in $[0;1]^n$, we denote by

$$Free_{MV}(n)_{\bar v} := \{[f] : g \in [f] \text{ if } f(\bar v) = g(\bar v); \text{ for } f; g \in Free_{MV}(n)\}$$

$$Free_{MV}(n)_{(\bar v)} := \{[(f;U)] : (g;V) \in [(f;U)] \text{ if } f(\bar x) = g(\bar x) \text{ for every } \bar x \in U \cap V$$

$$\text{where } f; g \in Free_{MV}(n) \text{ and } U;V \text{ are open sets such that } \bar v \in U \cap V\}.$$

Let $\bar{v}$ be a point in $[0;1]^n$, we denote by

$$Free_{MV}(n) \quad \bar{v} := f[f] : g \; 2 \; [f] \text{ if } f(\bar{v}) = g(\bar{v}); \text{ for } f;g \; 2 \; Free_{MV}(n)g$$

$$Free_{MV}(n) \quad (\bar{v}) := f[(f;U)] : (g;V) \; 2 \; [(f;U)] \text{ if } f(\bar{x}) = g(\bar{x}) \text{ for every } \bar{x} \; 2 \; U \setminus V$$

$$\text{where } f;g \; 2 \; Free_{MV}(n) \text{ and } U;V \text{ are open sets such that } \bar{v} \; 2 \; U \setminus Vg:$$

THEOREM (PANTI)

Fix $n > 0$ be a natural number and let

$$V = V(f\mathbf{L}_{i_1};:::\mathbf{L}_{i_k}g \; [\; f\mathbf{L}_{j_1}^{1};:::;\mathbf{L}_{j_l}^{1}g)$$

be a proper subvariety of MV. Let X be the set of rational points of the n-cube whose denominator divides at least one $i \; 2 \; fi_1;:::;i_kg$, and let Y be the set ot rational points of the n-cube whose denominator divides at least one $j \; 2 \; fj_1;:::;j_lg$. Then the free algebra over n generators in V is isomorphic to the nite product

$$Free_V(n) = \prod_{u \, 2 \, X n Y} Free_{MV}(n) \quad u \quad \prod_{v \, 2 \, Y} Free_{MV}(n) \quad (v):$$

Free algebras: the case of Gödel hoops

We define the Gödel chain $X = \langle X^1, \ldots, X^r \rangle$ if $X^1, \ldots, X^r$ are subsets of $\{x_1, \ldots, x_n\}$ such that $X^i \cap X^j = \emptyset$ if $i \neq j$ and $X^i \neq \emptyset$, $\forall i = 1, \ldots, r$.

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$$R_X = \left\{ x \in [0,1]^n : \begin{array}{lll} x_i = x_j & \text{if} & x_i, x_j \in X^{(k)}, \text{ for some } k \in \{1, \ldots, r\} \\ x_i < x_j & \text{if} & x_i \in X^{(k)}, x_j \in X^{(l)} \text{ for } k < l \\ x_i < x_j & \text{if} & x_i \in X^{(r)}, x_j \notin \bigcup_{k=1}^{r} X^{(k)} \end{array} \right\}$$

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Given two Gödel chains $X_1 = \langle X_1^1, \ldots, X_1^r \rangle$ and $X_2 = \langle X_2^1, \ldots, X_2^q \rangle$, we say that X_1 is a subchain of X_2 if $r \leq q$ and $X_1^i = X_2^i$ for $1 \leq i \leq r$.

Free algebras: the case of Gödel hoops

We define the **Gödel chain** $\mathbf{X} = \langle X^1, \ldots, X^r \rangle$ if $X^1, \ldots, X^r$ are subsets of $\{x_1, \ldots, x_n\}$ such that $X^i \cap X^j = \emptyset$ if $i \neq j$ and $X^i \neq \emptyset$, $\forall i = 1, \ldots, r$.

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Given two Gödel chains $\mathbf{X}_1 = \langle X_1^1, \ldots, X_1^r \rangle$ and $\mathbf{X}_2 = \langle X_2^1, \ldots, X_2^q \rangle$, we say that $\mathbf{X}_1$ is a subchain of $\mathbf{X}_2$ if $r \leq q$ and $X_1^i = X_2^i$ for $1 \leq i \leq r$.
We say that a set of Gödel chains defines a Gödel forest if no chain in the set is subchain of other chain.

Free algebras: the case of Gödel hoops

To each Gödel chain $\mathbf{X} = \langle X^1, \ldots, X^r \rangle$ we can associate a function $f_{\mathbf{X}}$:

$$f_{\mathbf{X}} = \begin{cases} x_j & \text{if } \bar{x} \in R_{\mathbf{X}}, \text{ and } x_j \in X^r \\ 1 & \text{otherwise.} \end{cases}$$

Free algebras: the case of Gödel hoops

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Theorem (Aguzzoli, Bova, Gerla)

A function $f : [0,1]^n_{\mathbf{G}} \to [0,1]_{\mathbf{G}}$ is in $Free_G(n)$ if and only if there is a Gödel forest $\bar{\mathbf{X}}$ containing the Gödel chains $\mathbf{X}_1, \dots, \mathbf{X}_m$ such that

$$f = \bigvee_{j=1}^{m} f_{\mathbf{X}_j}.$$

$Free_{G_l}(n)$

Theorem (Aguzzoli, Bova, Gerla)

A function $f : [0,1]^n_{\mathbf{G}_l} \to [0,1]_{\mathbf{G}_l}$ *is in* $Free_{G_l}(n)$ *if and only if there is a Gödel forest* $\bar{\mathbf{X}}$ *containing the chains* $\mathbf{X}_1, \ldots, \mathbf{X}_m$ *where no chain has height greater than* l *such that*

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Theorem

The algebra $Free_{G_l}(n)$ is isomorphic to the quotient of the algebra $Free_G(n)$ over the principal filter generated by the forest where every maximal chain has height $l+1$.

$Free_{MG}(1)$

Free$_{MG}$ (1)

Free_{MG} (1)

Free_{MG} (2)

Free_{MG} (2)

$Free_{MG}(n)$

$$g[0;1]^n_{\mathbf{MV}} := f\bar{x}\ 2\ [0;1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \quad i \quad ng:$$

$Free_{MG}(n)$

$$g[0,1]^n_{\mathbf{MV}} := \{\bar{x} \in [0,1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \quad i \quad n\}.$$

For every $\bar{z} = (z_1, \ldots, z_n) \in ([0,1]_{\mathbf{MV}} \quad [0,1]_{\mathbf{G}})^n$ we define the projections:

$$_{\mathbf{G}}(\bar{z}) := (z_{j_1}, \ldots, z_{j_m}) \in [0,1]^m_{\mathbf{G}}$$

and

$$_{\mathbf{MV}}(\bar{z}) := (z_{k_1}, \ldots, z_{k_{n \quad m}}) \in [0,1]^m_{\mathbf{MV}}.$$

Free$_{MG}(n)$

$$g[0;1]^n_{\mathbf{MV}} := f\bar{x}\ 2\ [0;1]^n_{\mathbf{MV}} : x_i = 1 \text{ for some } 1 \quad i \quad ng:$$

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If $\bar{x} = (x_1; : : : ; x_n)\ 2\ [0;1]^n_{\mathbf{MV}}$ we define:

$$1_{\bar{x}} := fi\ 2\ f1; : : : ; ng : x_i = 1g$$

$$\tilde{x} := f\bar{z}\ 2\ \mathrm{A}^n\ n\ [0;1]^n_{\mathbf{MV}} : \quad _{\mathbf{MV}}(\bar{z}) = \quad _{\mathbf{MV}}(\bar{x})g$$

and we say that $\tilde{x}$ is the cyllindrification of $\bar{x}$.

$\mathrm{Free}_{MG}(n)$

[redacted]

A function $F : ([0,1]_{MV} \quad [0,1]_G)^n \to A$ is in $\mathrm{Free}_{MG}(n)$ if and only if the following conditions hold:

- For every $x \in ([0,1]_{MV})^n$, $F(x) = f(x)$ for some $f \in \mathrm{Free}_{MV}(n)$.
- For every $x \in g[0,1]^n_{MV}$ such that $F(x) < 1$, $F(y) = F(x)$, for every $y \in x$.
- There is a unimodular triangulation of the rational polyhedra $U = \{x \in g[0,1]^n_{MV} : F(x) = 1\}$ such that for every $S \in$:
  - $1_y = 1_z$ for every $y, z \in S \in$.
  - there is a function $g \in \mathrm{Free}_G(|1_y|)$ (for any $y \in S$) such that

$$F(x) = g(\ _{[0,1]_G}(x))$$

for every $x \in y \in S$, where

$$S \ := \{z \in S : \ _{MV}(z) \text{ is in the interior of } \ _{MV}(S)\}.$$

An implicative filter (simply filter from now on) in a BL-algebra (or basic hoop) A is a subset $F \subseteq A$ satisfying that $1 \in F$ and if $x \in F$ and $x \to y \in F$ then $y \in F$.

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For every $x \in [0;1]^n_{MV}$, let F_x be the filter in $Free_{MG}(n)$ generated by a function $F_x \in Free_{MG}(n)$ such that:

- $F_x(x) = 1$,
- $F_x(y) = 1$ for every $y \in \tilde{x}$,
- $F_x(y) < 1$ for every $y \in [0;1]^n_{MV} \setminus \{x\}$.

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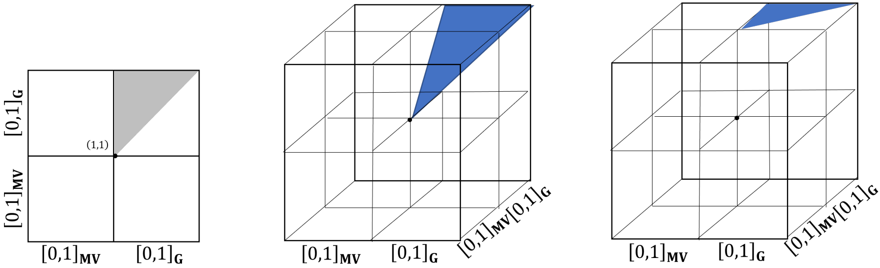
[redacted]

If x is a rational point contained in $[0,1]^n_{MV}$ such that $den(x) = m$ and $|x| = d$, and F_x is the filter in $Free_{MG}(n)$ generated by a function $F_x \in Free_{MG}(n)$, then the algebra $Free_{MG}(n)/F_x$ is isomorphic to $L_m \oplus Free_G(d)$.

Example

Let $F_{(1;1)}$ $\mathit{Free}_{MG}(2)$ be the filter localized in $(1;1)$. We know that a function F 2 $\mathit{Free}_{MG}(2)$ is contained in $F_{(1;1)}$ if and only if $\mathrm{F}(1;1) = 1$ and for every $(x;y)$ 2 $[0;1]^2_{\mathbf{G}}$ such that $x < y$, then either $\mathrm{F}(x;y) = y$ or $\mathrm{F}(x;y) = 1$.
If we consider the classes in $\mathit{Free}_{MG}(2)/F_{(1;1)}$, we have that it is isomorphic to $\mathbf{L}_2$ $\mathit{Free}_{G}(2)$, since two functions $\mathrm{F}_1;\mathrm{F}_2$ 2 $\mathit{Free}_{MG}(2)$.



For every $\bar{x}\ 2\ [0;1]^n_{\mathbf{MV}}$ and $\bar{y}\ 2\ ([0;1]_{\mathbf{MV}})^n\ n\ \mathrm{g}([0;1]_{\mathbf{MV}})^n$ let $F_{[\bar{x};\bar{y})}$ be the principal filter in $Free_{MG}(n)$ generated by a function $\mathsf{F}_{[\bar{x};\bar{y})}\ 2\ Free_{MG}(n)$ such that:

- $\mathsf{F}_{[\bar{x};\bar{y})}(\bar{z}) = 1$ for every $\bar{z} = \bar{x} + \ \bar{y}$, for some $\ 2\ [0;1)$,
- $\mathsf{F}_{[\bar{x};\bar{y})}(\bar{z}) < 1$ for every $\bar{z}\ 6 = \bar{x} + \ \bar{y}$, for some $\ 2\ [0;1)$,

For every $\bar{x} \in [0;1]^n_{\mathbf{MV}}$ and $\bar{y} \in ([0;1]_{\mathbf{MV}})^n \setminus \{([0;1]_{\mathbf{MV}})^n$ let $F_{[\bar{x};\bar{y})}$ be the principal filter in $Free_{MG}(n)$ generated by a function $\mathrm{F}_{[\bar{x};\bar{y})} \in Free_{MG}(n)$ such that:

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Definition

Given an MV algebra A, the radical of A, written $Rad(A)$ is the intersection of all maximal filters of A.
A subalgebra S of $\mathbf{L}^1_m$ is full if it has infinite elements and $S{=}Rad(S) = \mathbf{L}_m$.

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Theorem

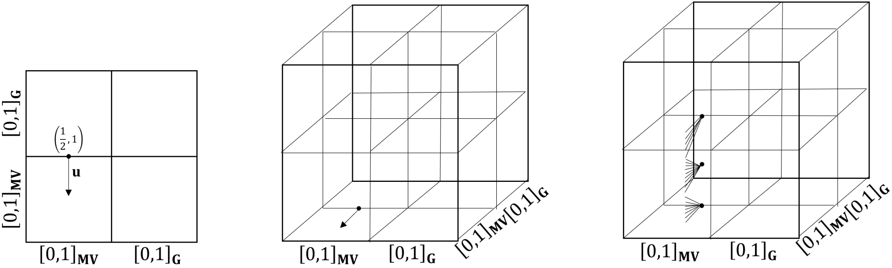
If $\bar{x}$ is a rational point contained in $[0;1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $j\bar{x}j = d$, and $\bar{y}$ is a rational point contained in $([0;1]_{\mathbf{MV}})^n$ n g$([0;1]_{\mathbf{MV}})^n$ then the algebra $Free_{MG}(n){=}F_{[\bar{x};\bar{y})}$ is isomorphic to $\mathbf{A}_i$ $Free_G(d)$, where $\mathbf{A}_i$ is a full subalgebra of $\mathbf{L}^{1}_{m}$, for some i 2 f0; : : : ; m 1g.

Example

Let $F_1 \quad Free_{MG}(2)$ be the prime filter localized in $(\frac{1}{2}; 1)$ and associated with the index $\mathbf{u} = f(0; \quad 1)g$. We know that a function F 2 $Free_{MG}(2)$ is contained in F_1 if and only if $\mathrm{F}(\frac{1}{2}; 1 \quad a) = 1$ for every a 2 $[0; \ )$.

We are now in the case when $F_{1MV} \not\subseteq MV_{(\frac{1}{2};1)}$.

If we consider the classes in $Free_{MG}(2)/F_1$, we have that it is isomorphic to a full subalgebra of $\mathbf{L}_2^{1}$, since two functions $\mathrm{F}_1; \mathrm{F}_2$ 2 $Free_{MG}(2)$ are in the same class in the quotient whenever for some $\ > 0$, $\mathrm{F}_1(\frac{1}{2}; 1 \quad a) = \mathrm{F}_2(\frac{1}{2}; 1 \quad a)$ for every a 2 $[0; \ )$.



GENERALIZATIONS

THEOREM

If $\bar{x}$ is a rational point contained in $[0;1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $j\bar{x}j = d$, and l is a natural number, with $l \quad n$, then the algebra $Free_{MG}(n)/F_{\bar{x};l}$ is isomorphic to $\mathbf{L}_m \quad Free_{G_l}(d)$.

Generalizations

Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and l is a natural number, with $l \leq n$, then the algebra $Free_{\mathcal{MG}}(n)/F_{\bar{x},l}$ is isomorphic to $\mathbf{L}_m \times Free_{G_l}(d)$.

Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and $\bar{y}$ is a rational point contained in $([0,1]_{\mathbf{MV}})^n \setminus g([0,1]_{\mathbf{MV}})^n$ then the algebra $Free_{\mathcal{MG}}(n)/F_{[\bar{x},\bar{y}),l}$ is isomorphic to $\mathbf{A}_i \times Free_{G_l}(d)$, where $\mathbf{A}_i$ is a full subalgebra of $\mathbf{L}^1_m$, for some $i \in \{0, \ldots, m-1\}$.

Generalizations

Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and l is a natural number, with $l \leq n$, then the algebra $Free_{MG}(n)/F_{\bar{x},l}$ is isomorphic to $\mathbf{L}_m \times Free_{G_l}(d)$.

Theorem

If $\bar{x}$ is a rational point contained in $[0,1]^n_{\mathbf{MV}}$ such that $den(\bar{x}) = m$ and $|\bar{x}| = d$, and $\bar{y}$ is a rational point contained in $([0,1]_{\mathbf{MV}})^n \setminus g([0,1]_{\mathbf{MV}})^n$ then the algebra $Free_{MG}(n)/F_{[\bar{x},\bar{y}),l}$ is isomorphic to $\mathbf{A}_i \times Free_{G_l}(d)$, where $\mathbf{A}_i$ is a full subalgebra of $\mathbf{L}_m^{1}$, for some $i \in \{0,\dots,m-1\}$.

Let $Free_{MG}(n)_{(\bar{x})}$ be the algebra of equivalence classes of pairs (F,U), with $F \in Free_{MG}(n)$ and U an open set in $[0,1]^n_{\mathbf{MV}}$ which contains $\bar{x}$. Two such pairs (F_1,U_1) and (F_2,U_2) are equivalent if $F_1 = F_2$ on $U_1 \cap U_2$, and the operations are inherited from $Free_{MG}(n)$.

$Free_V(n)$

THEOREM

Fix n 2 N and let $V = V(\mathbf{L}_{m_1} \quad \mathbf{G}_{i_1}; : : : ; \mathbf{L}_{m_r} \quad \mathbf{G}_{i_r}; \mathbf{L}^{1}_{t_1} \quad \mathbf{G}_{j_1}; : : : ; \mathbf{L}^{1}_{t_s} \quad \mathbf{G}_{j_s})$ be a proper subvariety of $M\,G$. Let X be the set of rational points of the cube $[0;1]^n_{\mathbf{MV}}$ whose denominator divides at least one of $m_1; : : : ; m_r$ and let Y be the set of rational points of the cube $[0;1]^n_{\mathbf{MV}}$ whose denominator divides at least one of $t_1; : : : ; t_s$. If A is the algebra in $M\,G$ de ned by the nite product

$$A = \prod_{\bar{x}\,2\,X} Free_{M\,G}(n) \quad {}_{\bar{x};l_x} \qquad \prod_{\bar{y}\,2\,Y} Free_{M\,G}(n) \quad {}_{(\bar{y});l_y}$$

where $l_x = \min f n; O_G(\bar{x}) g$, $l_y = \min f n; O_G(\bar{y}) g$ and $\ _i(\bar{x})$ is the image in A of the i-th projection $\bar{x}_i$ 2 $Free_{M\,G}(n)$, then the subalgebra $Free_V(n)$ of A generated by $f \ _i(\bar{x}) : i < n g$ is the free algebra over n generators in V, where the elements $\ _i(\bar{x})$ are the free generators and

$$O_G(\bar{x}) := \max f i_j : \mathbf{L}_m \quad \mathbf{G}_{i_j} \ 2\ V \text{ and } den(\bar{x}) \text{ divides } m g:$$

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Thank you for your attention!