Hereditary Structural Completeness over K4: Rybakov's Theorem Revisited

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Background

Definition A rule Γ/φ is said to be admissible for a deductive system $\vdash$ iff the set of tautologies of $\vdash$ is closed under applications of Γ/φ.
It is derivable for $\vdash$ iff $\Gamma \vdash \varphi$.

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Whilst every derivable rule for a given deductive system is admissible the converse can fail.

This gap has motivated an in depth study of admissibility, including Friedman [1975], Rybakov [1984], Iemhoff [2001] and Jeřábek [2010].

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Investigations by Prucnal [1972] and Dzik & Wroǹski [1973] among others suggested that whilst a full characterisation of SC intermediate and modal logics was out of reach a herediarily strucutrally complete (HSC) characterisation might be possible.

Definition If every finitary extension of $\vdash$ is structurally complete then we say $\vdash$ is HSC.

Citkin's Theorem

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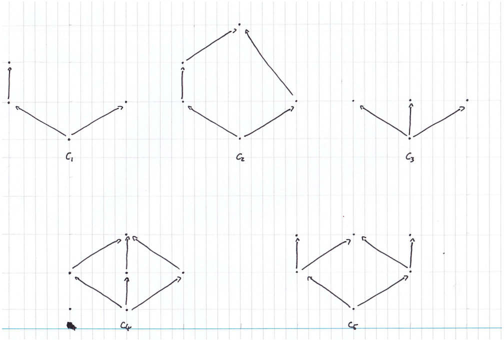
Citkin's Theorem [1978] In order for an intermediate logic Λ to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(C_i) : 1 \leq i \leq 5$.



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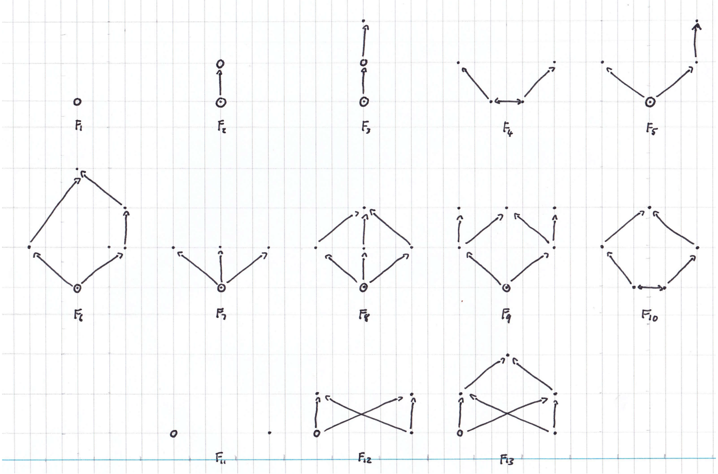
This strategy can also be applied to the modal case and Rybakov's result.

However, more than simply provide a new proof, this approach illuminates a mistake in Rybakov's characterisation. It is too restrictive and misses an infinite collection of HSC transitive modal logics.

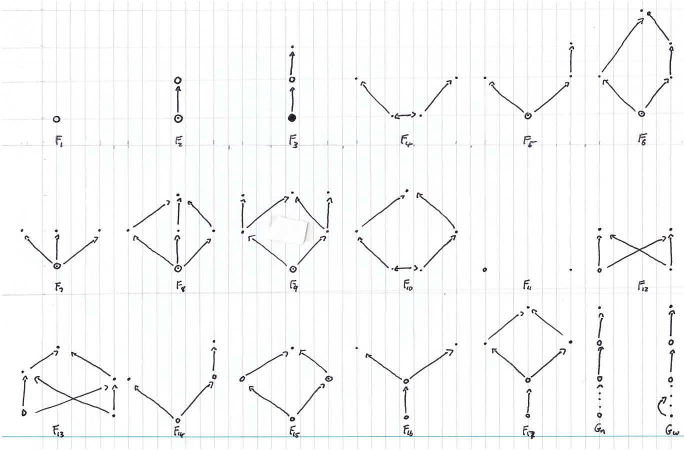
We want to both correct and prove the characterisation.

The Two Characterisations

Rybakov's Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 13$.



Revised Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 17$ and $Log(G_n)$ for some $n \in \omega$.



Proof Strategy

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Definition A variety $\mathcal{A}$ is primitive iff every sub-quasivarity of $\mathcal{A}$ is a variety.

Theorem A normal modal logic Λ over K4 is HSC iff its corresponding variety $\mathcal{A}$ is primitive.

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Definition An algebra A is weakly projective in a variety $\mathcal{A}$ iff $\forall B \in \mathcal{A}$ iff $A \in \mathbb{H}(B)$ then $A \in \mathbb{IS}(B)$.

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Lemma Let $\mathcal{A}$ be a variety of K4-algebras.

(i) If $\mathcal{A}$ is primitive then the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$.

(ii) Suppose all sub-vareties of $\mathcal{A}$ have the FMP. If the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$ then $\mathcal{A}$ is primitive.

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Definition A transitive space is a triple $\mathcal{X} := (X, \tau, R)$ where (X, R) is a Kripke frame, (X, τ) is a Stone space and such that

(i) $R[x]$ is closed for all $x \in X$;

(ii) $R^{-1}[U]$ is clopen for all clopen $U \subseteq X$;

(iii) R is a transitive relation.

Theorem The category of K4-algebras and category of transitive spaces are dually equivalent.

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Algebra	Topology
FSI	Rooted
Sub-algebra	Quotient Space
Quotient Algebra	Closed Upset
Direct Product	Disjoint Union

Explaining the Mistake

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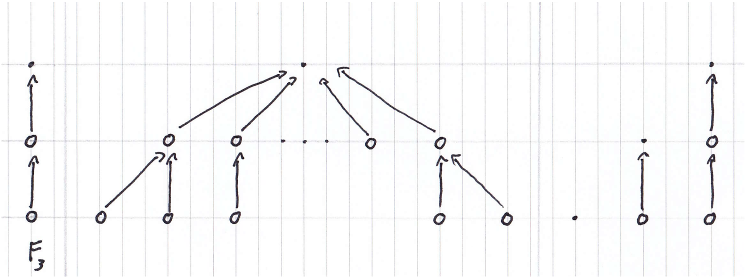

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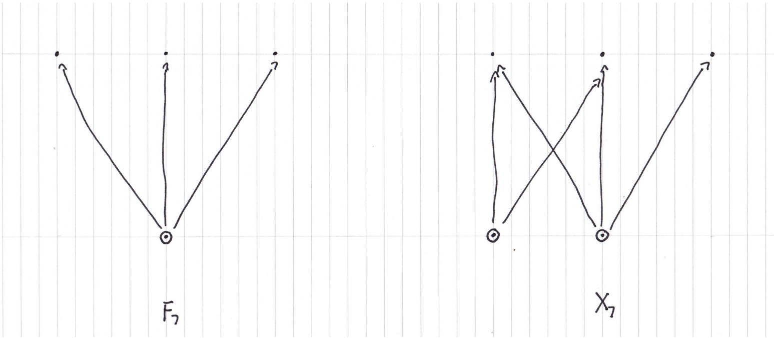

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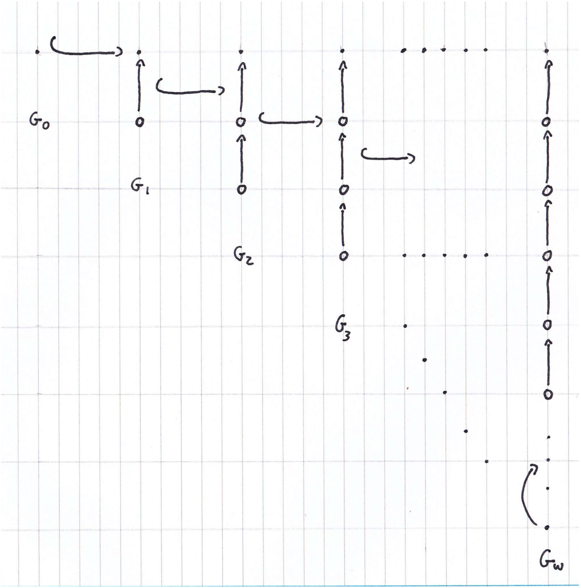

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We first establish a detailed description of the finitely generated, non-trivial SI members of the varieties.

This requires establishing a group of results demonstrating certain frame substructures never appear in our spaces.

The Difficult Direction

Theorem Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in V$ be finitely generated, non-trivial and SI. Then the frame underlying A_* is a sequential composition of frames $\bigoplus_{\alpha \leq \beta} Q_\alpha$ for some $\beta \in Ord$ and such that:

$$Q_\alpha \text{ is } \begin{cases} \text{a single cluster} & \text{if } \alpha = \beta \text{ or } \alpha \text{ is a limit ordinal} \\ \text{a single cluster, a two cluster anti-chain or } H & \text{if } \alpha = 0 \\ \text{a single cluster or a two cluster anti-chain} & \text{otherwise} \end{cases}$$

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Moreover: Any maximal clusters are single reflexive points

If Q_α is a two cluster anti-chain then clusters in $Q_{\alpha+1}$ are improper.

If A_* contains an irreflexive point then $\beta = \lambda + n$ for some limit ordinal λ, $n \neq 0$ and $\exists 0 < m \leq n : \forall \alpha < \lambda + m \; Q_\alpha$ contains no irreflexive points, $\forall k \geq m \; Q_{\lambda+k}$ is a single irreflexive point and if $m < n$ then $Q_{\lambda+m-1}$ is a single cluster.

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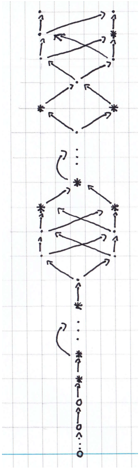
Theorem Let $\mathcal{A}$ be one of our varieties, let $A \in V$ be finitely generated, non-trivial and SI. The frame underlying A_* has the following structure:



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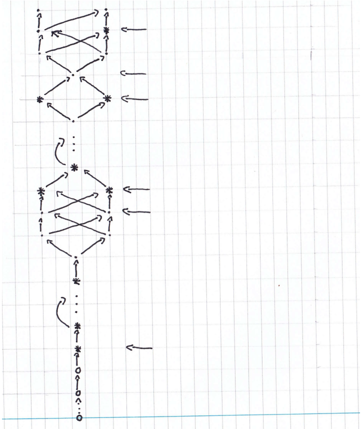
Theorem All our varieties have the FMP.

Proof Sketch: We follow a variation on the drop point technique of K. Fine.



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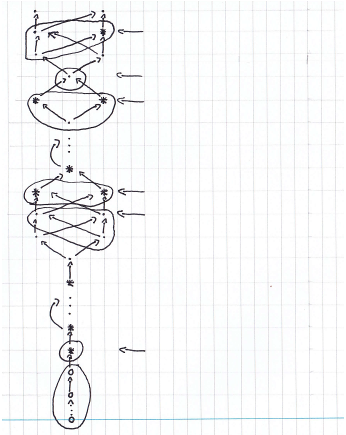

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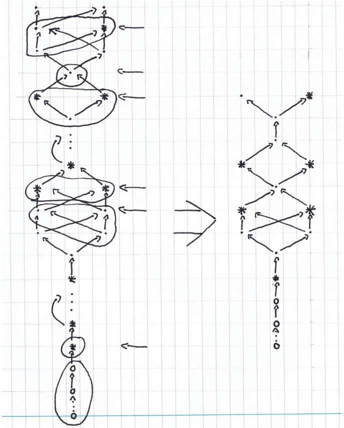

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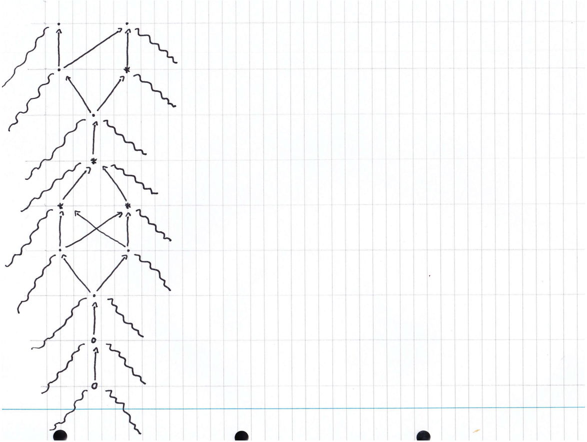

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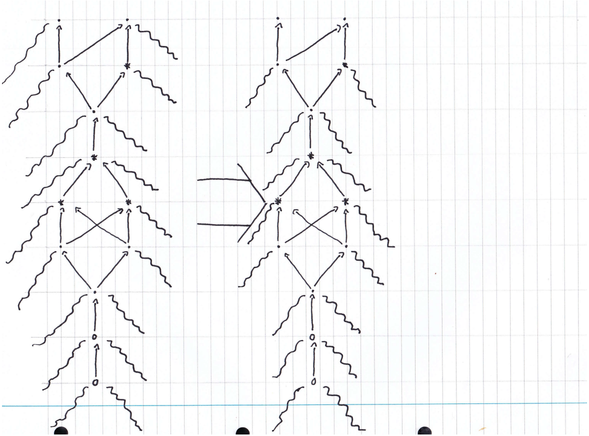

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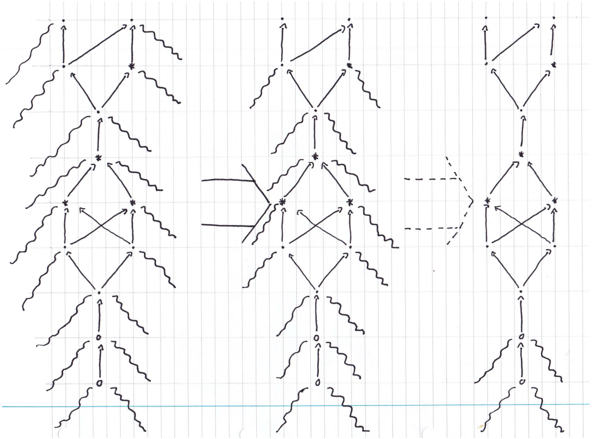

Summary

Combing all our results we have a complete characterisation of primitive K4-algebras.

Theorem A variety of K4-algebras $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n > 0$.

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Theorem A variety of K4-algebras $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n > 0$.

Consequently we also have a complete characterisation of HSC logics over K4.

Revised Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 17$ and $Log(G_n)$ for some $n \in \omega$.

Further Study

Extend the strategy to situations with a comparable set-up. Candidates include:

1. Modal logics over wK4;
2. All modal logics;
3. Intuitionistic modal logic;
4. Multi-modal logic.

Thanks

Thank you all for listening.

Additional thanks goes to Nick and Tommaso, the supervisors of my master's thesis from which this talk is based.