Axiomatization of logics with two-layered modal syntax: the protoalgebraic case

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Two-layered classical modal logics of probability

The original idea: interpret modality '$\Box\varphi$' as 'φ is probable', and define:

$$\Box\varphi \text{ is true} \quad \text{iff} \quad P(\varphi) \geq \alpha$$

Two-layered syntax consisting of:

- classical inner formulas describing the events
- atomic outer formulas of the form $\Box\varphi$, for each inner φ
- outer formulas built from atomic ones using connectives of classical logic

Two-layered fuzzy modal logics of probability

A fuzzy-logic idea: interpret modality '$\Box\varphi$' as 'φ is probable', and define:

truth value of $\Box\varphi$ is equal to $P(\varphi)$

Two-layered syntax consisting of:

- classical inner formulas describing the events
- atomic outer formulas of the form $\Box\varphi$, for each inner φ
- outer formulas built from atomic ones using connectives of Łukasiewicz logic

Probability Kripke frames and Kripke models

A *probability Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a finitely additive probability measure defined on a sublattice of $\mathcal{P}(W)$

A *Kripke model* $\mathbf{M}$ over a frame $\mathbf{F} = \langle W, \mu \rangle$ is a tuple $\mathbf{M} = \langle \mathbf{F}, \langle e_w \rangle_{w \in W} \rangle$ where:

- e_w is a classical evaluation of inner formulas
- for each inner formula φ, the domain of μ contains the set

$$\varphi^{\mathbf{M}} = \{w \mid e_w(\varphi) = 1\}$$

Truth definition

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$$||\Box\varphi||_{\mathbf{M}} = \mu(\{w \mid e_w(\varphi) = 1\}) = \mu(\varphi^{\mathbf{M}})$$

The truth values of other outer formulas are computed as :

$$\begin{aligned}
||\neg_{\text{Ł}}\varphi||_{\mathbf{M}} &= 1 - ||\varphi||_{\mathbf{M}} \\
||\varphi \to_{\text{Ł}} \psi||_{\mathbf{M}} &= \min\{1, 1 - ||\varphi||_{\mathbf{M}} + ||\psi||_{\mathbf{M}}\} \\
||\varphi \oplus \psi||_{\mathbf{M}} &= \min\{1, ||\varphi||_{\mathbf{M}} + ||\psi||_{\mathbf{M}}\} \\
||\varphi \ominus \psi||_{\mathbf{M}} &= \max\{0, ||\varphi||_{\mathbf{M}} - ||\psi||_{\mathbf{M}}\}
\end{aligned}$$

Axiomatization (Godo, Esteva, Hájek)

Let $T \cup \{A\}$ be a finite set of outer formulas. TFAE:

- $||A||_{\mathbf{M}} = 1$ for each Kripke model $\mathbf{M}$ where $||B||_{\mathbf{M}} = 1$ for each $B \in T$
- there is a proof of A from premises T in the axiomatic system consisting of:
  - the axioms and rules of classical logic for inner formulas
  - axioms and rules of finitary Łukasiewicz logic for outer formulas
  - modal axioms

$$\neg_{Ł}\Box(\bar{0})$$

$$\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$$

$$\neg_{Ł}\Box(\varphi) \to_{Ł} \Box(\neg\varphi)$$

$$\Box(\varphi \vee \psi) \to_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))$$

  - a unary modal rule:

$$\varphi \rhd \Box\varphi$$

Changing the measure

A *necessity* *Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a necessity measure on a subset of $\mathcal{P}(W)$

In the axiomatization we just replace the modal axioms by:

$$\neg_{Ł}\Box(\overline{0})$$
$$\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$$
$$(\Box\varphi \wedge_{Ł} \Box\psi) \to_{Ł} \Box(\varphi \wedge \psi)$$

More variations

Variations considered in the literature:

- changing the measure
- changing the ‘outer’ logic: replacing the Łukasiewicz logic by any other logic
- changing the ‘inner’ logic: e.g. replacing classical logic
  - by Łukasiewicz logic to speak about probability of ‘fuzzy’ events or
  - by Dunn–Belnap logic to speak about inconsistent events
- adding more modalities, also non-unary ones
- any combination of the above four options

The goal of this contribution

Advance the study of a general theory of two-layered logics

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In particular, we have

- identified the common aspects of existing completeness proofs
- obtained general/abstract completeness theorems
- instantiated them to recover (known) particular results

The first step: Cintula, Noguera. Modal Logics of Uncertainty with Two-Layer Syntax: A General Completeness Theorem. In Proceedings of WOLLIC 2014

Languages, variables, and formulas

Propositional case	*Two-layered case*
language $\mathcal{L}$	language $\mathfrak{T} = \langle \mathcal{L}_{\mathrm{i}}, \{\Box\}, \mathcal{L}_{\mathrm{o}} \rangle$ is a triple of prop. languages
variables Var (denumerable)	inner variables V (countable)
formulas $Fm_{\mathcal{L}}^{Var}$	formulas $Fm_{\mathfrak{T}}^{V} = Fm_{\mathcal{L}_{\mathrm{i}}}^{V} \cup Fm_{\mathcal{L}_{\mathrm{o}}}^{\alpha(\mathfrak{T},V)}$ where,

- inner formulas $Fm_{\mathcal{L}_{\mathrm{i}}}^{V}$
- *atomic* outer formulas $\alpha(\mathfrak{T},V)$ of the form $\Box\varphi$ for $\varphi \in Fm_{\mathcal{L}_{\mathrm{i}}}^{V}$
- outer formulas $Fm_{\mathcal{L}_{\mathrm{o}}}^{\alpha(\mathfrak{T},V)}$

Propositional/two-layered logics

A (finitary) logic on a set of formulas Fm is a relation $\vdash$ between sets of Fm-formulas and Fm-formulas st.:

- $A \vdash A$
- If $T \vdash A$, then $T \cup S \vdash A$
- If $T \vdash B$ for each $B \in S$ and $S \vdash A$, then $T \vdash A$
- (If $T \vdash A$, then there is a *finite* $T' \subseteq T$ such that $T' \vdash A$)

Propositional logic in $\mathcal{L}$: a logic on $Fm_{\mathcal{L}}$ such that

- If $T \vdash A$, then $\sigma[T] \vdash \sigma(A)$ for each propositional substitution σ

Two-layered logic in $\mathfrak{T}$: a logic on $Fm_{\mathfrak{T}}$ such that

- If $T \vdash A$, then $\sigma[T] \vdash \sigma(A)$ for each two-layered substitution σ

Algebra-based semantics of propositional logics

$Fm_{\mathcal{L}}^{Var}$: the domain of the absolutely free $\mathcal{L}$-algebra $\boldsymbol{Fm}_{\mathcal{L}}^{Var}$ with generators Var

$\boldsymbol{A}$-evaluation: a homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}}^{Var} \to \boldsymbol{A}$

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One algebra may give rise to many propositional logics; e.g.

$\Gamma \models_{Ł}^{1} \varphi$	iff	$e[\Gamma] \subseteq \{1\}$ implies $e(\varphi) = 1$	for each $[0,1]_{Ł}$-eval. e
$\Gamma \models_{Ł}^{1,f} \varphi$	iff	$\Gamma' \models_{Ł}^{1} \varphi$	for some finite $\Gamma' \subseteq \Gamma$
$\Gamma \models_{Ł}^{+} \varphi$	iff	$0 \notin e[\Gamma]$ implies $e(\varphi) > 0$	for each $[0,1]_{Ł}$-eval. e
$\Gamma \models_{Ł}^{\leq} \varphi$	iff	$\inf(e[\Gamma]) \leq e(\varphi)$	for each $[0,1]_{Ł}$-eval. e

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$\Gamma \models^{\leq}_{Ł} \varphi$	iff	$\inf(e[\Gamma]) \leq e(\varphi)$	for each $[0,1]_{Ł}$-eval. e

For an $\mathcal{L}$-matrix $\mathbf{A} = \langle A, F\rangle$ (where A is an $\mathcal{L}$-algebra and $F \subseteq A$), we set:

$\Gamma \models_{\mathbf{A}} \varphi$ iff $e[\Gamma] \subseteq F$ implies $e(\varphi) \in F$ for each A-eval. e

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For a class $\mathbb{K}$ of $\mathcal{L}$-matrices, we set:

$$\models_{\mathbb{K}} = \bigcap_{\mathbf{A} \in \mathbb{K}} \models_{\mathbf{A}}$$

From probability Kripke frame to $\mathfrak{T}$-frames

A probability Kripke frame is a system $\mathbf{F} = \langle W, \mu_\Box \rangle$

- W is a set (of possible worlds)
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- $\mu_\Box$ is a *partial* mapping $\mu_\Box\colon \mathcal{P}(W) \to [0,1]_{\text{Ł}}$ with some additional properties

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- e_w is a classical evaluation of inner formulas
- for every inner formula φ, the set

$$\varphi^{\mathbf{M}} = \{w \in W \mid e_w(\varphi) = 1\}$$

is in the domain of $\mu_\Box$.

From probability Kripke frame to $\mathfrak{T}$-frames

An ($\mathbf{I}$-based $\mathbf{O}$-measured) $\mathfrak{T}$-frame is a system $\mathbf{F} = \langle W, \mathbf{I}, \mathbf{O}, \mu_\Box \rangle$

- W is a set (of possible worlds)
- $\mathbf{I}$ is an $\mathcal{L}_\mathrm{i}$-matrix and $\mathbf{O}$ is an $\mathcal{L}_\mathrm{o}$-matrix
- $\mu_\Box$ is a *partial* mapping $\mu_\Box \colon I^W \to O$

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An $\mathfrak{T}$-*model* $\mathbf{M}$ over an $\mathfrak{T}$-frame $\mathbf{F}$ is a tuple $\mathbf{M} = \langle \mathbf{F}, \langle e_w \rangle_{w \in W} \rangle$ where:

- e_w is an $\boldsymbol{I}$-evaluation of inner formulas
- for every inner formula φ, the element

$$\varphi^{\mathbf{M}} = \langle e_w(\varphi) \rangle_{w \in W}$$

is in the domain of $\mu_\Box$ (elements of the domain are called *measurable*)

Note: in general we need to allow different matrices in different worlds

Truth definition

Given an $\mathfrak{T}$-model $\mathbf{M} = \langle\langle W, \mathbf{I}, \mathbf{O}, \mu_\Box\rangle, \langle e_w\rangle_{w\in W}\rangle$, we define the truth value of

- inner formulas in a given world w using the $\boldsymbol{I}$-evaluation e_w
- atomic outer formulas uniformly in $\mathbf{M}$ as:

$$||\Box\varphi||_\mathbf{M} = \mu_\Box(\varphi^\mathbf{M})$$

- non-atomic outer formulas using the operations from $\boldsymbol{O}$

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We say that $\mathbf{M}$ *is a model* a formula A, $\mathbf{M} \models A$ in symbols, if:

- $e_w(A) \in F_{\mathbf{I}}$ for each $w \in W$ (for inner formulas A)
- $||A||_{\mathbf{M}} \in F_{\mathbf{O}}$ (for outer formulas A)

Semantical consequence

For an $\mathfrak{T}$-frame $\mathbf{F}$, we set:

$$T \vDash_{\mathbf{F}} A \quad \text{iff} \quad \text{for each } \mathfrak{T}\text{-model } \mathbf{M} \text{ over } \mathbf{F}\text{:} \quad \mathbf{M} \models T \text{ implies } \mathbf{M} \models A$$

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$\vDash_{\mathsf{K}}$ is a two-layered logic in $\mathfrak{T}$

Modal rules

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Given a set R of modal rules/axioms, we set

$$\mathrm{R}(\mathbf{I}, \mathbf{O}) = \{\mathbf{F} \mid \mathbf{F} \text{ is an } \mathbf{I}\text{-based } \mathbf{O}\text{-measured } \mathfrak{T}\text{-frame and } \Gamma \models_{\mathbf{F}} \Psi \text{ for each } \Gamma \rhd \Psi \in \mathrm{R}\}$$

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E.g. $\mathrm{FP}(\mathbf{2}, [0,1]_{Ł})$ is the class of probability Kripke frames; for FP consisting of

$$\rhd \neg_{Ł} \Box \overline{0} \qquad \rhd \Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi) \qquad \rhd \neg_{Ł} \Box\varphi \to_{Ł} \Box(\neg\varphi)$$

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Main question: Can we axiomatize $\models_{\mathrm{R}(\mathbf{I},\mathbf{O})}$?

There is actually one more question …

Which classes of frames can be "axiomatized" as $\mathsf{R}(\mathbf{I}, \mathbf{O})$?

I.e., given a class K of frames, is there a set of rules R such that

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Example: $\emptyset(\mathbf{I}, \mathbf{O})$ is the class of all $\mathbf{I}$-based $\mathbf{O}$-measured frames and for K being the class of all total ones we have:

$$\models_{\mathsf{K}} = \models_{\emptyset(\mathbf{I},\mathbf{O})}?$$

Answer to our main question: Yes, we can!

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But we have to assume that $\models_{\mathbf{I}}$ and $\models_{\mathbf{O}}$ are "nice" ...

Given an $\mathcal{L}$-matrix $\mathbf{A}$ and a set Γ of formulas, we define the relation $\theta^{\mathbf{A}}_{\Gamma}$:

$$\langle \varphi, \psi \rangle \in \theta^{\mathbf{A}}_{\Gamma} \qquad \text{iff} \qquad e(\varphi) = e(\psi) \text{ for each } \boldsymbol{A}\text{-evaluation } e \text{ st. } e[\Gamma] \subseteq F_{\mathbf{A}}$$

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We say that $\models_{\mathbf{A}}$ is equivalential if there is a set of binary (primitive or definable) connectives E (*assume for simplicity that* $E = \{\leftrightarrow\}$) st. for each Γ:

$$\langle \varphi, \psi \rangle \in \theta^{\mathbf{A}}_{\Gamma} \qquad \text{iff} \qquad \Gamma \models_{\mathbf{A}} \varphi \leftrightarrow \psi$$

Axiomatizing $\models_{R(\mathbf{I},\mathbf{O})}$ — nice version

A strongly complete axiomatization of $\models_{R(\mathbf{I},\mathbf{O})}$

Let $\models_{\mathbf{I}}$ and $\models_{\mathbf{O}}$ be equivalential logics. Then, for each a set of inner formulas Γ and outer formulas $T \cup \{\Phi\}$,

$\Gamma, T \models_{R(\mathbf{I},\mathbf{O})} \Phi$ iff there is a tree-proof of Φ from the premises $\Gamma \cup T$ in the axiomatic system consisting of:

- the axioms and rules of $\models_{\mathbf{I}}$ for inner formulas
- axioms and rules of $\models_{\mathbf{O}}$ for outer formulas
- all substitution instances of modal axioms/rules R
- congruence rules CONG:

$$\{\varphi \leftrightarrow_{\mathrm{i}} \psi \rhd \Box\varphi \leftrightarrow_{\mathrm{o}} \Box\psi \mid \varphi, \psi \text{ inner formulas}\}$$

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A preparation: we say that $\models_{\mathbf{A}}$ is "weakly Fregean" if

$$\langle \varphi, \psi \rangle \in \theta^{\mathbf{A}}_{\emptyset} \quad \text{iff} \quad \varphi \models_{\mathbf{A}} \psi \text{ and } \psi \models_{\mathbf{A}} \varphi$$

Note: Dunn–Belnap is weakly Fregean.

Axiomatizing $\models_{\mathrm{R}(\mathbf{I},\mathbf{O})}$ — finitary version

A complete finitary axiomatization of $\models_{\mathrm{R}(\mathbf{I},\mathbf{O})}$ for finites sets of premises

Let $\models_{\mathbf{I}}$ and $\models_{\mathbf{O}}$ be equivalential logics st. $\models_{\mathbf{I}}$ is locally finite and let V and R be finite. Then, for each finite sets of inner formulas Γ and outer formulas $T \cup \{\Phi\}$,

$\Gamma, T \models_{\mathrm{R}(\mathbf{I},\mathbf{O})} \Phi$ iff there is a (finite sequence) proof of Φ from the premises $\Gamma \cup T$ in the axiomatic system consisting of:

- the axioms and rules of the finitary variant of $\models_{\mathbf{I}}$ for inner formulas
- axioms and rules of the finitary variant of $\models_{\mathbf{O}}$ for outer formulas
- all substitution instances of modal axioms/rules R
- congruence rules CONG: :

$$\{\varphi \leftrightarrow_{\mathrm{i}} \psi \rhd \Box\varphi \leftrightarrow_{\mathrm{o}} \Box\psi \mid \varphi, \psi \text{ inner formulas}\}$$

Axiomatizing $\models_{\mathsf{R}(\mathbf{I},\mathbf{O})}$ — “weakly Fregean” version

A complete axiomatization of $\models_{\mathsf{R}(\mathbf{I},\mathbf{O})}$ of outer premises

Let $\models_{\mathbf{I}}$ be a “weakly Fregean” logic, $\models_{\mathbf{O}}$ be an equivalential logic, and R contains axioms only. Then, for each set of outer formulas $T \cup \{\Phi\}$,

$T \models_{\mathsf{R}(\mathbf{I},\mathbf{O})} \Phi$ iff there is a tree-proof of Φ from the premises T in the axiomatic system consisting of:

- the axioms and rules of $\models_{\mathbf{I}}$ for inner formulas
- axioms and rules of $\models_{\mathbf{O}}$ for outer formulas
- all substitution instances of modal axioms R
- congruence axioms CONG:

$$\{ \rhd \Box\varphi \leftrightarrow_{\mathrm{o}} \Box\psi \mid \varphi,\psi \text{ inner formulas st. } \varphi \models_{\mathbf{I}} \psi \text{ and } \psi \models_{\mathbf{I}} \varphi \}$$

Axiomatizing $\models_{\mathsf{R}(\mathbf{I},\mathbf{O})}$ — the most abstract version (for now)

A strongly complete axiomatization of $\models_{\mathsf{R}(\mathbf{I},\mathbf{O})}$

Let $\models_{\mathbf{O}}$ be an equivalential logic. Then, for each a set of inner formulas Γ and outer formulas $T \cup \{\Phi\}$,

$\Gamma, T \models_{\mathsf{R}(\mathbf{I},\mathbf{O})} \Phi$ iff there is a tree-proof of Φ from the premises $\Gamma \cup T$ in the axiomatic system consisting of:

- the axioms and rules of $\models_{\mathbf{I}}$ for inner formulas
- axioms and rules of $\models_{\mathbf{O}}$ for outer formulas
- all substitution instances of modal axioms/rules R
- congruence rules CONG:

$$\{\Delta \rhd \Box\varphi \leftrightarrow_{\mathrm{o}} \Box\psi \mid \Delta \cup \{\varphi, \psi\} \text{ inner formulas and } \langle\varphi, \psi\rangle \in \theta^{\mathbf{I}}_{\Delta}\}$$