Sahlqvist theory for fragments of intuitionistic logic

Damiano Fornasiere*, Tommaso Moraschini

Department of Philosophy, University of Barcelona

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- Intuitionistic Kripke frames, i.e., posets $\mathbb{X} = \langle X, \leqslant \rangle$;
- Heyting algebras, i.e., structures $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1 \rangle$ that include a bounded lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and satisfy

$$a \wedge b \leqslant c \iff a \leqslant b \to c, \text{ for every } a, b, c \in A.$$

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In addition, every Heyting algebra $\boldsymbol{A}$ embeds into $\mathsf{Up}(\boldsymbol{A}_*)$.

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- Correspondence: Is there a sentence $\mathsf{tr}(\varphi)$ in the language of posets s.t. for every poset $\mathbb{X}$,

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- a Sahlqvist antecedent (SA) if it is constructed from atoms, negative formulas, and 0 and 1 using only $\wedge$ and $\vee$;
- a Sahlqvist implication (SI) if it is positive, or of the form $\neg\varphi$ for a SA φ, or of the form $\varphi \to \psi$ for a SA φ and a positive ψ;
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Given an IPC formula φ, two questions arise:

- Canonicity: Is it true that if $A \models \varphi$, then $\mathsf{Up}(A_*) \models \varphi$?
- Correspondence: Is there a sentence $\mathsf{tr}(\varphi)$ in the language of posets s.t. for every poset $\mathbb{X}$,

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Remark. Sahlqvist formulas are of the form $\varphi = \varphi_1 \vee \ldots \vee \varphi_n$. For example, $x \vee \neg x$ and $(x \to y) \vee (y \to x)$ are Sahlqvist.

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Definition

A Sahlqvist quasiequation is an expression of the form

$$\Phi = (\varphi_1 \wedge y \leqslant z) \,\&\, \ldots \,\&\, (\varphi_n \wedge y \leqslant z) \Longrightarrow (y \leqslant z),$$

where $\varphi_1, \ldots, \varphi_n$ are Sahlqvist formulas.

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For instance, up to equivalence in Heyting algebras,

- The excluded middle $x \vee \neg x$ can be rendered as

$$x \wedge y \leqslant z \quad \& \quad \neg x \wedge y \leqslant y \Longrightarrow y \leqslant z;$$

- The prelinearity axiom $(x_1 \to x_2) \vee (x_2 \to x_1)$ can be rendered as

$$(x_1 \to x_2) \wedge y \leqslant z \quad \& \quad (x_2 \to x_1) \wedge y \leqslant y \Longrightarrow y \leqslant z.$$

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Remark

For every Heyting algebra $\boldsymbol{A}$ it holds

$$\boldsymbol{A} \vDash \Phi \quad \text{iff} \quad \boldsymbol{A} \vDash \varphi_1 \vee \ldots \vee \varphi_n.$$

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- The "bounded top width n" axiom

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can be rendered as the Sahlqvist quasiequation

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- In the algebraic models of the $\langle \wedge, \neg \rangle$-fragment of IPC there are only three nonequivalent equations, while there are infinitely many nonequivalent Sahlqvist quasiequations.

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For every Heyting algebra $\boldsymbol{A}$ it holds

$$\boldsymbol{A} \vDash \Phi \quad \text{iff} \quad \boldsymbol{A} \vDash \varphi_1 \leqslant z \,\&\, \ldots \,\&\, \varphi_n \leqslant z \Rightarrow z \approx 1.$$

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Example

The Sahlqvist quasiequation

$$\neg x \wedge y \leqslant z \,\&\, \neg\neg x \wedge y \leqslant z \Longrightarrow y \leqslant z$$

is not equivalent to its "context free" version

$$\neg x \leqslant z \,\&\, \neg\neg x \leqslant z \Longrightarrow z \approx 1$$

over the algebraic models of the $\langle \wedge, \neg \rangle$-fragment of IPC.

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Example. The meaning of $\mathsf{tr}(x \vee \neg x)$ is "$\mathbb{X}$ is discrete" and that of $\mathsf{tr}((x \to y) \vee (y \to x))$ is "$\mathbb{X}$ is a root system".

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Sahlqvist theorem for fragments of IPC with $\wedge$.

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- For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra, if $\boldsymbol{A} \vDash \Phi$ then $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$;
- Φ has a first-order correspondent $\mathsf{tr}(\Phi)$, i.e., for every poset $\mathbb{X}$ it holds $\mathsf{Up}(\mathbb{X}) \vDash \Phi \iff \mathbb{X} \vDash \mathsf{tr}(\Phi)$.

Sahlqvist Canonicity for fragments of IPC with $\wedge$.

Let Φ be a Sahlqvist quasiequation in the language of a fragment L of IPC including $\wedge$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

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Let Φ be a Sahlqvist quasiequation in the language of a fragment L of IPC including $\wedge$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

$$\text{if } \boldsymbol{A} \models \Phi, \text{ then } \mathrm{Up}(\boldsymbol{A}_*) \models \Phi.$$

Proof sketch (the case $\wedge, \neg, 0$ of pseudocomplemented semilattices).

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Since $\mathrm{Up}(\boldsymbol{B}_*)$ validates Φ, so does $\mathrm{Up}(\boldsymbol{A}_*)$. □

Corollary.

Let Φ be a Sahlqvist quasiequation in the language of a fragment L of IPC including $\wedge$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

$$\boldsymbol{A} \models \Phi \text{ iff } \boldsymbol{A}_* \models \mathsf{tr}(\Phi).$$

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Proof.

- $\boldsymbol{A} \vDash \Phi$ implies $\mathrm{Up}(\boldsymbol{A}_*) \vDash \Phi$ by canonicity;

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- Therefore: $\boldsymbol{A} \vDash \Phi$ iff $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$;

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- Therefore: $\boldsymbol{A} \vDash \Phi$ iff $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$;
- Moreover, $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$ iff $\boldsymbol{A}_* \vDash \mathsf{tr}(\Phi)$ in view of the correspondence part of the Intuitionistic Sahlqvist Theorem;

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Proof.

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- $\mathrm{Up}(\boldsymbol{A}_*) \vDash \Phi$ implies $\boldsymbol{A} \vDash \Phi$ because $\boldsymbol{A}$ embeds into $\mathrm{Up}(\boldsymbol{A}_*)$;
- Therefore: $\boldsymbol{A} \vDash \Phi$ iff $\mathrm{Up}(\boldsymbol{A}_*) \vDash \Phi$;
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- The two items above yield

$$\boldsymbol{A} \vDash \Phi \text{ iff } \boldsymbol{A}_* \vDash \mathrm{tr}(\Phi).$$

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Similarly, $\boldsymbol{A} \vDash \Psi \iff$ the order of $\boldsymbol{A}_*$ is the identity, for every pseudocomplemented semilattice $\boldsymbol{A}$.

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Remark

The formula btw_n cannot be rendered as an equation!

Thank you very much for your attention!