Belief functions over Belnap–Dunn logic

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Inconsistencies: shortcomings of the available information
(Dunn: ”... too much of a good thing”)

- issue that has to be lived with until they can be resolved
- many attempts in the literature to devlop paraconsistent logics for these scenarios
- we build our work on the Belnap-Dunn four valued logic

Uncertainty

- probability – the most prominent representation of uncertainty
- defined on a boolean algebra
- each value precisely known (subjective probability)
- values of complex events are determined by the values of simple ones

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Alternatives:

- avoiding precise values (interval probabilities, ...)
- probability measures on weaker structures
- weaker uncertainty measures (inner outer measures, belief functions,...)

1. Representing incomplete/contradictory probabilistic information
   - Belnap-Dunn Logic
   - Non-standard probabilities
2. Dempster-Shafer theory
   - Mass functions, belief functions and plausibility functions
   - Representation of evidence
3. Dempster-Shafer theory and BD logic
   - Belief functions
   - Plausibility?
   - Combination of evidence

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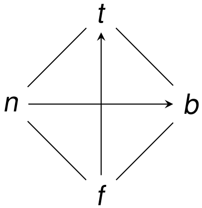
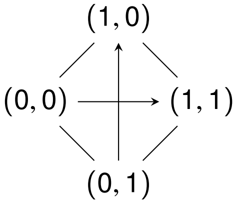
Belnap-Dunn square $(\mathbf{4}, \wedge, \vee, \neg)$ is a de Morgan algebra.

- $(\mathbf{4}, \wedge, \vee)$ is a lattice
- each element represents the available positive and/or negative information
  - n: no information
  - f: false (is bottom)
  - t: true (is top)
  - b: contradictory information
- $\neg$ is an involutive de Morgan negation.



Belnap-Dunn square 4



Independence of positive and negative information.

Belnap-Dunn Logic: models [Dunn 76]

Language. $L_{\mathsf{BD}} \ni \varphi := p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$

BD Models. $M = \langle W,\ v^{+}, v^{-}\ : \mathrm{Prop} \to \mathcal{P}(W) \rangle$
$v^{+}(p)$: states containing information supporting p
$v^{-}(p)$: states containing information refuting p

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Semantics. Two satisfaction relations $\vDash^{+}, \vDash^{-}$

$$
\begin{array}{ll}
w \vDash^{+} p \text{ iff } w \in v^{+}(p) & w \vDash^{-} p \text{ iff } w \in v^{-}(p) \\
w \vDash^{+} \neg\phi \text{ iff } w \vDash^{-} \phi & w \vDash^{-} \neg\phi \text{ iff } w \vDash^{+} \phi \\
w \vDash^{+} \phi \wedge \phi' \text{ iff } w \vDash^{+} \phi \text{ and } w \vDash^{+} \phi' & w \vDash^{-} \phi \wedge \phi' \text{ iff } w \vDash^{-} \phi \text{ or } w \vDash^{-} \phi' \\
w \vDash^{+} \phi \vee \phi' \text{ iff } w \vDash^{+} \phi \text{ or } w \vDash^{+} \phi' & w \vDash^{-} \phi \vee \phi' \text{ iff } w \vDash^{-} \phi \text{ and } w \vDash^{-} \phi'
\end{array}
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$$w \vDash^{+} \phi \vee \phi' \text{ iff } w \vDash^{+} \phi \text{ or } w \vDash^{+} \phi' \qquad w \vDash^{-} \phi \vee \phi' \text{ iff } w \vDash^{-} \phi \text{ and } w \vDash^{-} \phi'$$

Can we introduce probability in this framework?

Non-standard probabilities

Frame semantics (Klein, Majer, Rafiee Rad 2021)

- independence of positive and negative probabilistic information
- BD model extended with a (classical) probability measure.

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A **probabilistic BD model** is a tuple $M = \langle W, v^+, v^-, m\rangle$, s.t. $\langle W, v^+, v^-\rangle$ is a BD model and $\mu : \mathcal{P}(W) \to [0,1]$ is a probability measure on W

Positive probability

$$p^+(\varphi) := \mu(||\varphi||^+)$$

Negative probability of φ: $p^-(\varphi) := \mu(||\varphi||^-)$

Remark $p^-(\varphi) = p^+(\neg\varphi)$, otherwise $p^+(\varphi)$ and $p^-(\varphi)$ independent,

Non-standard probabilities: axioms

Theorem [Klein et al]

The following axiomatization of non-standard probabilities is complete with respect to the class of probabilistic frames.

(A1) normalization	$0 \leq p^+(\varphi) \leq 1$
(A2) monotonicity	if $\varphi \vdash_{BD} \psi$ then $p^+(\varphi) \leq p^+(\psi)$
(A3) import-export	$p^+(\varphi \wedge \psi) + p^+(\varphi \vee \psi) = p^+(\varphi) + p^+(\psi)$.

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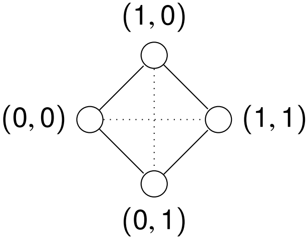
Remarks

- $p^-(\varphi) = p^+(\neg\varphi)$
- weaker than classical Kolmogorovian axioms. (A3 instead of additivity).
- In general $p^+(\neg\varphi) \neq 1 - p^+(\varphi)$
- one can have $0 < p^+(\varphi \wedge \neg\varphi), p^+(\varphi \vee \neg\varphi) < 1$

Non-standard probabilities: intuitive representation

Continuous extension of Belnap-Dunn square, which we can see as the product bilattice $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$
with $\mathbf{L}_{[0,1]} = ([0,1], \min, \max)$.

- $(p^+(\varphi), p^-(\varphi))$: positive and negative probabilistic support of φ.
- $(0,0)$: no information concerning φ is available
- $(1,1)$: maximally conflicting information
- vertical dashed line: "classical" case



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Dempster Shafer belief functions

- bel(A) represents **total evidence** supporting A
- defined on a boolean algebra of events
- weaker than probability
- complex formulas are not determined by the simpler ones
  $\text{bel}(A \vee B) \geq \text{bel}(A) + \text{bel}(B)$ for A, B disjoint
- provides a lower bound for ‘true‘ probability

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Example

Scenario. A patient has disease a, b or c.
A doctor says “given the evidence she has disease a or b with a high certainty (e.g. 0.7).” However the doctor gives only a low certainty to each of a, b (e.g. 0.1).

Mass function

– evidence assigned *exactly* to a particular event

.

Definition

A function m : $\mathcal{P}(S) \rightarrow [0, 1]$ is a mass function if

- m($\emptyset$)
- $\sum_{A \in \mathcal{P}(S)} \mathrm{m}(A) = 1$.

Evidence via mass functions

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Belief function via mass function:

$$\mathrm{bel}(A) = \sum_{B \subseteq A} \mathrm{m}(B)$$

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Probability via mass function:

$$p(A) = \sum_{B \subseteq A} \mathrm{m}(B) = \sum_{s \in A} \mathrm{m}(\{s\})$$

All information is encoded in singletons.

Explicit representation of belief functions

Definition

$f : \mathcal{P}(S) \rightarrow [0,1]$ is a belief function if $f(\emptyset) = 0$, $f(S) = 1$ and

$$f\left(\bigvee_{1\le i\le k} A_i\right) \ge \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot f\left(\bigwedge_{j\in J} A_j\right). \qquad (1)$$

holds for every $k \ge 1$, and for every $A_1, \ldots, A_k \in \mathcal{P}(S)$.

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Theorem

For every belief function bel there is a mass function $\mathrm{m}_{\mathrm{bel}} : \mathcal{P}(S) \to [0,1]$ such that, for every $A \in \mathcal{P}(S)$,

$$\mathrm{bel}(A) = \sum_{B \le A} \mathrm{m}_{\mathrm{bel}}(B)$$

Plausibility functions

- dual to belief
- pl(A) represents evidence which is *compatibe* with A
- gives an upper bound for ‘true‘ probability

Plausibility from belief

$$\mathrm{pl}(A) = 1 - \mathrm{bel}(\neg A)$$

Plausibility via mass function:

$$\mathrm{bel}(A) = \sum_{B \cap A \neq \emptyset} \mathrm{m}(B)$$

Explicit definition

$$f\left(\bigwedge_{1 \leq i \leq k} A_i\right) \leq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \emptyset}} (-1)^{|J|+1} \cdot f\left(\bigvee_{j \in J} A_j\right). \tag{2}$$

Representation of evidence. Example

Example

Scenario. A patient has disease a, b or c.
A doctor says "the patient has disease a or b with certainty 0.7."
The doctor gives no information about disease c.

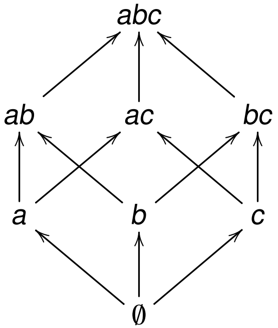
- $\mathrm{m} : \mathcal{P}(S) \to [0,1]$ is computed based on the evidence
- $\mathrm{bel}(A) = \sum_{B \leq A} \mathrm{m}(B)$: the evidence supporting a
- $\mathrm{pl}(A) = 1 - \mathrm{bel}(\neg A) = \sum_{B \cap A \neq \emptyset} \mathrm{m}(B)$: the evidence not contradicting A
- $\mathrm{bel}(A) \leq \mathrm{pl}(A)$.

Representation of evidence. An example

Scenario. A patient has disease *a*, *b* or *c*.
A doctor says "the patient has disease *a* or *b* with certainty 0.7."
It is assumed it is impossible for the patient to have two of them.

Representation

- $S = \{a, b, c\}$ and $\mathrm{m}, \mathrm{bel}, \mathrm{pl} : \mathcal{P}(S) \to [0, 1]$
- $\mathrm{m}(\{a, b\}) = 0.7$ and $\mathrm{m}(S) = 0.3$.



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We get:

$\mathrm{bel}(\{a\}) = \mathrm{bel}(\{b\}) = \mathrm{bel}(\{c\}) = 0$

$\mathrm{bel}(\{a, b\}) = \sum_{X \subseteq \{a,b\}} \mathrm{m}(X) = 0.7$ $\quad$ $\mathrm{pl}(\{a, b\}) = 1 - \mathrm{bel}(\{c\}) = 1$

$\mathrm{pl}(\{a\}) = \mathrm{pl}(\{b\}) = 1$ $\quad$ $\mathrm{pl}(\{c\}) = 1 - \mathrm{bel}(\{a, b\}) = 0.3$

- $\mathrm{m}(\{a, b\})$: the 'probability' that the disease is in the set $\{a, b\}$ without being able to say to which subset it belongs.

Dempster-Shafer combination rule

Let m_1 and m_2 be two mass functions on $\mathcal{P}(S)$. Dempster-Shafer combination rule computes their aggregation $\mathrm{m}_{1\oplus 2} : \mathcal{P}(S) \rightarrow [0,1]$ as follows.

$$\mathrm{m}_{1\oplus 2}(X) \mapsto \begin{cases} 0 & \text{if } X = \varnothing \\ \dfrac{\sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 = X\}}{N} & \text{otherwise.} \end{cases}$$

Normalization factor:

$$\begin{aligned} N &= \sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\} \\ &= 1 - \sum\{\mathrm{m}_1(X_1)\cdot \mathrm{m}_2(X_2) \mid X_1 \cap X_2 = \varnothing\} \end{aligned}$$

What happens with contradictory evidence?

Scenario

A patient has disease *a*, *b* or *c*.
Doctor 1:
"the patient has *a* with certainty 0.9 and *b* with certainty 0.1."
Doctor 2:
"the patient has *c* with certainty 0.9 and *b* with certainty 0.1."

Representation

$S = \{a, b, c\}$
$\mathrm{m}_1(\{a\}) = 0.9$ and $\mathrm{m}_1(\{b\}) = 0.1$.
$\mathrm{m}_2(\{c\}) = 0.9$ and $\mathrm{m}_2(\{b\}) = 0.1$.

DS combination rule ignores contrtadictory information

$$\mathrm{m}_{1\oplus 2}(\{b\}) = 1, \mathrm{m}_{1\oplus 2}(\{a\}) = \mathrm{m}_{1\oplus 2}(\{c\}) = 0$$

because $\{a\} \cap \{b\} = \{a\} \cap \{c\} = \emptyset$.

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Belief function on BD

Extending belief with a (classical) belief function

Definition belief on BD-models

$M = \langle W, v^+, v^-, Bel\rangle$, a BD model plus $Bel : \mathcal{P}(W) \to [0,1]$.

$$\mathrm{bel}^+(\phi) := Bel(|\phi|^+) \quad \text{and} \quad \mathrm{bel}^-(\phi) := Bel(|\phi|^-)$$

bel^+: belief function on the associated Lindenbaum algebra $\mathcal{L}_{\mathrm{BD}}$.
bel^-: belief function on $\mathcal{L}_{\mathrm{BD}}^{op}$.

Remark. if $\bot$ and $\top$ are not in the language bel^+ (resp. pl^+) are general belief (resp. plausibility) functions.

Non-standard probabilities

Models: $(W, v^+, v^-, \mathrm{m} : W \to [0,1])$
$p^+(\phi) = \sum_{s \in |\phi|^+} \mathrm{m}(s)$ and $p^-(\phi) = \sum_{s \in |\phi|^-} \mathrm{m}(s)$

Immediate generalisation for belief.

Non-standard beliefs

Models: $(W, v^+, v^-, m : \mathcal{P}(W) \to [0,1])$
$\mathrm{bel}^+(\phi) = \sum_{X \subseteq |\phi|^+} \mathrm{m}(X)$ and $\mathrm{bel}^-(\phi) = \sum_{X \subseteq |\phi|^-} \mathrm{m}(X)$

- $\mathrm{bel}^+(\phi)$: belief that ϕ is true
- $\mathrm{bel}^-(\phi)$: belief that ϕ is false
- bel^+ satisfies the axioms of belief functions

Two-dimensional interpretation

Two dimensional reading allows for various combinations of positive/negative belief/plausibility:

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- classically bel(ϕ) $=$ 1 – pl($\neg\phi$).
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$$(\text{bel}^{+}(\phi), \text{pl}^{-}(\phi))$$

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- consider both belief and plausibility independently:

$$(\text{bel}^{+}(\phi), \text{bel}^{-}(\phi)), (\text{pl}^{+}(\phi), \text{pl}^{-}(\phi))$$

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- If we require $\mathrm{bel}(X) \leq \mathrm{pl}(X)$, for $X \in \mathcal{P}(W)$,
  then bel and pl must be defined from different mass functions.
  a piece of evidence might support belief and plausibility in a different way $\rightarrow$ gives rise to two mass functions
  (e.g., circumstantial evidence vs. direct evidence)

Combination of evidence

Let $\mathcal{L}$ be a finite distributive lattice.

Without $\bot$ and $\top$

$$\begin{aligned} \mathrm{m}_{1\oplus 2} &: \mathcal{L} \to [0,1] \\ & x \mapsto \sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 = x\}. \end{aligned}$$

With $\bot$ and $\top$

$$\begin{aligned} \mathrm{m}_{1\oplus 2} &: \mathcal{L} \to [0,1] \\ & x \mapsto \begin{cases} 0 & \text{if } x = \bot \\ \dfrac{\sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 = x\}}{\sum\{\mathrm{m}_1(x_1)\cdot \mathrm{m}_2(x_2) \mid x_1 \wedge x_2 \neq \bot\}} & \text{otherwise.} \end{cases} \end{aligned}$$

Examples. The two doctors

Scenario. A patient has disease *a*, *b* or *c*.
Doctor 1: *a* with certainty 0.9 and *b* with certainty 0.1.
Doctor 2: *c* with certainty 0.9 and *b* with certainty 0.1.
Representation. $\mathrm{m}_1, \mathrm{m}_2 \ : \ \mathcal{DM}_3 \to [0, 1]$

$$\mathrm{m}_1(x) = \begin{cases} 0.9 & \text{if } x = a \\ 0.1 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases} \qquad \mathrm{m}_2(x) = \begin{cases} 0.9 & \text{if } x = c \\ 0.1 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases}$$

Dempster-Shafer combination rule gives

$$\mathrm{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \text{if } x = a \wedge c \\ 0.09 & \text{if } x = a \wedge b \text{ or } x = b \wedge c \\ 0.01 & \text{if } x = b \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{bel}_{1\oplus 2}(a) = \mathrm{bel}_{1\oplus 2}(c) = 0.9$ and $\mathrm{bel}_{1\oplus 2}(b) = 0.19$

Representation. $\mathrm{m}_1, \mathrm{m}_2 : \mathcal{DM}_3 \to [0,1]$

$$\mathrm{m}_1(x) = \begin{cases} 0.9 & \text{if } x = a \wedge \neg b \wedge \neg c \\ 0.1 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{m}_2(x) = \begin{cases} 0.9 & \text{if } x = \neg a \wedge \neg b \wedge c \\ 0.1 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

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Future work

Reasoning with inconsistent / incomplete uncertain information

- other uncertinty measures (upper/lower probabilities, ...)
- qualitative probability
- various aggregation methods
- two layered framework

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