Multi-type modal extensions of the Lambek calculus for structural control

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Typelogical grammars

[Moot & Retoré]: book

Goal: develop a *compositional* and *modular* account of grammatical form and meaning in natural languages:

formal grammar is presented as a **logic.**

The **basic judgement**

$$x_1 : A_1, \ldots, x_n : A_n \vdash x : A$$

reads: the (structured configuration of) linguistic expressions x_1 of type $A_1, \ldots, x_n$ of type A_n can be categorized as a well-formed expression x of type A.

- Form: **residuated families** of type-forming operations (**logical level**) + different means to control the grammatical resource management (**structural level**)
- Meaning: **algebraic**, computational, relational, and categorial semantics

Parsing as deduction

[Ajdukiewicz 35, Bar-Hillel 64]: AB-grammars, [Lambek 58]: string of words, [Lambek 61]: bracketed strings (phrases)

- Parts of speech (noun, verb...) $\rightsquigarrow$ logical formulas - types.
- Grammaticality judgement $\rightsquigarrow$ logical deduction - computation.

$$\underset{\texttt{time}}{np} \cdot \underset{\texttt{flies}}{(np\backslash s)} \cdot \underset{\texttt{like}}{(((np\backslash s)\backslash(np\backslash s))/np)} \cdot \underset{\texttt{an}}{(np/n)} \cdot \underset{\texttt{arrow}}{n} \vdash s$$

Lexicon

- transitive verb `love`: $(np\backslash s)/np$
  - `kids (love games)`
- conjunction words `and`/`but`: *chameleon* word $(X\backslash X)/X$
  - $X = s$: $\texttt{(kids like sweets)}_s$ `but` $\texttt{(parents prefer liquor)}_s$
  - $X = np\backslash s$: `kids` $\texttt{(like sweets)}_{np\backslash s}$ `but` $\texttt{(hate vegetables)}_{np\backslash s}$
- relative pronoun `that`: $(n\backslash n)/(s/np)$, i.e. it looks for a noun n to its left and an *incomplete* sentence to its right (s/np: it misses a np, the *gap* at the right)

Deriving a sentence (in Natural Deduction - Gentzen)

$$\dfrac{\dfrac{\text{Alice}}{np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{np/n} \qquad \dfrac{\text{key}}{n}}{np}\,/E}{np\backslash s}\,/E}{s}\,\backslash E$$

Deriving a sentence (in Natural Deduction - Gentzen)

$$\dfrac{\dfrac{\text{Alice}}{np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{np/n} \quad \dfrac{\text{key}}{n}}{np}\,/E}{np\backslash s}\,/E}{s}\,\backslash E$$

$$\dfrac{\dfrac{\text{Alice}}{np \vdash np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np \vdash (np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{np/n \vdash np/n} \quad \dfrac{\text{key}}{n \vdash n}}{np/n \cdot n \vdash np}\,/E}{(np\backslash s)/np \cdot (np/n \cdot n) \vdash np\backslash s}\,/E}{np \cdot ((np\backslash s)/np \cdot (np/n \cdot n)) \vdash s}\,\backslash E$$

Deriving a sentence (in Natural Deduction - Gentzen)

$$\dfrac{\dfrac{\text{Alice}}{np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{np/n} \qquad \dfrac{\text{key}}{n}}{np}\,/E}{np\backslash s}\,/E}{s}\,\backslash E$$

$$\dfrac{\dfrac{\text{Alice}}{np \vdash np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np \vdash (np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{np/n \vdash np/n} \qquad \dfrac{\text{key}}{n \vdash n}}{np/n \cdot n \vdash np}\,/E}{(np\backslash s)/np \cdot (np/n \cdot n) \vdash np\backslash s}\,/E}{np \cdot ((np\backslash s)/np \cdot (np/n \cdot n)) \vdash s}\,\backslash E$$

$$\dfrac{\text{Alice} \vdash np \qquad \dfrac{\text{found} \vdash (np\backslash s)/np \qquad \dfrac{\text{the} \vdash np/n \qquad \text{key} \vdash n}{\text{the} \cdot \text{key} \vdash np}\,/E}{\text{found} \cdot (\text{the} \cdot \text{key}) \vdash np\backslash s}\,/E}{\text{Alice} \cdot (\text{found} \cdot (\text{the} \cdot \text{key})) \vdash s}\,\backslash E$$

Modal Lambek calculus 1/2

[Moortgat at al. 97], [Morrill 17], [Sadrzadeh at al. 21]: Language expansion + axiomatic extensions

LC lacks the required expressivity for realistic grammar development.

The extended Lambek calculi $LC_\diamond$ enrich the type language with modalities for **structural control**.

- `Alice (found (the key))` $\vdash s \rightsquigarrow LC$
- `key (that (Alice (found there)))` $\vdash n \rightsquigarrow LC_\diamond$
  - `that`: $(n\backslash n)/(s/\diamond\blacksquare np)$, `there`: $(np\backslash s)\backslash(np\backslash s)$
  - **licensing** (controlled associativity and (mixed) commutativity)
- `(Kids love videogames) but (parents hate videogames)` $\vdash s \rightsquigarrow LC$
- `((Kids love) but (parents hate)) videogames` $\vdash s \rightsquigarrow LC_\diamond$
  - `but`: $((s/\diamond\blacksquare np)\backslash\blacksquare(s/np))/(s/\diamond\blacksquare np)$
  - **licensing** (controlled associativity OR controlled contraction)
  - **blocking** (to avoid `Kids love videogames but parents hate`)

Licensing Associativity via SC √

[Moortgat 96, Kurtonina & Moortgat 97], [Morrill 17]: structural control

$$\frac{\text{found}}{(np\backslash s)/np} \qquad \frac{[\_ \vdash \Box np]^2}{\langle \_ \rangle \vdash np}\ \Box E$$

$$\frac{(np\backslash s)/np \qquad \langle \_ \rangle \vdash np}{\text{found} \cdot \langle \_ \rangle \vdash np\backslash s}\ /E$$

$$\frac{\text{alice}}{np} \qquad \frac{np \qquad \text{found} \cdot \langle \_ \rangle \vdash np\backslash s}{\text{alice} \cdot (\text{found} \cdot \langle \_ \rangle) \vdash s}\ \backslash E$$

$$\frac{\text{alice} \cdot (\text{found} \cdot \langle \_ \rangle) \vdash s}{(\text{alice} \cdot \text{found}) \cdot \langle \_ \rangle \vdash s}\ cA$$

$$\frac{[\_ \vdash \Diamond\Box np]^1 \qquad (\text{alice} \cdot \text{found}) \cdot \langle \_ \rangle \vdash s}{(\text{alice} \cdot \text{found}) \cdot \_ \vdash s}\ \Diamond E^2$$

$$\frac{(\text{alice} \cdot \text{found}) \cdot \_ \vdash s}{\text{alice} \cdot \text{found} \vdash s/\Diamond\Box np}\ /I^1$$

$$\frac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \frac{(n\backslash n)/(s/\Diamond\Box np) \qquad \text{alice} \cdot \text{found} \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{alice} \cdot \text{found}) \vdash n\backslash n}\ /E$$

$$\frac{\text{key}}{n} \qquad \frac{n \qquad \text{that} \cdot (\text{alice} \cdot \text{found}) \vdash n\backslash n}{\text{key} \cdot (\text{that} \cdot (\text{alice} \cdot \text{found})) \vdash n}\ \backslash E$$

$$\lambda x.((\textsc{key}\ x) \wedge ((\textsc{found}\ x)\ \textsc{alice}))$$

Licensing Mixed Commutativity via SC √

$$\dfrac{\dfrac{\text{found}}{(np\backslash s)/np} \qquad \dfrac{[\,\_ \vdash \Box np]^2}{\langle\,\_\,\rangle \vdash np}\ \Box E}{\text{found} \cdot \langle\,\_\,\rangle \vdash np\backslash s}\ /E$$

$$\dfrac{\text{found} \cdot \langle\,\_\,\rangle \vdash np\backslash s \qquad \dfrac{\text{there}}{(np\backslash s)\backslash(np\backslash s)}}{(\text{found} \cdot \langle\,\_\,\rangle) \cdot \text{there} \vdash np\backslash s}\ \backslash E$$

$$\dfrac{\dfrac{\text{alice}}{np} \qquad (\text{found} \cdot \langle\,\_\,\rangle) \cdot \text{there} \vdash np\backslash s}{\dfrac{\dfrac{\text{alice} \cdot ((\text{found} \cdot \langle\,\_\,\rangle) \cdot \text{there}) \vdash s}{\text{alice} \cdot ((\text{found} \cdot \text{there}) \cdot \langle\,\_\,\rangle) \vdash s}\ cMC}{(\text{alice} \cdot (\text{found} \cdot \text{there})) \cdot \langle\,\_\,\rangle \vdash s}\ cA}\ \backslash E$$

$$\dfrac{[\,\_ \vdash \Diamond\Box np]^1 \qquad (\text{alice} \cdot (\text{found} \cdot \text{there})) \cdot \langle\,\_\,\rangle \vdash s}{\dfrac{(\text{alice} \cdot (\text{found} \cdot \text{there})) \cdot \,\_ \vdash s}{\text{alice} \cdot (\text{found} \cdot \text{there}) \vdash s/\Diamond\Box np}\ /I^1}\ \Diamond E^2$$

$$\dfrac{\dfrac{\text{key}}{n} \qquad \dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \text{alice} \cdot (\text{found} \cdot \text{there}) \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{alice} \cdot (\text{found} \cdot \text{there})) \vdash n\backslash n}\ /E}{\text{key} \cdot (\text{that} \cdot (\text{alice} \cdot (\text{found} \cdot \text{there}))) \vdash n}\ \backslash E$$

$$\lambda x.((\textsc{key}\ x) \wedge ((\textsc{there}\ (\textsc{found}\ x))\ \textsc{alice}))$$

Modal Lambek calculus 2/2

[De Marneffe et al. 21]: dependency structures, [Kogkalidis et al. 20]: d. modalities as blocking devices

Function-argument: opposition between a *function type* A/B (or $B\backslash A$) that combines with its *argument* B to produce an A.

Dependency structures: opposition between a *head* and its *dependents* (i.e. *complements* selected by the head, or *adjuncts* modifying the head)

- `(Alice left) unexpectedly` $\rightsquigarrow LC^{\diamond\Box}$
  - `left` is the head selecting for `Alice` as a *complement* with the subject role $(\diamond^{\mathrm{su}} np)\backslash s$
  - `unexpectedly` is an *adjunct* modifying the head $\Box^{\mathrm{adv}}(s\backslash s)$

Domain of locality: The dependency modalities have the effect of sealing off (i.e. **blocking**) a structure (i.e. a head with its dependents): .

Interaction postulates: In some cases, the domain of locality should be *permeable*, so dependency and structural control modalities can interact.

Starting point: display calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing Gentzen's comma in sequents $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\dfrac{\dfrac{\dfrac{Y \vdash X \,\check{\backslash}\, Z}{X \,\hat{\otimes}\, Y \vdash Z}}{Y \,\hat{\otimes}\, X \vdash Z}}{X \vdash Y \,\check{\backslash}\, Z} \qquad \dfrac{X \vdash \check{\neg}\, Y}{Y \vdash \check{\sim}\, X}$$

display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

Proper display calculi

[Wansing 98]: proper, [Belnap 82, 89]: display logic, [Mints 72, Dunn 73, 75]: structural connectives

Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation);
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters (Properness!);
5. **reduction strategy** exists when cut formulas are principal.

Theorem (Canonical!)

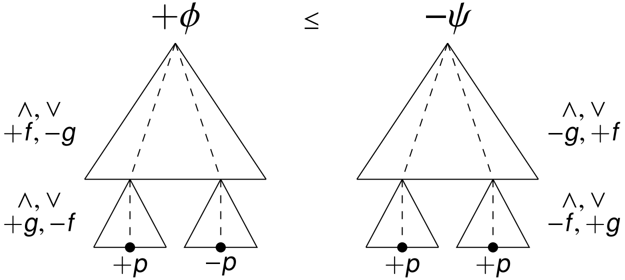
Cut elim. and subformula property hold for any **proper DC**.

Which logics are properly displayable?

[Ciabattoni et al. 15, Greco et al. 16]

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:
   ⇝ unified correspondence



Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheorem + ALBA-technology.

Examples

The definition of analytic inductive inequalities is uniform in each signature.

- Analytic inductive axioms

  $(A \to (B \vee C)) \to ((A \to B) \vee C)$

  $(\Diamond A \to \Box B) \to \Box(A \to B)$

- Sahlqvist but non-analytic axioms

  $A \to \Diamond\Box A$

  $(\Box A \to \Diamond B) \to (A \to B)$

The excluded middle is derivable using *Grishin's rule*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \mathbin{\hat{\wedge}} \top \vdash A}}{A \mathbin{\hat{\wedge}} \top \vdash \bot \mathbin{\check{\vee}} A}}{\top \vdash A \mathbin{\check{\to}} (\bot \mathbin{\check{\vee}} A)}}{\top \vdash (A \mathbin{\check{\to}} \bot) \mathbin{\check{\vee}} A}\,Gri}{}$$

$$\vdots$$

$$\top \vdash \neg A \vee A$$

For many... but not for all.

- The characterization theorem sets **hard boundaries** to the scope of proper display calculi.
- Interesting logics are **left out**:
  - First order logic
  - Non normal modal logics
  - Conditional logics
  - Dynamic epistemic logic
  - Inquisitive logic
  - Semi De Morgan logic
  - Bi-lattice logic
  - Rough algebras
  - ...

Can we **extend the scope** of proper display calculi?

Yes: proper display calculi $\rightsquigarrow$ proper **multi-type** calculi
(read: multi-sorted calculi)

Multi-type ($\rightsquigarrow$ multi-sorted) proper display calculi

[Greco et al. 14...]

Definition

A **proper mDC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Theorem (Canonical!)

Cut elim. and subformula property hold for any **proper mDC.**

The language of the modal Lambek calculus $LC^{\Diamond\Box}_{\Diamond}$

$$\mathrm{Fm} \ni A ::= \quad p$$

$$A \,/\, A \mid A \otimes A \mid A \backslash A \qquad \text{Lambek connectives}$$

$$\Diamond_i \alpha \mid \Diamond^c A \mid \Box^m A \qquad \text{s.c. and d. modalities}$$

$$\mathrm{Fm} \ni \alpha ::= \quad \blacksquare_i A$$

$$\mathrm{Str} \ni X ::= \quad A$$

$$X \,\check{/}\, X \mid X \,\hat{\otimes}\, X \mid X \,\check{\backslash}\, X \qquad \text{Lambek connectives}$$

$$\hat{\Diamond}_i \Gamma \mid \hat{\Diamond}^c X \mid \check{\blacksquare}^c \mid \check{\Box}^m X \mid \hat{\blacklozenge}^m \qquad \text{s.c. and d. modalities}$$

$$\mathrm{Str} \ni \Gamma ::= \quad \check{\blacksquare}_i X$$

Basic display calculus

- Identity and Cut rules (preorder)

$$\text{Id}\ \frac{}{A \vdash A} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{Cut}$$

- Display rules (residuation)

$$\otimes \dashv /\ \frac{X \vdash Z \check{/} Y}{\dfrac{X \hat{\otimes} Y \vdash Z}{Y \vdash X \check{\backslash} Z}}\ \otimes \dashv \backslash$$

- Logical rules (arity and tonicity)

$$\otimes_L\ \frac{A \hat{\otimes} B \vdash Y}{A \otimes B \vdash Y} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \hat{\otimes} Y \vdash A \otimes B}\ \otimes_R$$

$$\backslash_L\ \frac{X \vdash A \qquad B \vdash Y}{A \backslash B \vdash X \check{\backslash} Y} \qquad \frac{X \vdash A \check{\backslash} B}{X \vdash A \backslash B}\ \backslash_R$$

$$/_L\ \frac{A \vdash X \qquad Y \vdash B}{A / B \vdash X \check{/} Y} \qquad \frac{X \vdash B \check{/} A}{X \vdash B / A}\ /_R$$

Language expansion: dependency modalities

- Display rules (adjunction)

$$\text{adj}\ \frac{Y \vdash \check{\blacksquare} X}{\hat{\Diamond} Y \vdash X}$$

- Logical rules (arity and tonicity)

$$\Diamond_L\ \frac{\hat{\Diamond} X \vdash Y}{\Diamond X \vdash Y} \qquad \frac{X \vdash A}{\hat{\Diamond} X \vdash \Diamond A}\ \Diamond_R$$

$$\check{\blacksquare}_L\ \frac{A \vdash X}{\blacksquare A \vdash \check{\blacksquare} X} \qquad \frac{X \vdash \check{\blacksquare} A}{X \vdash \blacksquare A}\ \blacksquare_R$$

Language expansion: structural control operators

- Display rules (adjunction)

$$\text{adj}\ \frac{\Gamma \vdash \check{\blacksquare} X}{\hat{\Diamond}\Gamma \vdash X}$$

- Logical rules (arity and tonicity)

$$\Diamond_L\ \frac{\hat{\Diamond}\alpha \vdash X}{\Diamond\alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\hat{\Diamond}\Gamma \vdash \Diamond\alpha}\ \Diamond_R$$

$$\check{\blacksquare}_L\ \frac{A \vdash X}{\blacksquare A \vdash \check{\blacksquare} X} \qquad \frac{\Gamma \vdash \check{\blacksquare} A}{\Gamma \vdash \blacksquare A}\ \blacksquare_R$$

Axiomatic extensions via analytic structural rules

- Structural rules

$$A\ \frac{X \hat{\otimes} (Y \hat{\otimes} Z) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W} \qquad MC\ \frac{(X \hat{\otimes} Z) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W}$$

- Controlled structural rules

$$cA\ \frac{X \hat{\otimes} (Y \hat{\otimes} \hat{\Diamond}\Gamma) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond}\Gamma \vdash W} \qquad cMC\ \frac{(X \hat{\otimes} \hat{\Diamond}\Gamma) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond}\Gamma \vdash W}$$

Licensing rules: the case of Linear Logic

The full language of linear logic (in Girard's notation) is the following:

$$
\begin{array}{lll}
A ::= & p \mid & \\
 & 1 \mid \bot \mid A \otimes A \mid A \parr A \mid A \multimap A \mid & \text{multiplicatives} \\
 & !A \mid ?A \mid & \text{exponentials} \\
 & \top \mid 0 \mid A \,\&\, A \mid A \oplus A & \text{additives}
\end{array}
$$

Girard's rules for exponentials (in sequent format):

- left (right) dereliction and right (left) promotion rules:

$$
\frac{X, A \vdash Y}{X, !A \vdash Y} \qquad \frac{X \vdash A, Y}{X \vdash ?A, Y} \qquad \frac{!X \vdash A, ?Y}{!X \vdash !A, ?Y} \qquad \frac{!X, A \vdash ?Y}{!X, ?A \vdash ?Y}
$$

- left (right) weakening and left (right) contraction rules:

$$
\frac{X \vdash Y}{X, !A \vdash Y} \qquad \frac{X \vdash Y}{X \vdash ?A, Y} \qquad \frac{X, !A, !A \vdash Y}{X, !A \vdash Y} \qquad \frac{X \vdash ?A, ?A, Y}{X \vdash ?A, Y}
$$

Is Linear Logic properly displayable?

[Belnap 92]: **not** a **proper** display calculus:

$$\frac{Z \vdash A}{Z \vdash !A} \qquad \frac{A \vdash W}{!A \vdash W}$$

$$\frac{W \vdash A}{W \vdash ?A} \qquad \frac{A \vdash Z}{?A \vdash Z}$$

Z more general than X, Y on the previous slide, but still not arbitrary: they are still **exponentially restricted** as before.

Notice that the following sequents are derivable:

$!!A \Leftrightarrow !A$

$!A \vdash A$

$A \vdash B$ implies $!A \vdash !B$

$!\top \Leftrightarrow 1$

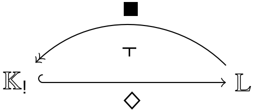
$!(A \& B) \Leftrightarrow !A \otimes !B$ analytic?

Linear logic: algebraic analysis

[Greco et al. 22]: to appear

$$
\begin{array}{ll}
!!a = !a & !\top = 1 \\
!a \leq a & !(a \& b) = !a \otimes !b \\
a \leq b \text{ implies } !a \leq !b &
\end{array}
$$

$! : \mathbb{L} \to \mathbb{L}$ **interior operator.** Then $! := \Diamond\blacksquare$, where



Fact: Range(!) has natural BA/HA-structure.

Upshot: natural semantics for the following **multi-type** language:

$$
\begin{array}{lll}
\text{Kernel} \ni \alpha & ::= & \blacksquare A \mid \mathtt{t} \mid \mathtt{f} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \\
\text{Linear} \ni A & ::= & p \mid \Diamond \alpha \mid \\
 & & 1 \mid \bot \mid A \otimes A \mid A \parr A \mid A \multimap A \mid \\
 & & \top \mid 0 \mid A \& A \mid A \oplus A
\end{array}
$$

Reverse-engineering linear logic 1/2

Interior operator axioms/rule recaptured:

$$
\begin{array}{c}
A \vdash A \\
\hline
\blacksquare A \vdash \check{\blacksquare} A \\
\hline
\hat{\Diamond} \blacksquare A \vdash A \\
\hline
\Diamond \blacksquare A \vdash A \\
\hline\hline
!A \vdash A
\end{array}
\qquad
\begin{array}{c}
A \vdash A \\
\hline
\blacksquare A \vdash \check{\blacksquare} A \\
\hline
\blacksquare A \vdash \blacksquare A \\
\hline
\hat{\Diamond} \blacksquare A \vdash \Diamond \blacksquare A \\
\hline
\blacksquare A \vdash \check{\blacksquare} \Diamond \blacksquare A \\
\hline
\blacksquare A \vdash \blacksquare \Diamond \blacksquare A \\
\hline
\hat{\Diamond} \blacksquare A \vdash \Diamond \blacksquare \Diamond \blacksquare A \\
\hline
\Diamond \blacksquare A \vdash \Diamond \blacksquare \Diamond \blacksquare A \\
\hline\hline
!A \vdash !!A
\end{array}
\qquad
\begin{array}{c}
A \vdash B \\
\hline
\blacksquare A \vdash \check{\blacksquare} B \\
\hline
\blacksquare A \vdash \blacksquare B \\
\hline
\hat{\Diamond} \blacksquare A \vdash \Diamond \blacksquare B \\
\hline
\Diamond \blacksquare A \vdash \Diamond \blacksquare B \\
\hline\hline
!A \vdash !B
\end{array}
$$

Reverse-engineering linear logic 2/2

Problem: the following axioms are **non-analytic**.

$$!\top = 1 \quad \rightsquigarrow \quad \Diamond\blacksquare\top = 1$$
$$!(A \,\&\, B) = !A \otimes !B \quad \rightsquigarrow \quad \Diamond\blacksquare(A \,\&\, B) = \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

Solution: $\blacksquare$ surjective and finitely meet-preserving $\Rightarrow$ axioms above semantically equivalent to the following analytic identities:

$$\Diamond \mathtt{t} = 1 \qquad \Diamond(\alpha \wedge \beta) = \Diamond\alpha \otimes \Diamond\beta$$

corresponding to the following analytic rules:

$$\text{nec / conec}\ \frac{\hat{1} \vdash X}{\hat{\Diamond}\,\hat{\mathtt{t}} \vdash X} \qquad \frac{\hat{\Diamond}\Gamma \,\hat{\otimes}\, \hat{\Diamond}\Delta \vdash X}{\hat{\Diamond}(\Gamma \,\hat{\wedge}\, \Delta) \vdash X}\ \text{reg / coreg}$$

Deriving $!(A \mathbin{\&} B) \Leftrightarrow\; !A \otimes !B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \mathbin{\&} B \vdash A}}{\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare} A}}{\blacksquare(A \mathbin{\&} B) \vdash \blacksquare A}}{\hat{\Diamond}\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A \mathbin{\&} B \vdash B}}{\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare} B}}{\blacksquare(A \mathbin{\&} B) \vdash \blacksquare B}}{\hat{\Diamond}\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare B}}
{\hat{\Diamond}\blacksquare(A \mathbin{\&} B) \mathbin{\hat{\otimes}} \hat{\Diamond}\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B}\;\text{reg}}
{\hat{\Diamond}(\blacksquare(A \mathbin{\&} B) \mathbin{\hat{\wedge}} \blacksquare(A \mathbin{\&} B)) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B}}
{\blacksquare(A \mathbin{\&} B) \mathbin{\hat{\wedge}} \blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare}(\Diamond\blacksquare A \otimes \Diamond\blacksquare B)}\;C}
{\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare}(\Diamond\blacksquare A \otimes \Diamond\blacksquare B)}}
{\hat{\Diamond}\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B}}
{\Diamond\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B}
$$

$$
\overline{\overline{\;!(A \mathbin{\&} B) \vdash\; !A \otimes !B\;}}
$$

Beyond analiticity: towards a general theory

Achievements in logic:

- Several examples of logics which are single-type not analytic but multi-type analytic:
  - DEL, Inquisitive logic, semi De Morgan logic
  - (Substructural) first order logic
  - Linear logic
  - ...
  - (D)LEs and their analytic inductive axiomatic extensions
- Main guideline: discovering and exploiting hidden adjunctions.
- Can we make this practice into a uniform theory?

Open problems:

- find a list of sufficient (and necessary) conditions to show that a multi-type presentation exists;
- provide a recipe to construct the multi-type presentation.

Structural control via the multi-type approach

General strategy:

- Define a multi-modal logic where linguistic composition is relativized to specific resource **management modes** (via a language expansion: structural control and dependency modalities).
- The extra expressivity is obtained in a controlled fashion via the addition of **interaction postulates** (via axiomatic extensions).
- Structural modalities can be used to **licence** (or to **block**) the access to different regimes of resource management.
- Dependency modalities can be used to **block** the access to different regimes of resource management.

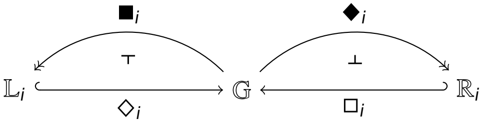
Ingredients:

- The sort of **general** elements that inhabit the more restrictive regime;
- The sorts of **special** elements that witness the licence of a more liberal regime;
- The sort(s) of **blocking** elements that provide the room to block structural transformations.

Heterogeneous structural control algebras 1/2

For each $i \in I$, $\mathbb{H} := (\mathbb{G}, \mathbb{L}_i, \mathbb{R}_i, \mathbb{B})$ is a structure such that

- $\mathbb{G} := (G, \leq_G, \mathcal{F}, \mathcal{G})$ is a fully residuated algebra;
- $(L_i, \leq_{L_i})$ and $(R_i, \leq_{R_i})$ are partial orders



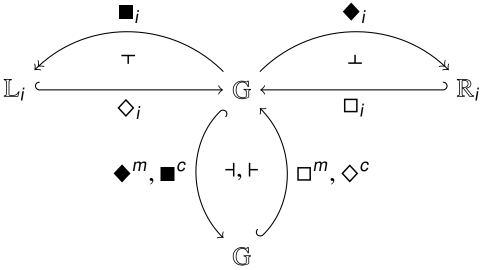
where the composition

$\Diamond_i \blacksquare_i$	defines	an interior operator on $\mathbb{G}$
$\Box_i \blacklozenge_i$	defines	a closure operator on $\mathbb{G}$
$\blacksquare_i \Diamond_i$	defines	identity on $\mathbb{L}_i$
$\blacklozenge_i \Box_i$	defines	identity on $\mathbb{R}_i$

Heterogeneous structural control algebras 2/2

For each $i \in I$, $\mathbb{H} := (\mathbb{G}, \mathbb{L}_i, \mathbb{R}_i, \mathbb{B})$ is a structure such that

- $\mathbb{G} := (G, \leq_G, \mathcal{F}, \mathcal{G})$ is a fully residuated algebra;
- $(L_i, \leq_{L_i})$ and $(R_i, \leq_{R_i})$ are a partial orders;
- for the blocking type we use dependency modalities:
  - $\mathbb{B} = \mathbb{G}$.



where the composition

$\blacksquare^c \Diamond^c$	defines	a closure operator
$\blacklozenge^m \Box^m$	defines	an interior operator

Conclusions

- Multi-type methodology: *Uniform* and *modular* **algorithmic proof theory** paired with *multi-sorted* **algebraic semantics.**
- To do: lift the approach to categories providing **semantics of proofs.**

- Work in the vicinity: Soft Sub-exponential (last invited talk today) + Module actions (generalizing vector spaces) + n-ary heterogenous modalities.