Ecumenical modal logic

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This work: what makes a **modality** classical or intuitionistic?

- with strong proof theoretical properties

Ecumenism in logic

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$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{init}}{\Rightarrow A, \neg A}\,\neg R}{\Rightarrow A \vee \neg A}\,\vee R \qquad \dfrac{\dfrac{\overset{?}{A \Rightarrow \bot}}{\Rightarrow \neg A}\,\neg R}{\Rightarrow A \vee \neg A}\,\vee R_2$$

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On the rules they use? On the proofs they allow?

A solution: They are not talking about the same connective(s) (Prawitz 2015)

Ecumenical connectives and rules

$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B} \rightarrow_c R \qquad \frac{}{\Gamma, \bot \Rightarrow C} \bot L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B} \rightarrow_i R$$

$$\frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B} \vee_c R \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \qquad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2} \vee_i R_j$$

$$\frac{\Gamma, \forall x. \neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x. A} \exists_c R \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x. A} \forall R \qquad \frac{\Gamma \Rightarrow A[a/x]}{\Gamma \Rightarrow \exists_i x. A} \exists_i R$$

Classical | **Shared** | **Intuitionistic**

(Pimentel, Pereira, de Paiva 2020)

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Ecumenism for modal logic

Classical Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \neg A \mid \Box A \mid \Diamond A$
- **Duality** by De Morgan laws and $\neg \Box A \leftrightarrow \Diamond \neg A$

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- Axioms: **classical** propositional logic and

$$\mathsf{k}: \ \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$

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- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Relational structures (W, R)

  a non-empty set W of *worlds*;

  a binary relation $R \subseteq W \times W$;

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \neg A \mid \Box A \mid \Diamond A$
- Independence of the modalities

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\mathsf{k}_4 &: (\Diamond A \to \Box B) \to \Box(A \to B) \\
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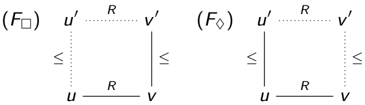
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of *worlds*;

  a binary relation $R \subseteq W \times W$;

  and a preorder $\leq$ on W.



Classical modal proof theory

Sequent system: classical sequent calculus and

$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A, \Delta}{\Box\Gamma \Rightarrow \Box A, \Diamond\Delta} \qquad \mathsf{k}_{\Diamond}\ \frac{\Gamma, A \Rightarrow \Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta}$$

where $\circ\Delta = \circ A_1, \ldots, \circ A_n$ if $\Delta = A_1, \ldots, A_n$ for $\circ \in \{\Box, \Diamond\}$

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Labelled sequent system: (Negri)

$$\Box_\mathsf{L} \ \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B, \Delta}{xRy, \Gamma, x : \Box A \Rightarrow z : B, \Delta} \qquad \Box_\mathsf{R} \ \frac{xRy, \Gamma \Rightarrow y : A, \Delta}{\Gamma \Rightarrow x : \Box A, \Delta} \ y \text{ is fresh}$$

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Completeness?

See Nicola's talk tomorrow and **https://prooftheory.blog**.

Ecumenical modalities

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Ecumenical standard translation:

$$[\Box A]^e_x = \forall y(R(x,y) \to_i [A]^e_y)$$

$$[\Diamond_i A]^e_x = \exists_i y(R(x,y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y(R(x,y) \wedge [A]^e_y)$$

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Formulas: $A ::=$
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Labelled modal rules:

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond_i A}\ \Diamond_i R$$

Classical **Shared** **Intuitionistic**

A pure and internal system

Getting rid of negation

LE

$\Gamma, \neg\Delta \vdash C$

Getting rid of negation

LE $\longrightarrow$ LCE

$\Gamma, \neg\Delta \vdash C$ $\qquad$ $\Gamma \vdash \Delta; C$

Getting rid of negation

LE $\longrightarrow$ LCE

$\Gamma, \neg\Delta \vdash C$ $\qquad$ $\Gamma \vdash \Delta; C$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R$$

Getting rid of negation

LE $\longrightarrow$ LCE

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R \qquad\qquad \frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \rightarrow_i B} \rightarrow_i R$$

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$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R \qquad\qquad \frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \rightarrow_i B} \rightarrow_i R$$

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \rightarrow_c B} \rightarrow_c R \qquad\qquad \frac{\Gamma, A \vdash B, \Delta; \cdot}{\Gamma \vdash A \rightarrow_c B, \Delta; \cdot} \rightarrow_c R$$

Getting rid of negation

LE $\longrightarrow$ LCE

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

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labEK

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A} \Diamond_c R$$

Getting rid of negation

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$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

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labEK $\longrightarrow$ Pure labEK

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A} \Diamond_c R \qquad\qquad \frac{xRy, \Gamma \vdash y : A, x : \Diamond_c A, \Delta; \cdot}{xRy, \Gamma \vdash x : \Diamond_c A, \Delta; \cdot} \Diamond_c R$$

Internal: nested sequents

Nested sequents generalise sequents from a multiset of formulas

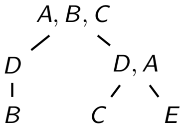
Sequent:

$$A, B, C$$

Internal: nested sequents

Nested sequents generalise sequents from a multiset of formulas to a tree of multisets of formulas.

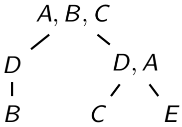
Nested sequent:



Internal: nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree

Nested sequent:

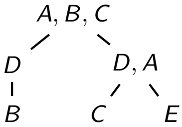


$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

Internal: nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree and can be interpreted as the modal $\Box$.

Nested sequent:



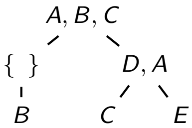
$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

$$A \vee B \vee C \vee \Box(D \vee \Box B) \vee \Box(D \vee A \vee \Box C \vee \Box E)$$

Internal: nested sequents

A context is obtained by removing a formula and replacing it by a hole

Sequent context:

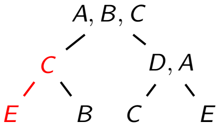


$$\Gamma\{\ \} = A, B, C, [\{\ \}, [B]], [D, A, [C], [E]]$$

Internal: nested sequents

A context is obtained by removing a formula and replacing it by a hole that can then be filled by another nested sequent.

Sequent context:

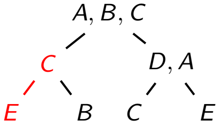


$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

Internal: nested sequents

This allows us to build rules than can be applied at any depth in the tree.

Sequent context:



$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

Pure: stoup vs. store

Separate sequent's RHS: inspired by the mechanism of LU (Girard 1991)

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In nested sequents, the distinction between store and stoup is materialized as:

$$\Lambda ::= \varnothing \mid A^{\bullet}, \Lambda \mid A^{\triangledown}, \Lambda \mid [\Lambda] \qquad \Gamma ::= A^{\circ}, \Lambda \mid [\Gamma], \Lambda \qquad \Delta ::= \Lambda \mid \Gamma$$

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Separate sequent's RHS: inspired by the mechanism of LU (Girard 1991)

In nested sequents, the distinction between store and stoup is materialized as:

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Formula interpretation:

$$\begin{array}{rcl} \mathtt{et}(\varnothing) & ::= & \top \\ \mathtt{et}(A^{\triangledown}, \Lambda) & ::= & \neg A \wedge \mathtt{et}(\Lambda) \\ \mathtt{et}(\Lambda, A^{\circ}) & ::= & \mathtt{et}(\Lambda) \rightarrow_i A \end{array} \qquad \begin{array}{rcl} \mathtt{et}(A^{\bullet}, \Lambda) & ::= & A \wedge \mathtt{et}(\Lambda) \\ \mathtt{et}([\Lambda_1], \Lambda_2) & ::= & \Diamond_i \mathtt{et}(\Lambda_1) \wedge \mathtt{et}(\Lambda_2) \\ \mathtt{et}(\Lambda, [\Gamma]) & ::= & \mathtt{et}(\Lambda) \rightarrow_i \Box \mathtt{et}(\Gamma) \end{array}$$

Pure: stoup vs. store

Polarities:
A formula is called negative if its main connective is classical or the negation

$$N ::= p_c \mid A \vee_c A \mid A \rightarrow_c A \mid \Diamond_c A \mid \neg A$$

Pure: stoup vs. store

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and positive otherwise.

$$P ::= p_i \mid \bot \mid A \wedge A \mid A \vee_i A \mid A \rightarrow_i A \mid \Box A \mid \Diamond_i A$$

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- Negative formulas can be stored.
- Positive formulas are handled in the stoup.

Putting it together

Between store and stoup:

$$\frac{\Gamma^{*}\{P^{\nabla}, P^{\circ}\}}{\Gamma^{\perp^{\circ}}\{P^{\nabla}\}}\ \text{dec} \qquad \frac{\Lambda\{N^{\nabla}, \perp^{\circ}\}}{\Lambda\{N^{\circ}\}}\ \text{sto}$$

Modal rules:

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R$$

$$\frac{\Delta_1^{\perp^{\circ}}\left\{\Diamond_c A^{\nabla}, \left[A^{\nabla}, \Delta_2^{\perp^{\circ}}\right]\right\}}{\Delta_1^{\perp^{\circ}}\left\{\Diamond_c A^{\nabla}, \left[\Delta_2^{\perp^{\circ}}\right]\right\}}\ \Diamond_c^{\nabla}$$

Classical

$$\frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond_i A}\ \Diamond_i R$$

$$\frac{\Lambda_1\{[A^{\circ}, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^{\circ}, [\Lambda_2]\}}\ \Diamond_i^{\circ}$$

Intuitionistic

Putting it together

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Classical | **Shared** | **Intuitionistic**

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Between store and stoup:

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Illustration

$$\vdash \neg\Diamond\neg A \to \Box A$$

is true for any A classically but not intuitionistically

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Ecumenically:
For P positive (classical or negated)

$$\vdash^{\mathsf{EK}} \neg\Diamond_i\neg P \to_i \Box P$$

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While for N negative (intuitionistic or neutral)

$$\not\vdash^{\mathsf{EK}} \neg\Diamond_c\neg N \to_c \Box N$$

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One advantage of the pure system is that this can see structurally without a case analysis on P or N.

Illustration

For P positive (classical or negated)

$$\vdash^{\mathsf{EK}} \neg\Diamond_i\neg P \rightarrow_i \Box P$$

Illustration

For N negative (intuitionistic or neutral)

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Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logic.

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Important remark:
It is true that we can prove $(A \vee_c B) \equiv \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this analysis relies on having three different operators, $\neg$, $\vee_c$ and $\wedge$.

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- relations between results on negative translations and Ecumenical systems

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- relations between results on negative translations and Ecumenical systems

Expanding this discussion to **modalities** is particularly interesting

- the challenges of constructive vs. intuitionistic modal logic