Applications of Real–Valued Logics to Probabilistic Logics

Matteo Mio

CNRS and ENS–Lyon, France

September 6, 2022

Definition (Program Logic)

A formal language to express properties of computer programs.

Definition (Program Logic)

A formal language to express properties of computer programs.

Examples of properties:

- The program P eventually terminates.

Definition (Program Logic)

A formal language to express properties of computer programs.

Examples of properties:

- The program P eventually terminates.
- ...

Program semantics: Programs $\leftrightarrow$ mathematical structures

Program semantics: Programs $\leftrightarrow$ mathematical structures

E.g., Directed Graphs (Kripke Frames, LTSs, ...)

Program semantics: Programs $\leftrightarrow$ mathematical structures

E.g., Directed Graphs (Kripke Frames, LTSs, ...)

Definition (Program Logic)

A formal language to express properties of directed graphs.

Program semantics: Programs $\leftrightarrow$ mathematical structures

E.g., Directed Graphs (Kripke Frames, LTSs, ...)

Definition (Program Logic)

A formal language to express properties of directed graphs.

Example of property:

- There are no infinite paths.

Program semantics: Programs $\leftrightarrow$ mathematical structures

E.g., Directed Graphs (Kripke Frames, LTSs, ...)

Definition (Program Logic)

A formal language to express properties of directed graphs.

Example of property:

- There are no infinite paths.

List of popular program logics:

- modal logic (K), modal μ–calculus
- Computation Tree Logic (CTL),
- Propositional Dynamic Logic (PDL),
- ...

Definition (Probabilistic Logic)

A formal language to express properties of probabilistic programs.

```
covid19 := coin_flip(head,tail);
if (covid19 == head) then:
        return "LATD is in 2021";
else:
        return "LATD is in 2022";
```

Definition (Probabilistic Logic)

A formal language to express properties of probabilistic programs.

```
covid19 := coin_flip(head,tail);
if (covid19 == head) then:
        return "LATD is in 2021";
else:
        return "LATD is in 2022";
```

Example of properties:

- The program P eventually terminates with probability $\geq \frac{1}{3}$.
- ...

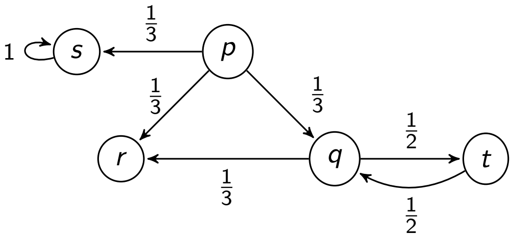
Program semantics: Probabilistic Programs $\leftrightarrow$ mathematical structures

Program semantics: Probabilistic Programs $\leftrightarrow$ mathematical structures

Probabilistic Transition Systems (Discrete Time Markov Chains)

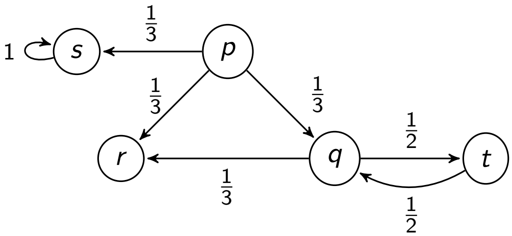
Program semantics: Probabilistic Programs $\leftrightarrow$ mathematical structures

Probabilistic Transition Systems (Discrete Time Markov Chains)



Program semantics: Probabilistic Programs $\leftrightarrow$ mathematical structures

Probabilistic Transition Systems (Discrete Time Markov Chains)

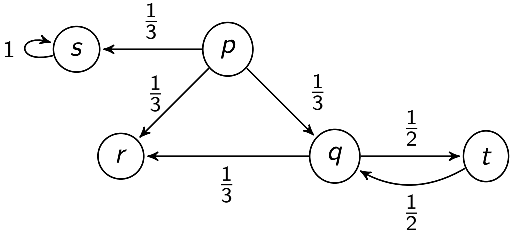


$M =$

	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Program semantics: Probabilistic Programs $\leftrightarrow$ mathematical structures

Probabilistic Transition Systems (Discrete Time Markov Chains)



$$M \quad = \quad \begin{array}{c|c|c|c|c|c|} & p & q & r & s & t \\ \hline p & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \hline q & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \hline r & 0 & 0 & 0 & 0 & 0 \\ \hline s & 0 & 0 & 0 & 1 & 0 \\ \hline t & 0 & \frac{1}{2} & 0 & 0 & 0 \end{array}$$

$$\tau_M : S \to \mathcal{D}^{\leq 1}(S)$$

where $S = \{p, q, r, s, t\}$

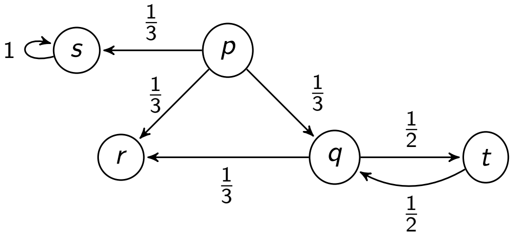
Some Probabilistic Logics:

- probabilistic CTL (pCTL),
- probabilistic Relational Hoare Logic,
- probabilistic μ–calculus,
- probabilistic PDL,
- ...

Some Probabilistic Logics:

- probabilistic CTL (pCTL),
- probabilistic Relational Hoare Logic,
- probabilistic μ–calculus,
- probabilistic PDL,
- ...

The semantics of pCTL is Boolean: $[\![\phi]\!] : S \to \{\text{true}, \text{false}\}$



Example: $[\![\text{“Terminates with probability} \geq \frac{1}{2}\text{''}]\!]\,(p) = \text{true}$

Important Facts about pCTL

Fact 1: Given a finite model M and a pCTL formula ϕ and a state $s \in M$,

$$\text{is } \llbracket \phi \rrbracket (s) = \text{true ?}$$

is <u>decidable</u>, and we have efficient algorithms.

Important Facts about pCTL

Fact 1: Given a finite model M and a pCTL formula ϕ and a state $s \in M$,

$$\text{is } \llbracket \phi \rrbracket (s) = \text{true } ?$$

is decidable, and we have efficient algorithms.

Fact 2: Is a formula ϕ equivalent to a formula ψ in all models?

$$\phi \equiv \psi$$

unknown to be decidable. Moreover, no axiomatisation of $\equiv$ exits.

Important Facts about pCTL

Fact 1: Given a finite model M and a pCTL formula ϕ and a state $s \in M$,

$$\text{is } [\![\phi]\!](s) = \text{true ?}$$

is decidable, and we have efficient algorithms.

Fact 2: Is a formula ϕ equivalent to a formula ψ in all models?

$$\phi \equiv \psi$$

unknown to be decidable. Moreover, no axiomatisation of $\equiv$ exits.

- Lehmann and Shelah, *Reasoning with time and chance*, Journal of Information and Control, 1982.

Failure of "Finite Model Property"

Property ϕ:

1. With probability > 0, the computation describes an <u>infinite</u> path $s_1, s_2 \ldots s_n \ldots$
2. such that, each state s_i stops with probability > 0.

Failure of "Finite Model Property"

Property ϕ:

1. With probability > 0, the computation describes an infinite path $s_1, s_2 \dots s_n \dots$
2. such that, each state s_i stops with probability > 0.

Observation 1: ϕ has an infinite model.

Failure of "Finite Model Property"

Property ϕ:

1. With probability > 0, the computation describes an <u>infinite</u> path $s_1, s_2 \ldots s_n \ldots$
2. such that, each state s_i stops with probability > 0.

Observation 1: ϕ has an infinite model.

$$s_1 \xrightarrow{\frac{3}{4}} s_2 \xrightarrow{\frac{15}{16}} s_3 \xrightarrow{\frac{63}{64}} \ldots \xrightarrow{\ldots} s_n \xrightarrow{1-\frac{1}{4^n}} \ldots \qquad \Big(\prod_{i=1}^{\infty}(1-\frac{1}{4^n}) \approx 0.7\Big)$$

Failure of "Finite Model Property"

Property ϕ:

1. With probability > 0, the computation describes an infinite path $s_1, s_2 \dots s_n \dots$
2. such that, each state s_i stops with probability > 0.

Observation 1: ϕ has an infinite model.

$$s_1 \xrightarrow{\frac{3}{4}} s_2 \xrightarrow{\frac{15}{16}} s_3 \xrightarrow{\frac{63}{64}} \dots \xrightarrow{\dots} s_n \xrightarrow{1-\frac{1}{4^n}} \dots \qquad \Big(\prod_{i=1}^{\infty}(1-\frac{1}{4^n}) \approx 0.7\Big)$$

Observation 2: ϕ is UNSAT in all finite models M.

General Research Program

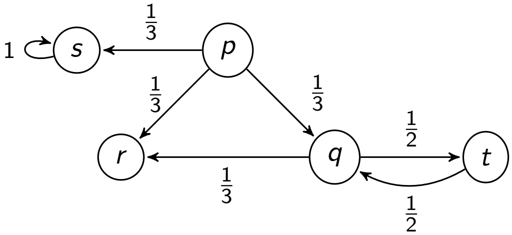
Try to design real-valued ($\mathbb{R}$) probabilistic logics attempting to have both:

1. Efficient Model Checking
2. Decidable theory ($\phi \equiv \psi$)
   - or at least complete (and well–behaved) proof systems.

General Research Program

Try to design real-valued ($\mathbb{R}$) probabilistic logics attempting to have both:

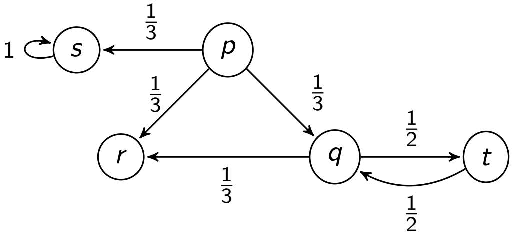
1. Efficient Model Checking
2. Decidable theory ($\phi \equiv \psi$)
   - or at least complete (and well–behaved) proof systems.



Semantics of formulas: $[\![\phi]\!] : S \to \mathbb{R}$ or $[\![\phi]\!] : S \to [0,1]$

Try to design real-valued ($\mathbb{R}$) probabilistic logics attempting to have both:

1. Efficient Model Checking
2. Decidable theory ($\phi \equiv \psi$)
   - or at least complete (and well–behaved) proof systems.



Semantics of formulas: $[\![\phi]\!] : S \to \mathbb{R}$ or $[\![\phi]\!] : S \to [0,1]$

Example of ϕ: "probability of terminating".

The rest of this Talk

1. Łukasiewicz μ-calculus

   `Matteo Mio, Alex Simpson`

   `“Łukasiewicz` μ`-calculus”, Fundamenta Informaticae 2017`

2. Riesz modal Logic

   `Robert Furber, Radu Mardare, Matteo Mio`

   `“Probabilistic logics based on Riesz spaces”, LMCS 2020`

3. Hypersequent Calculus for Riesz modal logic

   `Christophe Lucas, Matteo Mio`

   `“Proof Theory of Riesz Spaces and Modal Riesz Spaces.”, LMCS 2022`

Łukasiewicz μ-calculus logic

Łukasiewicz μ-calculus logic = Łukasiewicz logic + modality + fixed–points

Łukasiewicz μ-calculus logic

Łukasiewicz μ-calculus logic = Łukasiewicz logic + modality + fixed–points

---

Łukasiewicz logic (1930's): $\big([0,1], \neg, \vee, \wedge, \oplus, \odot\big)$

- $\neg x \stackrel{\text{def}}{=} 1 - x$
- $x \vee y \stackrel{\text{def}}{=} \max(x, y) \qquad x \wedge y \stackrel{\text{def}}{=} \min(x, y)$
- $x \oplus y \stackrel{\text{def}}{=} \min(x + y, 1) \qquad x \otimes y \stackrel{\text{def}}{=} \max(0, x + y - 1)$
- $r \cdot x$ for $r \in [0, 1]$

Łukasiewicz μ-calculus logic

Łukasiewicz μ-calculus logic = Łukasiewicz logic + modality + fixed–points

---

Łukasiewicz logic (1930's): $\big([0,1], \neg, \vee, \wedge, \oplus, \odot\big)$

- $\neg x \stackrel{\text{def}}{=} 1 - x$
- $x \vee y \stackrel{\text{def}}{=} \max(x, y)$ $\quad$ $x \wedge y \stackrel{\text{def}}{=} \min(x, y)$
- $x \oplus y \stackrel{\text{def}}{=} \min(x + y, 1)$ $\quad$ $x \otimes y \stackrel{\text{def}}{=} \max(0, x + y - 1)$
- $r \cdot x$ for $r \in [0, 1]$

The same operations can be interpreted pointwise on $[0, 1]^n$ or $S \to [0, 1]$

$$\llbracket \phi \vee \psi \rrbracket (s) = \max(\llbracket \phi \rrbracket (s), \llbracket \psi \rrbracket (s))$$

Łukasiewicz μ-calculus logic

Łukasiewicz μ-calculus logic = Łukasiewicz logic + modality + fixed–points

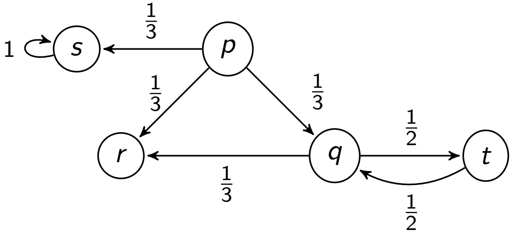
Łukasiewicz logic (1930's): $([0,1], \neg, \vee, \wedge, \oplus, \odot)$

- $\neg x \stackrel{\text{def}}{=} 1 - x$
- $x \vee y \stackrel{\text{def}}{=} \max(x, y) \qquad x \wedge y \stackrel{\text{def}}{=} \min(x, y)$
- $x \oplus y \stackrel{\text{def}}{=} \min(x + y, 1) \qquad x \otimes y \stackrel{\text{def}}{=} \max(0, x + y - 1)$
- $r \cdot x$ for $r \in [0, 1]$

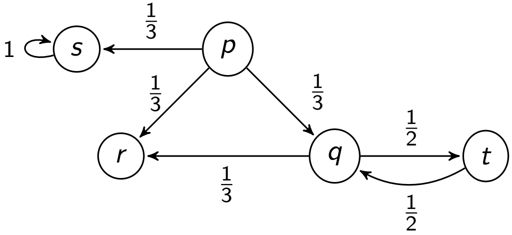
The same operations can be interpreted pointwise on $[0,1]^n$ or $S \to [0,1]$

$$[\![\phi \vee \psi]\!](s) = \max([\![\phi]\!](s), [\![\psi]\!](s))$$

Łukasiewicz μ-calculus formulas $:= x \mid \neg \mid \vee \mid \wedge \mid \oplus \mid \odot \mid r(\_) \mid \ldots$

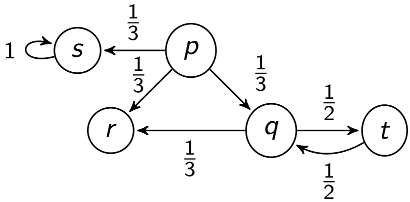


$$[\![\text{``makes a step''}]\!] = \begin{pmatrix} p \to 1 \\ q \to \frac{5}{6} \\ r \to 0 \\ s \to 1 \\ t \to \frac{1}{2} \end{pmatrix}$$

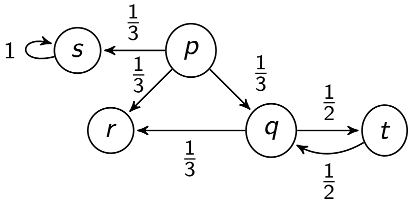


$$[\![\text{“makes a step”}]\!] = \begin{pmatrix} p \to 1 \\ q \to \frac{5}{6} \\ r \to 0 \\ s \to 1 \\ t \to \frac{1}{2} \end{pmatrix}$$

$$[\![\text{“stops”}]\!] = [\![\neg(\text{“makes a step”})]\!] = \begin{pmatrix} p \to 0 \\ q \to \frac{1}{6} \\ r \to 1 \\ s \to 0 \\ t \to \frac{1}{2} \end{pmatrix}$$

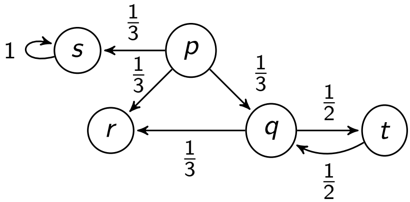


	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

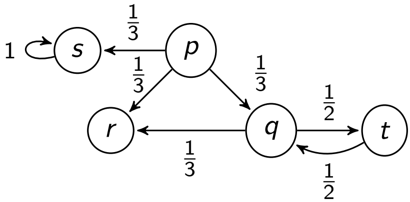
Modal Operator ($\circ$)



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Modal Operator ($\circ$)

$$[\![\circ\phi]\!] \quad = \quad \text{"expected value of } [\![\phi]\!] \text{ at next step."}$$

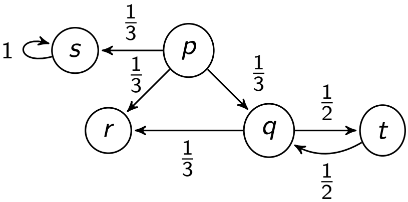


	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Modal Operator ($\circ$)

$$[\![\circ\phi]\!] \quad = \quad \text{“expected value of } [\![\phi]\!] \text{ at next step.”}$$

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Modal Operator ($\circ$)

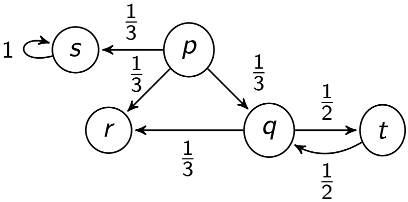
$[\![\circ\phi]\!]$ = "expected value of $[\![\phi]\!]$ at next step."

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$

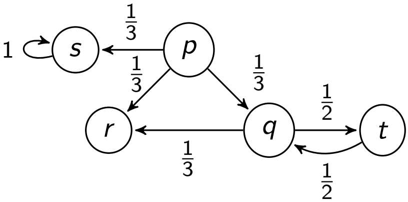
$$[\![\circ(\text{"makes a step"})]\!] =$$

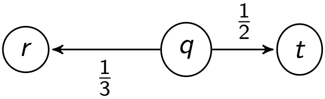
$$M \cdot \begin{pmatrix} p \to 1 \\ q \to \frac{5}{6} \\ r \to 0 \\ s \to 1 \\ t \to \frac{1}{2} \end{pmatrix} = \begin{pmatrix} p \to \frac{11}{18} \\ q \to \frac{1}{4} \\ r \to 0 \\ s \to 1 \\ t \to \frac{5}{12} \end{pmatrix}$$

How to calculate $[\![\circ\phi]\!]$ at a the state q (for example):

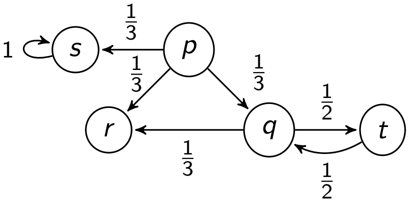


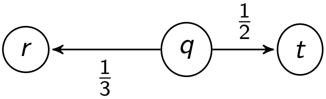
How to calculate $[\![\circ\phi]\!]$ at a the state q (for example):





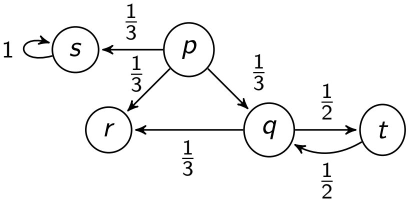
How to calculate $\llbracket \circ\phi \rrbracket$ at a the state q (for example):

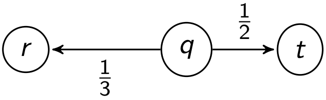




$$\llbracket \circ\phi \rrbracket (q) = \tfrac{1}{3}\big(\llbracket \phi \rrbracket (r)\big) + \tfrac{1}{2}\big(\llbracket \phi \rrbracket (r)\big)$$

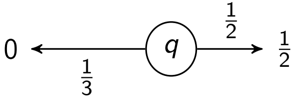
How to calculate $\llbracket \circ\phi \rrbracket$ at a the state q (for example):



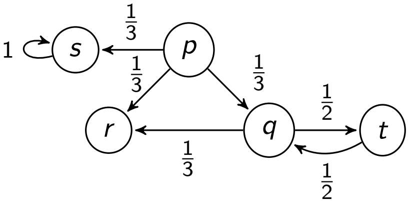


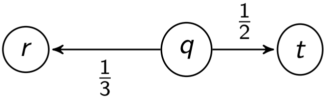
$$\llbracket \circ\phi \rrbracket\,(q) = \tfrac{1}{3}\big(\,\llbracket \phi \rrbracket\,(r)\big) + \tfrac{1}{2}\big(\,\llbracket \phi \rrbracket\,(r)\big)$$

Evaluate ϕ at all adjacent vertices:



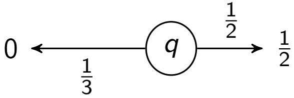
How to calculate $[\![\circ\phi]\!]$ at a the state q (for example):





$$[\![\circ\phi]\!]\,(q) = \tfrac{1}{3}\big(\,[\![\phi]\!]\,(r)\big) + \tfrac{1}{2}\big(\,[\![\phi]\!]\,(r)\big)$$

Evaluate ϕ at all adjacent vertices:



$$[\![\circ\phi]\!]\,(q) = \tfrac{1}{3}(0) + \tfrac{1}{2}(\tfrac{1}{2}) = \tfrac{1}{4}$$

The "Makes *one* step" formula: $\circ 1$

$$[\![\circ 1]\!] = M \cdot \begin{pmatrix} p \rightarrow 1 \\ q \rightarrow 1 \\ r \rightarrow 1 \\ s \rightarrow 1 \\ t \rightarrow 1 \end{pmatrix}$$

The "Makes *one* step" formula: $\circ 1$

$$[\![\circ 1]\!] = M \cdot \begin{pmatrix} p \to 1 \\ q \to 1 \\ r \to 1 \\ s \to 1 \\ t \to 1 \end{pmatrix}$$

The "Makes 2 steps" formula: $\circ(\circ 1)$

The "Makes *one* step" formula: $\circ 1$

$$[\![\circ 1]\!] = M \cdot \begin{pmatrix} p \to 1 \\ q \to 1 \\ r \to 1 \\ s \to 1 \\ t \to 1 \end{pmatrix}$$

The "Makes 2 steps" formula: $\circ(\circ 1)$

The "Makes *n* steps" formula: $\overbrace{\circ(\circ(\ldots(\circ(}^{\text{n}}1)\ldots)$

However we cannot express the "infinite horizon" formula:

"makes ∞–many steps"

However we cannot express the "infinite horizon" formula:

"makes ∞–many steps"

Standard approach: (Kozen's modal μ–calculus)
extend the logic with fixed–points (*à la* Tarski) operators:

However we cannot express the "infinite horizon" formula:

"makes ∞–many steps"

Standard approach: (Kozen's modal μ–calculus)
extend the logic with fixed–points (*à la* Tarski) operators:

$$\text{"make } \infty\text{–many steps"} \stackrel{\text{def}}{=} \nu f.\big(\circ (f)\big)$$

$\big[\big[\nu f.\big(\circ (f)\big)\big]\big]$ is the greatest solution of the equation

$$f = \circ(f)$$

However we cannot express the "infinite horizon" formula:

"makes ∞–many steps"

Standard approach: (Kozen's modal μ–calculus)
extend the logic with fixed–points (*à la* Tarski) operators:

$$\text{“make } \infty\text{–many steps”} \stackrel{\text{def}}{=} \nu f.\big(\circ(f)\big)$$

$\big[\!\big[\nu f.\big(\circ(f)\big)\big]\!\big]$ is the greatest solution of the equation

$$f = \circ(f)$$

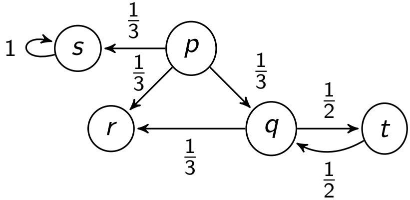
Can be calculated as the limit of "approximations from the top":

$$1 \;\geq\; \circ(1) \;\geq\; \circ(\circ(1)) \;\geq\; \dots \;\geq\; \overbrace{\circ(\circ(\dots(\circ(}^{\text{n}}1)\dots) \;\geq\; \dots$$

Łukasiewicz μ-calculus :=

1. $x \mid 0 \mid 1 \mid \cancel{\times} \mid \vee \mid \wedge \mid \oplus \mid \odot \mid r(x) \mid \dots$
2. $\dots \mid \circ\phi \mid \dots$
3. $\dots \mid \mu x.\phi(x) \mid \nu x.\phi(x)$

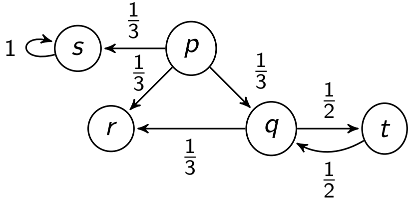
Semantics: $[\![\phi]\!] : S \to [0,1]$



Łukasiewicz μ-calculus :=

1. $x \mid 0 \mid 1 \mid \cancel{\neg} \mid \vee \mid \wedge \mid \oplus \mid \odot \mid r(x) \mid \dots$
2. $\dots \mid \circ\phi \mid \dots$
3. $\dots \mid \mu x.\phi(x) \mid \nu x.\phi(x)$

Semantics: $[\![\phi]\!] : S \to [0,1]$



Remark: without Łukasiewicz mult. connectives ($\oplus \mid \odot$) in:

- Michael Huth, Marta Z. Kwiatkowska: Quantitative Analysis and Model Checking. LICS 1997.
- Luca de Alfaro: Quantitative Verification and Control via the mu-Calculus. CONCUR 2003: 102-126.
- Annabelle McIver, Carroll Morgan: Results on the quantitative mu-calculus qMμ. ACM Trans. Comput. Log. 8(1): 3 (2007).

Main facts regarding Łukasiewicz μ-calculus.

Proposition. For any two states s and t, TFAE:

1. s is probabilistic–bisimilar to t,
2. $[\![\phi]\!](s) = [\![\phi]\!](t)$, for every (fixed–point free) formula ϕ.

Main facts regarding Łukasiewicz μ-calculus.

Proposition. For any two states s and t, TFAE:

1. s is probabilistic–bisimilar to t,
2. $\llbracket\phi\rrbracket(s) = \llbracket\phi\rrbracket(t)$, for every (fixed–point free) formula ϕ.

Proposition. For every pCTL formula $\llbracket F\rrbracket : S \to \{0, 1\}$,
there exists a Łukasiewicz μ-calculus formula ϕ such that

$$\llbracket\phi\rrbracket = \llbracket F\rrbracket .$$

In other words: Łukasiewicz μ-calculus can interpret the Boolean semantics of pCTL.

How to interpret Boolean Semantics:

Definition (Variant of Baaz Delta)

For $f : S \to [0,1]$, we defined $\nabla(f) : S \to [0,1]$ as:

$$\big(\ \nabla(f)\ \big)(s) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } f(s) = 0 \\ 1 & \text{otherwise} \end{cases}$$

How to interpret Boolean Semantics:

Definition (Variant of Baaz Delta)

For $f : S \to [0,1]$, we defined $\nabla(f) : S \to [0,1]$ as:

$$(\ \nabla(f) \ )(s) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } f(s) = 0 \\ 1 & \text{otherwise} \end{cases}$$

Proposition. $[\![\nabla(f)]\!] = \mu x.(x \oplus f)$.
Least solution of: $x = x \oplus f$

$$0 \ \leq \ 0 \oplus f \ \leq \ f \oplus f \ \leq \ \dots \ \leq \ \overbrace{f \ \oplus \cdots \oplus f}^{n} \ \leq \ \dots$$

Remark. Łukasiewicz μ-calculus can define discontinuous operators.

The rest of this Talk

1. Łukasiewicz μ-calculus

   `Matteo Mio, Alex Simpson`

   `“Łukasiewicz` μ`-calculus”, Fundamenta Informaticae 2017`

2. Riesz modal Logic

   `Robert Furber, Radu Mardare, Matteo Mio`

   `“Probabilistic logics based on Riesz spaces”, LMCS 2020`

3. Hypersequent Calculus for Riesz modal logic

   `Christophe Lucas, Matteo Mio`

   `“Proof Theory of Riesz Spaces and Modal Riesz Spaces.”, LMCS 2022`

Why Łukasiewicz ?

- Expressive collection of arithmetic operations (**USED**).

Why Łukasiewicz ?

- Expressive collection of arithmetic operations (**USED**).
- Well studied, a lot of results available
  - Stone-Weierstrass theorem for (rational) Łukasiewicz logic (**USED**),
  - Axiomatisations,
  - Duality Theory,
  - Proof Theory,
  - ...

Why Łukasiewicz ?

- Expressive collection of arithmetic operations (**USED**).
- Well studied, a lot of results available
  - Stone-Weierstrass theorem for (rational) Łukasiewicz logic (**USED**),
  - Axiomatisations,
  - Duality Theory,
  - Proof Theory,
  - ...

Question: can we use the above to axiomatise the Łukasiewicz μ-calculus?

Why Łukasiewicz ?

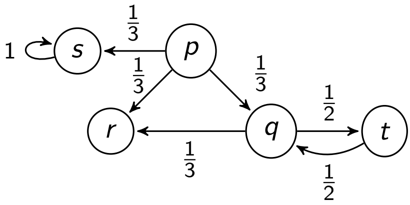
- Expressive collection of arithmetic operations (**USED**).
- Well studied, a lot of results available
  - Stone-Weierstrass theorem for (rational) Łukasiewicz logic (**USED**),
  - Axiomatisations,
  - Duality Theory,
  - Proof Theory,
  - ...

Question: can we use the above to axiomatise the Łukasiewicz μ-calculus?

Łukasiewicz μ-calculus :=

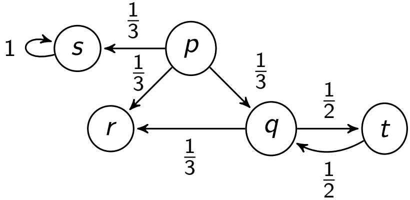
1. $x \mid 0 \mid 1 \mid \cancel{\neg} \mid \vee \mid \wedge \mid \oplus \mid \odot \mid r(x) \mid \ldots$
2. $\ldots \mid \circ\phi \mid \ldots$
3. $\ldots \mid \mu x.\phi(x) \mid \nu x.\phi(x)$

Axiomatisation of the modality $(\circ)$.



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Axiomatisation of the modality (∘).

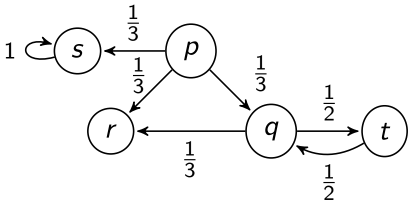


	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Modal Operator (∘)

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$

Axiomatisation of the modality ($\circ$).



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

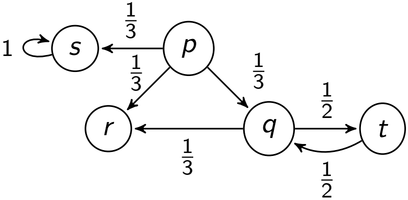
Modal Operator ($\circ$)

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$

Linear: $\circ(f+g) = \circ(f) + \circ(g)$

$r \cdot \circ(f) = \circ(r \cdot f) \qquad r \in \mathbb{R}$

Axiomatisation of the modality ($\circ$).



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

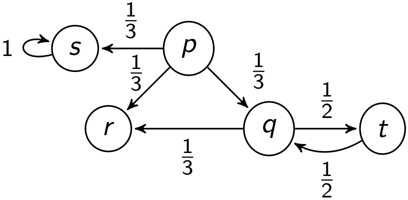
Modal Operator ($\circ$)

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$

Linear: $\circ(f+g) = \circ(f) + \circ(g)$

$r \cdot \circ(f) = \circ(r \cdot f) \qquad r \in \mathbb{R}$

Positive: if $f \geq 0$ then $\circ(f) \geq 0$



	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

Modal Operator ($\circ$)

$$[\![\circ\phi]\!] \stackrel{\text{def}}{=} M \cdot [\![\phi]\!]$$

Linear: $\circ(f+g) = \circ(f) + \circ(g)$

$r \cdot \circ(f) = \circ(r \cdot f) \qquad r \in \mathbb{R}$

Positive: if $f \geq 0$ then $\circ(f) \geq 0$

1–Dec: $\circ(1) \leq 1$

Language used:
Partially ordered vector spaces.

Riesz Representation Theorem

Soundness and completeness: (Finite dimensional)

One-to-one correspondence:

$$\begin{array}{ccc} & & op : \mathbb{R}^n \to \mathbb{R}^n \ \ \text{(Lin., Pos. and 1-dec)} \\ & & \Updownarrow \\ M & \Longleftrightarrow & \tau_M : \underline{n} \to \mathcal{D}^{\leq 1}(\underline{n}) \end{array}$$

where $\underline{n} = \{1, \ldots, n\}$.

Riesz Representation Theorem

Soundness and completeness: (Finite dimensional)
One-to-one correspondence:

$$\begin{array}{ccc} & & op : \mathbb{R}^n \to \mathbb{R}^n \;\; \text{(Lin., Pos. and 1-dec)} \\ & & \Updownarrow \\ M & \Longleftrightarrow & \tau_M : \underline{n} \to \mathcal{D}^{\leq 1}(\underline{n}) \end{array}$$

where $\underline{n} = \{1, \ldots, n\}$.

Soundness and completeness: (Infinite dimensional)
One-to-one correspondence:

$$\begin{array}{ccc} & & op : C(X) \to C(X) \;\; \text{(Lin., Pos. and 1-dec)} \\ & & \Updownarrow \\ \mathcal{M} & \Longleftrightarrow & \tau_{\mathcal{M}} : X \xrightarrow{cont} \mathcal{R}^{\leq 1}(X) \end{array}$$

where:

- X is a Compact Hausdorff space,
- $\mathcal{R}^{\leq 1}$ is the Radon sub–probability monad.

Riesz Modal Logic – Syntax and Semantics

Syntax:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Riesz Modal Logic – Syntax and Semantics

Syntax:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Semantics: $[\![A]\!]_M : S \to \mathbb{R}$

Given $M =$

	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

, $[\![A]\!]_M = \begin{pmatrix} 5 \\ -7 \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$

Semantics:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Semantics:

$$[\![A]\!]_M = \begin{pmatrix} a_p \\ a_q \\ a_r \\ a_s \\ a_t \end{pmatrix} \quad , \quad [\![B]\!]_M = \begin{pmatrix} b_p \\ b_q \\ b_r \\ b_s \\ b_t \end{pmatrix} , [\![A \sqcup B]\!]_M = \begin{pmatrix} \max(a_p, b_p) \\ \max(a_q, b_q) \\ \max(a_r, b_r) \\ \max(a_s, b_s) \\ \max(a_t, b_t) \end{pmatrix}$$

Riesz Modal Logic – Syntax and Semantics

Semantics:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Semantics:

$$[\![A]\!]_M = \begin{pmatrix} a_p \\ a_q \\ a_r \\ a_s \\ a_t \end{pmatrix} \quad , \quad [\![B]\!]_M = \begin{pmatrix} b_p \\ b_q \\ b_r \\ b_s \\ b_t \end{pmatrix} , [\![A + B]\!]_M = \begin{pmatrix} a_p + b_p \\ a_q + b_q \\ a_r + b_r \\ a_s + b_s \\ a_t + b_t \end{pmatrix}$$

Riesz Modal Logic – Syntax and Semantics

Syntax:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Semantics: $[\![\circ A]\!]_M = M \cdot [\![A]\!]_M$

$$M = \begin{array}{c|c|c|c|c|c|} & p & q & r & s & t \\ \hline p & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \hline q & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \hline r & 0 & 0 & 0 & 0 & 0 \\ \hline s & 0 & 0 & 0 & 1 & 0 \\ \hline t & 0 & \frac{1}{2} & 0 & 0 & 0 \end{array} \quad [\![A]\!]_M = \begin{pmatrix} 5 \\ -7 \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \quad [\![\circ A]\!]_M = M \cdot \begin{pmatrix} 5 \\ -7 \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Riesz Modal Logic – Syntax and Semantics

Syntax:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Semantics: $[\![A]\!]_M : S \to \mathbb{R}$

Given $M =$

	p	q	r	s	t
p	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
q	0	0	$\frac{1}{3}$	0	$\frac{1}{2}$
r	0	0	0	0	0
s	0	0	0	1	0
t	0	$\frac{1}{2}$	0	0	0

, $\quad [\![A]\!]_M = \begin{pmatrix} 5 \\ -7 \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$

Riesz Modal Logic – Syntax and Semantics

Syntax:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Continuous Semantics

Given $\tau_M : X \stackrel{cont}{\to} \mathcal{R}^{\leq 1}(X)$ $\qquad$ $[\![A]\!]_M \in C(X)$

$$[\![\circ A]\!]_M (x) \stackrel{\text{def}}{=} \int_X [\![A]\!]_M \, d\mu \qquad \text{where } \mu = \tau_M(x)$$

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x + y) \sqcap \underline{1}$
$x \otimes y$	$(x + y - \underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x + y) \sqcap \underline{1}$
$x \otimes y$	$(x + y - \underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 2: The axioms of $(\circ)$ can be directly expressed:

Linear: $\circ(r_1 x + r_2 y) = r_1 \circ(x) \ + \ r_2 \circ(y)$

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x + y) \sqcap \underline{1}$
$x \otimes y$	$(x + y - \underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 2: The axioms of $(\circ)$ can be directly expressed:

Linear: $\circ(r_1 x + r_2 y) = r_1 \circ(x) \; + \; r_2 \circ(y)$
Positive: $x \geq 0 \Rightarrow \circ(x) \geq 0$

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x+y) \sqcap \underline{1}$
$x \otimes y$	$(x+y-\underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 2: The axioms of $(\circ)$ can be directly expressed:

Linear: $\circ(r_1 x + r_2 y) = r_1 \circ(x) \; + \; r_2 \circ(y)$
Positive: $x \geq 0 \Rightarrow \circ(x) \geq 0$

also expressible as: $\circ(x \sqcup 0) \geq 0$ i.e., $\circ(x \sqcup 0) \sqcap 0 = 0$

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x+y) \sqcap \underline{1}$
$x \otimes y$	$(x+y-\underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 2: The axioms of $(\circ)$ can be directly expressed:

Linear: $\circ(r_1 x + r_2 y) = r_1 \circ(x) \ + \ r_2 \circ(y)$
Positive: $x \geq 0 \Rightarrow \circ(x) \geq 0$

also expressible as: $\circ\,(x \sqcup 0) \geq 0$ i.e., $\circ\,(x \sqcup 0) \sqcap 0 = 0$

1-Dec: $\circ(\underline{1}) \leq 1$

Observation 1: The Riesz modal logic can interpret the Łukasiewicz modal logic:

Łukasiewicz	Riesz
$x \oplus y$	$(x + y) \sqcap \underline{1}$
$x \otimes y$	$(x + y - \underline{1}) \sqcup \underline{0}$

So we can define a "Riesz modal μ–calculus" which can interpret the Łukasiewicz μ–calculus.

Observation 2: The axioms of $(\circ)$ can be directly expressed:

Linear: $\circ(r_1 x + r_2 y) = r_1 \circ(x) \; + \; r_2 \circ(y)$
Positive: $x \geq 0 \Rightarrow \circ(x) \geq 0$

also expressible as: $\circ\,(x \sqcup 0) \geq 0$ i.e., $\circ\,(x \sqcup 0) \sqcap 0 = 0$

1-Dec: $\circ(\underline{1}) \leq 1$

also expressible as: $\circ\,(\underline{1}) \sqcup 1 = 1$

Algebraisation of the Riesz Modal Logic

Definition

A modal Riesz space is a structure $(R, +, 0, 1, r(\_), \sqcup, \sqcap, \circ)$ such that:

1. $(R, +, 0, r(\_), \sqcup, \sqcap)$ is a Riesz space,
2. $\circ$ satisfies the three axioms: Linearity, Positivity and 1-Dec.

Algebraisation of the Riesz Modal Logic

Definition

A modal Riesz space is a structure $(R, +, 0, 1, r(\_), \sqcup, \sqcap, \circ)$ such that:

1. $(R, +, 0, r(\_), \sqcup, \sqcap)$ is a Riesz space,
2. $\circ$ satisfies the three axioms: Linearity, Positivity and 1-Dec.

Definition

A modal Riesz space $(R, +, 0, 1, r(\_), \sqcup, \sqcap, \circ)$ is uniformly–complete *unital* and Archimedean if:

1. The element 1 is a strong unit of R,
2. R is an Archimedean uniformly–complete Riesz space.

Algebraisation of the Riesz Modal Logic

Definition

A modal Riesz space is a structure $(R, +, 0, 1, r(\_), \sqcup, \sqcap, \circ)$ such that:

1. $(R, +, 0, r(\_), \sqcup, \sqcap)$ is a Riesz space,
2. $\circ$ satisfies the three axioms: Linearity, Positivity and 1-Dec.

Definition

A modal Riesz space $(R, +, 0, 1, r(\_), \sqcup, \sqcap, \circ)$ is uniformly–complete *unital* and Archimedean if:

1. The element 1 is a strong unit of R,
2. R is an Archimedean uniformly–complete Riesz space.

Hint: duality theory of Riesz Spaces (Yosida):

$$\text{Archimedean uniformly–complete unital Riesz space} \Longleftrightarrow C(X).$$

for X a compact Hausdorff space.

Results

Theorem 1 (Duality): Let

- $\mathbb{A}$ be the category of uniformly–complete *unital* Archimedean modal Riesz spaces with their homomorphisms.
- $\mathbb{M}$ be the category of topological Markov chains with their coalgebra morphisms.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau_x"'] & Y \arrow[d] \\ \mathcal{R}^{\leq 1}(X) \arrow[r, "\mathcal{R}^{\leq 1} f"'] & \mathcal{R}^{\leq 1}(Y) \end{tikzcd}$$

$\mathbb{A}$ is dual to $\mathbb{M}$: $\mathbb{A} \simeq \mathbb{M}^{\mathsf{op}}$.

Results

Theorem 1 (Duality): Let

- $\mathbb{A}$ be the category of uniformly–complete *unital* Archimedean modal Riesz spaces with their homomorphisms.
- $\mathbb{M}$ be the category of topological Markov chains with their coalgebra morphisms.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau_x"'] & Y \arrow[d] \\ \mathcal{R}^{\leq 1}(X) \arrow[r, "\mathcal{R}^{\leq 1} f"'] & \mathcal{R}^{\leq 1}(Y) \end{tikzcd}$$

$\mathbb{A}$ is dual to $\mathbb{M}$: $\mathbb{A} \simeq \mathbb{M}^{\mathsf{op}}$.

Theorem 2 (Completeness): Let A and B be Riesz modal logic formulas without variables. Then TFAE:

1. $[\![A]\!]_M = [\![B]\!]_M$, for all topological Markov chain M,
2. $\{\text{Axioms of modal Riesz spaces}\} \vdash A = B$.

Why Riesz?

- Expressive collection of arithmetic operations (**USED**).
- Well studied, a lot of results available
  - Stone-Weierstrass theorem for Riesz spaces (**USED**),
  - Axiomatisations (**USED**),
  - Duality Theory (**USED**),
  - Proof Theory,
  - ...

The rest of this Talk

1. Łukasiewicz μ-calculus

   `Matteo Mio, Alex Simpson`

   `“Łukasiewicz` μ`-calculus”, Fundamenta Informaticae 2017`

2. Riesz modal Logic

   `Robert Furber, Radu Mardare, Matteo Mio`

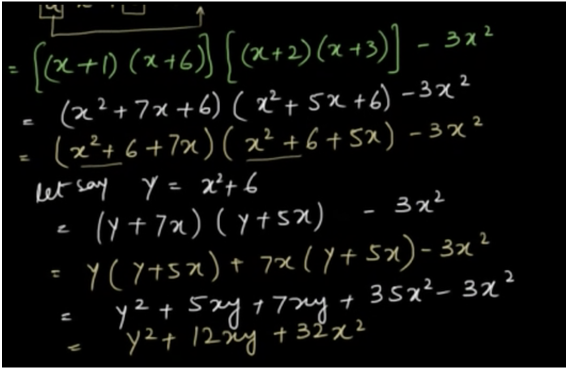
   `“Probabilistic logics based on Riesz spaces”, LMCS 2020`

3. Hypersequent Calculus for Riesz modal logic

   `Christophe Lucas, Matteo Mio`

   `“Proof Theory of Riesz Spaces and Modal Riesz Spaces.”, LMCS 2022`

The axiomatisation of the Riesz modal logic is equational:



What is an analytical Proof System?

Analytic proof

From Wikipedia, the free encyclopedia

In mathematics, an **analytic proof** is a proof of a theorem in analysis that only makes use of methods from analysis, and which does not predominantly make use of algebraic or geometrical methods. The term was first used by Bernard Bolzano, who first provided a non-analytic proof of his intermediate value theorem and then, several years later provided a proof of the theorem which was free from intuitions concerning lines crossing each other at a point, and so he felt happy calling it analytic (Bolzano 1817).

What is an analytical Proof System?

Analytic proof

From Wikipedia, the free encyclopedia

In mathematics, an **analytic proof** is a proof of a theorem in analysis that only makes use of methods from analysis, and which does not predominantly make use of algebraic or geometrical methods. The term was first used by Bernard Bolzano, who first provided a non-analytic proof of his intermediate value theorem and then, several years later provided a proof of the theorem which was free from intuitions concerning lines crossing each other at a point, and so he felt happy calling it analytic (Bolzano 1817).

Equational logic is not analytical:

$$\frac{A = B \qquad\qquad B = C}{A = C}\ \textit{Transitivity}$$

Equational logic is not very useful in establishing <u>in</u>equalities.

Lemma

$\circ(A \sqcup B) \neq \circ A \sqcup \circ B$

Equational logic is not very useful in establishing <u>in</u>equalities.

Lemma

$\circ(A \sqcup B) \neq \circ A \sqcup \circ B$

Proof by counter–example: Let

- $A = \circ\underline{1}$
- $B = \underline{1} - \circ\underline{1}$

and consider the model M:

$$\begin{tikzcd} p \arrow[loop above, "\frac{1}{3}"] \arrow[rr, "\frac{2}{3}"] & & q \end{tikzcd} \qquad\qquad M = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

Equational logic is not very useful in establishing <u>in</u>equalities.

Lemma

$\circ(A \sqcup B) \neq \circ A \sqcup \circ B$

Proof by counter–example: Let

- $A = \circ\underline{1}$
- $B = \underline{1} - \circ\underline{1}$

and consider the model M:

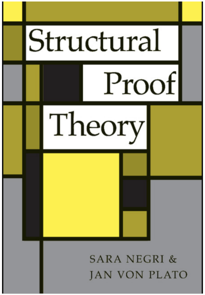
$$\begin{tikzcd} p \arrow[loop above, "\frac{1}{3}"] \arrow[rr, "\frac{2}{3}"] & & q \end{tikzcd} \qquad M = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

$\underline{1}$	A	B	$A \sqcup B$	$\circ(A \sqcup B)$	$\circ A$	$\circ B$	$\circ A \sqcup \circ B$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$

□

Gerhard Gentzen (1909–1945)

Sequent Calculus: $\Gamma \vdash \Delta$



Christophe Lucas, Matteo Mio

"Proof Theory of Riesz Spaces and Modal Riesz Spaces.", LMCS 2022

Christophe Lucas

"Proof Theory of Riesz Modal Logic", PhD thesis, ENS-Lyon, 2022

$$\cfrac{\cfrac{G \mid \vdash \Gamma_1, \vec{r}.A \quad G \mid \vdash \Gamma_2, \vec{s}.\overline{A}}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}.A, \vec{s}.\overline{A}}\ \text{M}}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CAN}, \textstyle\sum \vec{r} = \sum \vec{s}$$

Figure 2.6: Derivability of the CUT rule.

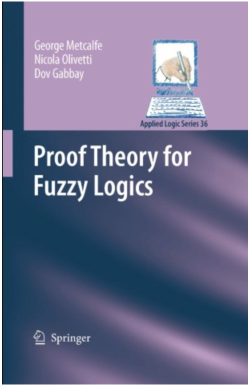
$$\cfrac{\cfrac{\cfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad \cfrac{\cfrac{\cfrac{}{\vdash}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 42}}{G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}\ \text{CUT}}{G \mid \vdash \Gamma, \vec{r}.(A + \overline{A})}\ + \quad \cfrac{\cfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad \cfrac{\cfrac{\cfrac{}{\vdash}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 42}}{G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}\ \text{CUT}}{G \mid \vdash \Gamma, \vec{r}.(\overline{A} + A)}\ +}{\cfrac{\cfrac{G \mid \vdash \Gamma, \Gamma}{G \mid \vdash \Gamma \mid \vdash \Gamma}\ \text{S}}{G \mid \vdash \Gamma}\ \text{C}}\ \text{CUT}$$

Figure 2.7: Derivability of the CAN rule.

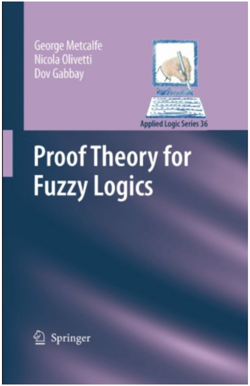
Example 18. Example of derivation of the hypersequent $\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)$ which consists of only one sequent.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash}\ \text{INIT}}{\vdash 2.y, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.y, 2.\overline{x}, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.\overline{y} \mid \vdash 2.y, 2.\overline{x}}\ \text{S}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.y, 1.\overline{x}}\ \text{T(multiplication by 2)}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 2.x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.2x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.(2x + 2\overline{y}) \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)}\ \sqcup$$

Based on earlier work on lattice–ordered abelian groups.



Based on earlier work on lattice–ordered abelian groups.



lattice–ordered abelian groups:

$$A, B ::= x \mid \underline{0} \quad \mid A + B \mid -A \qquad \mid A \sqcap B \mid A \sqcup B$$

modal Riesz spaces:

$$A, B ::= x \mid \underline{0} \mid \underline{1} \mid A + B \mid -A \mid rA \mid A \sqcap B \mid A \sqcup B \mid \circ A$$

Metcalfe Olivetti Gabbay Hypersequent calculus

Lattice ordered abelian groups:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B$$

Sequents: $\overbrace{A_1, \dots, A_n}^{\Gamma} \vdash \overbrace{B_1, \dots, B_m}^{\Delta}$

Hypersequents: $\underbrace{\Gamma_1 \vdash \Delta_1 \mid \ \dots \ \mid \Gamma_1 \vdash \Delta_1}_{H}$

Metcalfe Olivetti Gabbay Hypersequent calculus

Lattice ordered abelian groups:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B$$

Sequents: $\overbrace{A_1, \ldots, A_n}^{\Gamma} \vdash \overbrace{B_1, \ldots, B_m}^{\Delta}$

Hypersequents: $\underbrace{\Gamma_1 \vdash \Delta_1 \mid \ \ldots \ \mid \ \Gamma_1 \vdash \Delta_1}_{H}$

		interpretation
commas	$A_1, \ldots, A_n$	$A_1 + A_2 + \cdots + A_n$
sequents	$\Gamma \vdash \Delta$	$(\Delta) - (\Gamma)$
hypersequents	H	$(\Gamma_1 \vdash \Delta_1) \sqcup (\Gamma_2 \vdash \Delta_2) \sqcup \cdots \sqcup (\Gamma_n \vdash \Delta_n)$

Metcalfe Olivetti Gabbay Hypersequent calculus

Lattice ordered abelian groups:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B$$

Sequents: $\overbrace{A_1, \ldots, A_n}^{\Gamma} \vdash \overbrace{B_1, \ldots, B_m}^{\Delta}$

Hypersequents: $\underbrace{\Gamma_1 \vdash \Delta_1 \mid \ \ldots \ \mid \ \Gamma_1 \vdash \Delta_1}_{H}$

		interpretation
commas	$A_1, \ldots, A_n$	$A_1 + A_2 + \cdots + A_n$
sequents	$\Gamma \vdash \Delta$	$(\Delta) - (\Gamma)$
hypersequents	H	$(\Gamma_1 \vdash \Delta_1) \sqcup (\Gamma_2 \vdash \Delta_2) \sqcup \cdots \sqcup (\Gamma_n \vdash \Delta_n)$

Example: $A \vdash B \mid B \vdash A, C$ is interpreted as $(B - A) \sqcup (A + C - B)$

$$\frac{H \quad | \quad \Gamma, A, B \vdash \Delta}{H \quad | \quad \Gamma, A + B \vdash \Delta} +_{\text{left}}$$

$$\frac{H \quad | \quad \Gamma, A, B \vdash \Delta}{H \quad | \quad \Gamma, A + B \vdash \Delta} +_{\text{left}} \qquad \frac{H \quad | \quad \Gamma, A \vdash \Delta}{H \quad | \quad \Gamma \vdash -A, \Delta} -_{\text{right}}$$

$$\frac{H \mid \Gamma, A, B \vdash \Delta}{H \mid \Gamma, A + B \vdash \Delta} +_{\text{left}} \qquad \frac{H \mid \Gamma, A \vdash \Delta}{H \mid \Gamma \vdash -A, \Delta} -_{\text{right}}$$

$$\frac{H \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{H \mid \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2} S \qquad \frac{H \mid \Gamma_1 \vdash \Delta_1 \quad H \mid \Gamma_2 \vdash \Delta_2}{H \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} M$$

$$\frac{H \quad | \quad \Gamma, A, B \vdash \Delta}{H \quad | \quad \Gamma, A + B \vdash \Delta} +_{\text{left}} \qquad \frac{H \quad | \quad \Gamma, A \vdash \Delta}{H \quad | \quad \Gamma \vdash -A, \Delta} -_{\text{right}}$$

$$\frac{H \quad | \quad \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{H \quad | \quad \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2} \; S \qquad \frac{H \quad | \quad \Gamma_1 \vdash \Delta_1 \qquad H \quad | \quad \Gamma_2 \vdash \Delta_2}{H \quad | \quad \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \; M$$

Theorem (Metcalfe Olivetti Gabbay)

A Hypersequent H is derivable if and only if $H \geq 0$.

$$\frac{H \mid \Gamma, A, B \vdash \Delta}{H \mid \Gamma, A + B \vdash \Delta} +_{\text{left}} \qquad \frac{H \mid \Gamma, A \vdash \Delta}{H \mid \Gamma \vdash -A, \Delta} -_{\text{right}}$$

$$\frac{H \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{H \mid \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2} S \qquad \frac{H \mid \Gamma_1 \vdash \Delta_1 \quad H \mid \Gamma_2 \vdash \Delta_2}{H \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} M$$

Theorem (Metcalfe Olivetti Gabbay)

A Hypersequent H is derivable if and only if $H \geq 0$.

Remark: $A \geq B \iff A - B \geq 0$.

Adding the modality ($\circ$) to the Hypersequent calculus

Lattice ordered abelian groups with $\circ$:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

Adding the modality ($\circ$) to the Hypersequent calculus

Lattice ordered abelian groups with $\circ$:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

$$\frac{F_1, \ldots, F_n \vdash G_1, \ldots, G_m, 1, \ldots, 1}{\circ F_1, \ldots, \circ F_n \vdash \circ G_1, \ldots, \circ G_m, 1, \ldots, 1} \ \circ\text{–rule}$$

Adding the modality (∘) to the Hypersequent calculus

Lattice ordered abelian groups with ∘:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

$$\frac{F_1, \ldots, F_n \vdash G_1, \ldots, G_m, 1, \ldots, 1}{\circ F_1, \ldots, \circ F_n \vdash \circ G_1, \ldots, \circ G_m, 1, \ldots, 1} \circ\text{–rule}$$

Soundness of ∘–rule:

$$-(F_1 + \cdots + F_n) + (G_1, \ldots, G_m, 1, \ldots, 1) \qquad \geq \quad 0$$

Adding the modality (∘) to the Hypersequent calculus

Lattice ordered abelian groups with ∘:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

$$\frac{F_1, \ldots, F_n \vdash G_1, \ldots, G_m, 1, \ldots, 1}{\circ F_1, \ldots, \circ F_n \vdash \circ G_1, \ldots, \circ G_m, 1, \ldots, 1} \;\circ\text{–rule}$$

Soundness of ∘–rule:

$$\begin{array}{lcll} -(F_1 + \cdots + F_n) + (G_1, \ldots, G_m, 1, \ldots, 1) & \geq & 0 & \\ \circ\Big(-(F_1 + \cdots + F_n) + (G_1, \ldots, G_m, 1, \ldots, 1)\Big) & \geq & 0 & \text{(Positivity)} \end{array}$$

Adding the modality ($\circ$) to the Hypersequent calculus

Lattice ordered abelian groups with $\circ$:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

$$\frac{F_1, \dots, F_n \vdash G_1, \dots, G_m, 1, \dots, 1}{\circ F_1, \dots, \circ F_n \vdash \circ G_1, \dots, \circ G_m, 1, \dots, 1} \; \circ\text{–rule}$$

Soundness of $\circ$–rule:

$$\begin{array}{lccr}
-(F_1 + \cdots + F_n) + (G_1, \dots, G_m, 1, \dots, 1) & \geq & 0 & \\
\circ\Big(-(F_1 + \cdots + F_n) + (G_1, \dots, G_m, 1, \dots, 1)\Big) & \geq & 0 & \text{(Positivity)} \\
-(\circ F_1 + \cdots + \circ F_n) + (\circ G_1, \dots, \circ G_m, \circ 1, \dots, \circ 1) & \geq & 0 & \text{(Linearity)}
\end{array}$$

Adding the modality ($\circ$) to the Hypersequent calculus

Lattice ordered abelian groups with $\circ$:

$$A, B ::= \underline{0} \mid A + B \mid -A \mid A \sqcap B \mid A \sqcup B \mid \quad \underline{1} \mid \circ A$$

$$\frac{F_1, \dots, F_n \vdash G_1, \dots, G_m, 1, \dots, 1}{\circ F_1, \dots, \circ F_n \vdash \circ G_1, \dots, \circ G_m, 1, \dots, 1}\ \circ\text{–rule}$$

Soundness of $\circ$–rule:

$$\begin{array}{rcll}
-(F_1 + \cdots + F_n) + (G_1, \dots, G_m, 1, \dots, 1) & \geq & 0 & \\
\circ\Big(-(F_1 + \cdots + F_n) + (G_1, \dots, G_m, 1, \dots, 1)\Big) & \geq & 0 & \text{(Positivity)} \\
-(\circ F_1 + \cdots + \circ F_n) + (\circ G_1, \dots, \circ G_m, \circ 1, \dots, \circ 1) & \geq & 0 & \text{(Linearity)} \\
-(\circ F_1 + \cdots + \circ F_n) + (\circ G_1, \dots, \circ G_m, 1, \dots, 1) & \geq & 0 & \text{(1–dec)}
\end{array}$$

Theorem (Mio, Lucas 2022)

The hypersequent calculus:

1. *is sound and complete for the theory of modal Riesz spaces,*
2. *and has CUT–elimination.*

Theorem (Mio, Lucas 2022)

The hypersequent calculus:

1. *is sound and complete for the theory of modal Riesz spaces,*
2. *and has CUT–elimination.*

Applications of the proof theory:

Theorem

The equational theory of modal Riesz spaces is decidable.

$$F \stackrel{?}{=} G$$

Theorem

Free modal Riesz spaces are Archimedean.

This Talk

Motivating **Open Problem (∼1982)** in the field of probabilistic logics:

$$\phi \stackrel{?}{=} \psi$$

1. Łukasiewicz μ-calculus

   `Matteo Mio, Alex Simpson`

   `“Łukasiewicz` μ`-calculus”, Fundamenta Informaticae 2017`

2. Riesz modal Logic

   `Robert Furber, Radu Mardare, Matteo Mio`

   `“Probabilistic logics based on Riesz spaces”, LMCS 2020`

3. Hypersequent Calculus for Riesz modal logic

   `Christophe Lucas, Matteo Mio`

   `“Proof Theory of Riesz Spaces and Modal Riesz Spaces.”, LMCS 2022`

What's next?

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad\qquad \nu x.F(x)$$

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad \nu x.F(x)$$

- Axiomatisation

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad\qquad \nu x.F(x)$$

- Axiomatisation
    - Well–known "Park induction" rules
      (Tarski's theorem, pre/post fixed points).

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad \nu x.F(x)$$

- Axiomatisation
  - Well–known "Park induction" rules (Tarski's theorem, pre/post fixed points).
- Proof Theory

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad \nu x.F(x)$$

- Axiomatisation
  - Well–known "Park induction" rules (Tarski's theorem, pre/post fixed points).
- Proof Theory
  - Cyclic Proof systems.

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad \nu x.F(x)$$

- Axiomatisation
  - Well–known "Park induction" rules
    (Tarski's theorem, pre/post fixed points).
- Proof Theory
  - Cyclic Proof systems.
- Models

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad \nu x.F(x)$$

- Axiomatisation
  - Well–known "Park induction" rules (Tarski's theorem, pre/post fixed points).
- Proof Theory
  - Cyclic Proof systems.
- Models
  - Theorem: every modal Riesz space can be embedded in a Dedekind complete modal Riesz space.

What's next?

Big challenge are fixed–points:

$$\mu x.F(x) \qquad\qquad \nu x.F(x)$$

- Axiomatisation
  - Well–known "Park induction" rules (Tarski's theorem, pre/post fixed points).
- Proof Theory
  - Cyclic Proof systems.
- Models
  - Theorem: every modal Riesz space can be embedded in a Dedekind complete modal Riesz space.
- ...

THANKS

Motivating **Open Problem (∼1982)** in the field of probabilistic logics:

$$\phi \stackrel{?}{=} \psi$$

1. Łukasiewicz μ-calculus

   `Matteo Mio, Alex Simpson`

   `“Łukasiewicz` μ`-calculus”, Fundamenta Informaticae 2017`

2. Riesz modal Logic

   `Robert Furber, Radu Mardare, Matteo Mio`

   `“Probabilistic logics based on Riesz spaces”, LMCS 2020`

3. Hypersequent Calculus for Riesz modal logic

   `Christophe Lucas, Matteo Mio`

   `“Proof Theory of Riesz Spaces and Modal Riesz Spaces.”, LMCS 2022`