Ł-logic, MV-algebras and C*-algebraic truth-degrees

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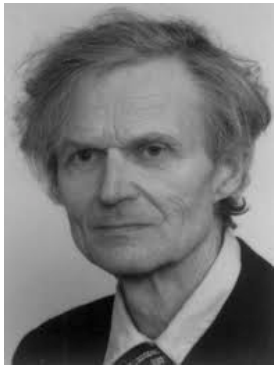
classical Polish logicians: a selection


Adj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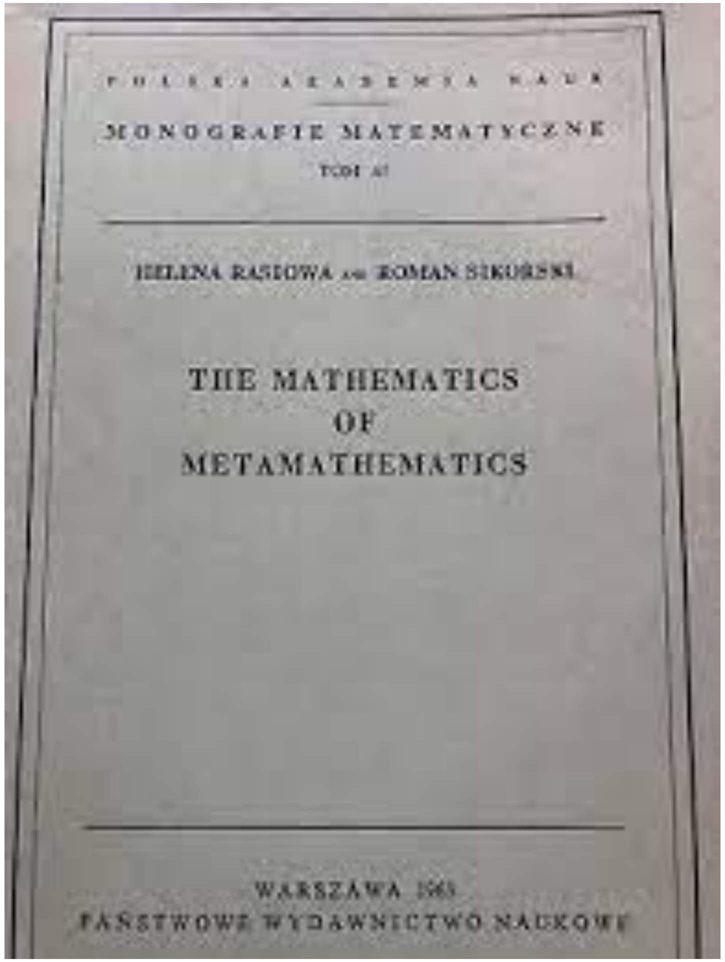
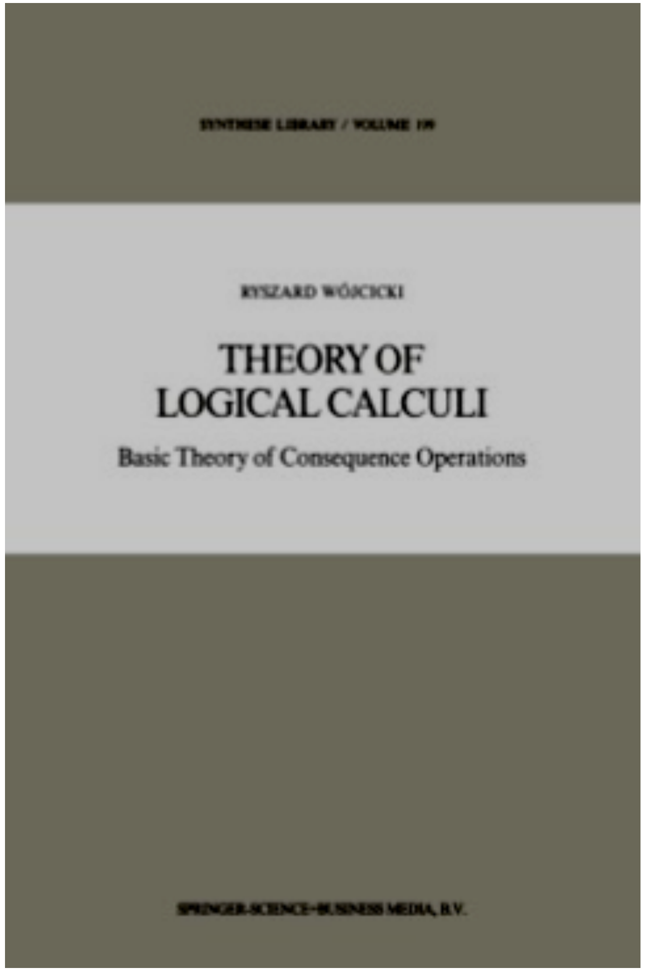
What is a logic? (minimalist Polish style)





What is a logic? (minimalist Polish style)

A set D of **truth-degrees** equipped with a (topological, order-theoretic, algebraic) *structure*, with operations and constants, including a constant 1 for “true”. A set S of symbols for these operations and constants.
A set V of symbols for the *variables.*

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In many lucky cases there is an algorithm to **infer** semantic consequences in a natural way. This usually provides a notion of *syntactic L-consequence.*

The oldest example

The two-element set {0,1} of **truth-degrees** equipped with the natural discrete topology, the natural order, and the (continuous) function $\rightarrow$ such that $x \rightarrow y = 1$ iff $x \leq y$. It follows that $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

The logic $\mathbf{L}_{\{0,1\}}$ with $\neg x = x \rightarrow 0$, $1 = \neg 0$, $x \oplus y = \neg x \rightarrow y$, is boolean **logic.**

The algebras of logically equivalent formulas are boolean **algebras.**

They are defined by the following equations:

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$
$$x \oplus 0 = x$$
$$\neg\neg x = x$$
$$x \oplus \neg 0 = \neg 0$$
$$y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$$
$$x \oplus x = x$$

irreducible ideal
=
maximal ideal

The second oldest example: (D.M., JSL 85, 2020)

The unit real interval $[0,1]$ of **truth-degrees** equipped with the natural topology, the natural order, and a continuous function $\rightarrow$ such that $x \rightarrow y = 1$ iff $x \leq y$. We also assume $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

The logic $\mathbf{L}_{[0,1]}$ with $\neg x = x \rightarrow 0$, $1 = \neg 0$, $x \oplus y = \neg x \rightarrow y$, is Łukasiewicz **logic.**

The algebras of logically equivalent formulas are MV-**algebras.**

They are defined by the following equations:

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irreducible ideal
=
prime ideal

compare the two axiomatizations

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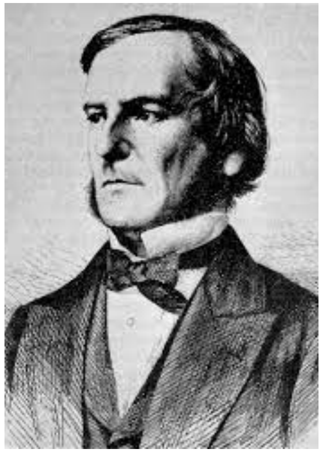

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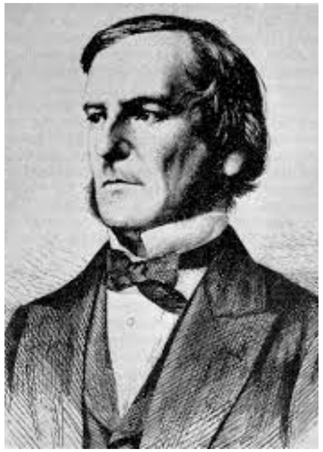

rules from these axioms?
E. Pimentel et al.
APAL 173, 2022

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$$\frac{\text{boolean logic}}{\text{boolean algebras}} = \frac{\text{Łukasiewicz logic Ł}}{\text{MV-algebras}}$$

The parallel analogy

boolean logic / boolean algebras	=	Łukasiewicz logic Ł / MV-algebras

has many faces

The Carathéodory-Stone theorem states that boolean logic and algebras code {0,1}-random variables on boolean spaces

DEFINITION A **state** s of a boolean algebra A is a normalized [0,1]-valued map on A additive on disjunctions of incompatible elements.

(Horn, Kelley and Tarski call s is a **finite-additive probability measure**)

THEOREM (Carathéodory extension theorem+Stone duality)

Any state s of a boolean algebra A can be uniquely extended to regular Borel probability measure μ on the maximal spectral space X of A, i.e., the Stone space of A. The state s can be recovered as

$$s(f) = \int_X f \, \mathrm{d}\mu \qquad \textit{for every } f \in A.$$

The correspondence μ→s is an affine homeomorphism.

https://terrytao.wordpress.com/2009/01/03/254a-notes-0a-an-alternate-approach-to-the-caratheodory-extension-theorem/

finitely additive (states)

≈

countably additive (Borel measures)

more generally,
the Kroupa-Panti
(Riesz- Stone-Weierstrass)
theorem states that Łukasiewicz logic and MV- algebras code Kolmogorov probability on all compact Hausdorff spaces:

Łukasiewicz logic codes [0,1]-random variables

THEOREM (Kroupa-Panti) *There is an affine homeomorphism between the regular Borel probability measures on the maximal spectral space of every **MV-algebra** A, and the states of A.*

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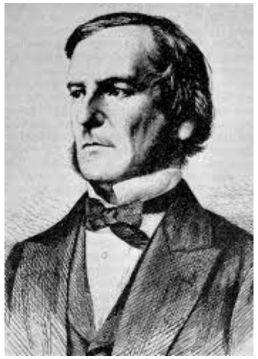
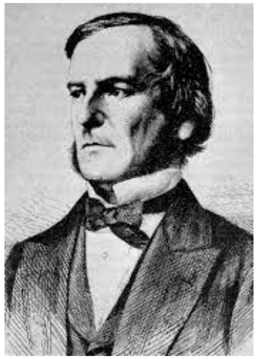


THEOREM (Carathéodory extension+Stone duality, repeated) *There is an affine homeomorphism between the regular Borel probability measures on the maximal spectral space of every **boolean algebra** A, and the states of A.*

Łukasiewicz logic codes [0,1]-random variables

THEOREM *MV-algebraic state = limit of a convex combination of* [0,1]-*homomorphisms.*

[0,1]-*homomorphism = extremal state = quotient by a maximal ideal.*



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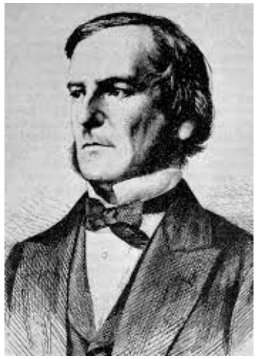
Theorem (D.M., Adv. Applied Math. 125 (2021) 102153)

MV-algebras and their states ≈ Kolmogorov probability on all compact Hausdorff spaces.



Special Case: *Boolean algebras and their states ≈ Kolmogorov probability on all totally disconnected compact Hausdorff spaces.*

Probability is just a first main research area where the scope of boolean algebras and logic is naturally and vastly generalized to MV-algebras and Łukasiewicz logic.

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A second main research area is Computation

as is well known, boolean logic is the universal prototype of NP-problems

Cook's formula codes a Turing machine running on an instance of an NP-problem

$$\bigwedge_t \bigwedge_i \textsc{justone}(\textsc{symb}(t,i,0),\ldots,\textsc{symb}(t,i,k))$$

$$\bigwedge_t \textsc{justone}(\textsc{head}(t,0),\ldots,\textsc{head}(t,2m))$$

$$\bigwedge_t \textsc{justone}(\textsc{state}(t,0),\ldots,\textsc{state}(t,h))$$

$$\bigvee_t \textsc{state}(t,h)$$

$$\bigwedge_t \bigwedge_{i\neq i'} \bigwedge_j (\textsc{head}(t,i') \wedge \textsc{symb}(t,i,j)) \rightarrow \textsc{symb}(t+1,i,j)$$

$$\textsc{state}(0,0) \wedge \textsc{head}(0,m) \ \wedge \bigwedge_{i=0}^{m-1} \textsc{symb}(0,i,0) \ \wedge \bigwedge_{i=m+n}^{2m} \textsc{symb}(0,i,0) \ \wedge$$
$$\wedge\, \textsc{symb}(0,m,j_1) \wedge \textsc{symb}(0,m+1,j_2) \wedge \ldots \wedge \textsc{symb}(0,m+n-1,j_n)$$

$$\bigwedge_t \bigwedge_q \bigwedge_i \bigwedge_j (\textsc{state}(t,q) \wedge \textsc{head}(t,i) \wedge \textsc{symb}(t,i,j)) \rightarrow \textsc{symb}(t+1,i,b(q,j))$$

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$$\bigwedge_t \bigwedge_q \bigwedge_i \bigwedge_j (\textsc{state}(t,q) \wedge \textsc{head}(t,i) \wedge \textsc{symb}(t,i,j)) \rightarrow \bigvee_{r\in\mathcal{S}(q,j)} \textsc{state}(t+1,r)$$

example: the first Cook formula

$$\bigwedge_t \bigwedge_i \textsc{justone}(\textsc{symb}(t, i, 0), \ldots, \textsc{symb}(t, i, k))$$

Read: "at each instant *t*, for each tape square *i*, there is just one symbol printed in *i* at time *t*, among the *k* many possible symbols in the problem alphabet".

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The formula JUSTONE($X_1,\ldots,X_n$) stands for

$$(X_1 \vee \ldots \vee X_n) \wedge (\wedge_{i \neq j}(\neg X_i \vee \neg X_j))$$

The first problem (P ¿=? NP) in the list of seven “Millennium Problems” amounts to asking whether the boolean satisfiability problem SAT has a polytime complexity.

What is so special of SAT among all NP-complete problems?

(Integer Programming, Colorability, Knapsack, Traveling Salesman, Clique,...)

logic is (truth-)functional

the truth-value of a formula only depends on the truth-values of its immediate subformulas

FUNCTIONALITY LEMMA: *Each* function *from the variables into* {0,1} *uniquely extends to a* homomorphism *of the algebra of formulas into the algebra of truth-values*.

a fragment of a PostScript file

00510074008400aa00d1ff4cffaf0012002c004200500051008400be012503da

ff680018003b0098009c009f00a100c100ec018201b4ff68ff76ffd0ffe10002

0018001c00530053007d01b401e103af0486ff9cffeafffe001f0028002a0052

0060009300a300aa00af00af00c001000145016b01740193019502400028202b4

04850517fefd0006002900470047004800 6f008800b400b900c400f200f901ef

02180310037403c5ff35fff3000b004b004c005200550065007600760087 0087

008e00ab00bb0106013001430150017d0194019501d3022a025502580277 0278

This is not so easy reading as Cook’s formula.

Too far from human language.

This is an example of a non-truth-functional coding

Also Łukasiewicz logic is universal for NP-problems

THEOREM (D.M., Theoretical Computer Science, 52, 1987) *The satisfiability problem of Łukasiewicz logic is* NP-*complete*.

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extra bonus: Łukasiewicz logic computes on Elliott classification

Apologies for introducing, in this final part of my talk, a modicum of functional analysis and Elliott classification.

I’m confident these are known to you.

Morever all this material is easily presentable.

My purpose is to show that the NP-complete algorithmic machinery of **Łukasiewicz** logic can be fruitfully applied to compute on an important class of **C*-algebras.**

classical functional analysis

A ****-algebra*** is an algebra B over $\mathbf{C}$ with a linear map * satisfying $y^{**}=y$, $(zy)^*=y^*z^*$, and $(\mu z)^*=\bar{\mu} z^*$ for all $\mu\in\mathbf{C}$ and $y, z\in B$.

Stefan Banach

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A ***C*-algebra*** *is* a *-algebra B with a norm making B into a **Banach** space such that $\|1_A\| = 1$, $\|yz\| \leq \|y\| \, \|z\|$, and $\|zz^*\| = \|z\|^2$.

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Why C*-algebras

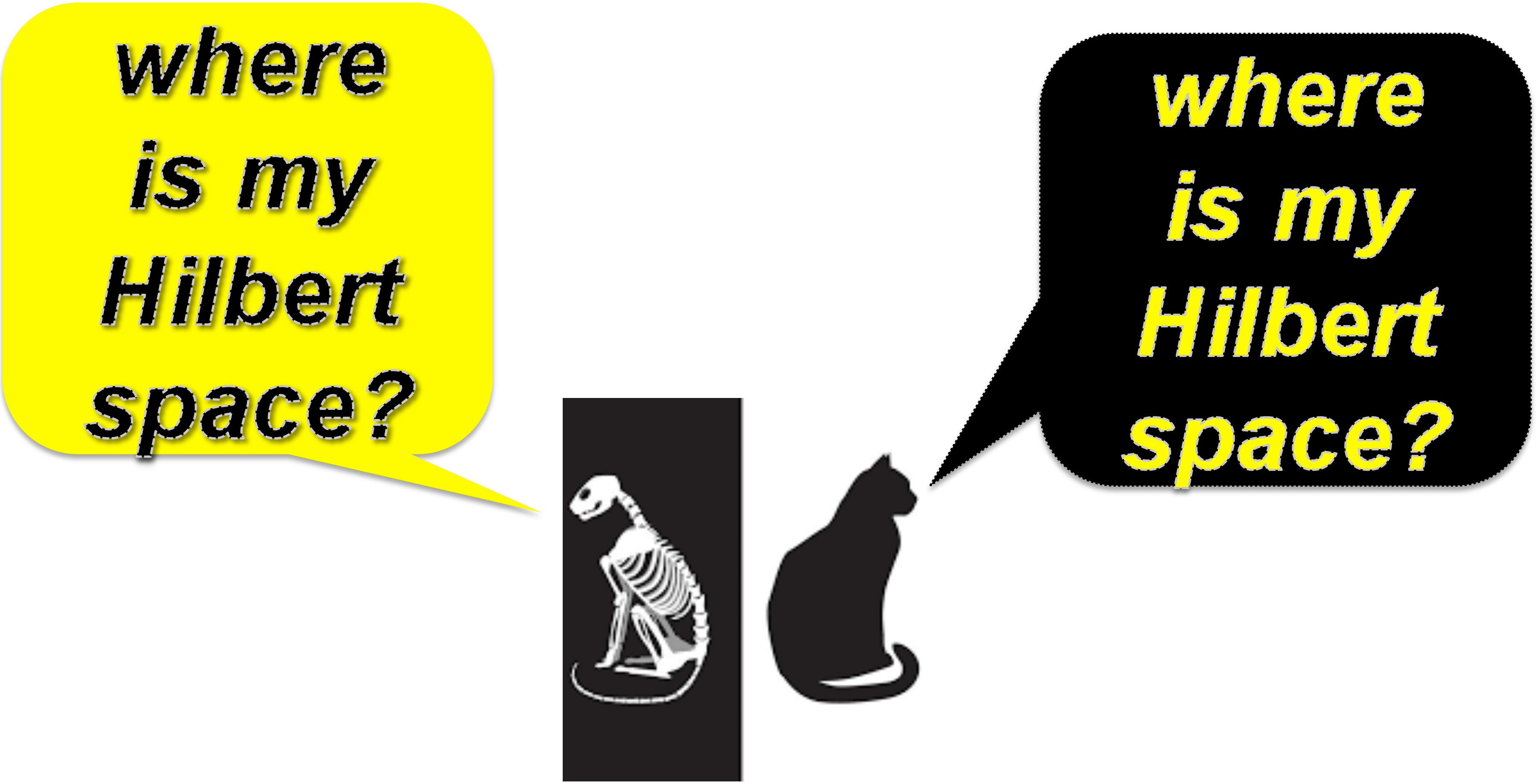
“... *the Hilbert space formulation of quantum mechanics scored impressive successes when dealing with non-relativistic systems involving only a finite number of degrees of freedom. As the years passed, however, quantum statistical mechanics and relativistic quantum field theory were grudgingly recognized to lie somewhere beyond the reach of this formalism. [...] The observables should be constructed from the self-adjoint elements of a C*-algebra,...*”

G.EMCH, Mathematical and Conceptual Foundations of 20th Century Physics, North Holland, 1984, p. 361

Gerard Emch

One can no longer speak of **the** Hilbert space of the system

Schrödinger’s cat in a C*-algebra



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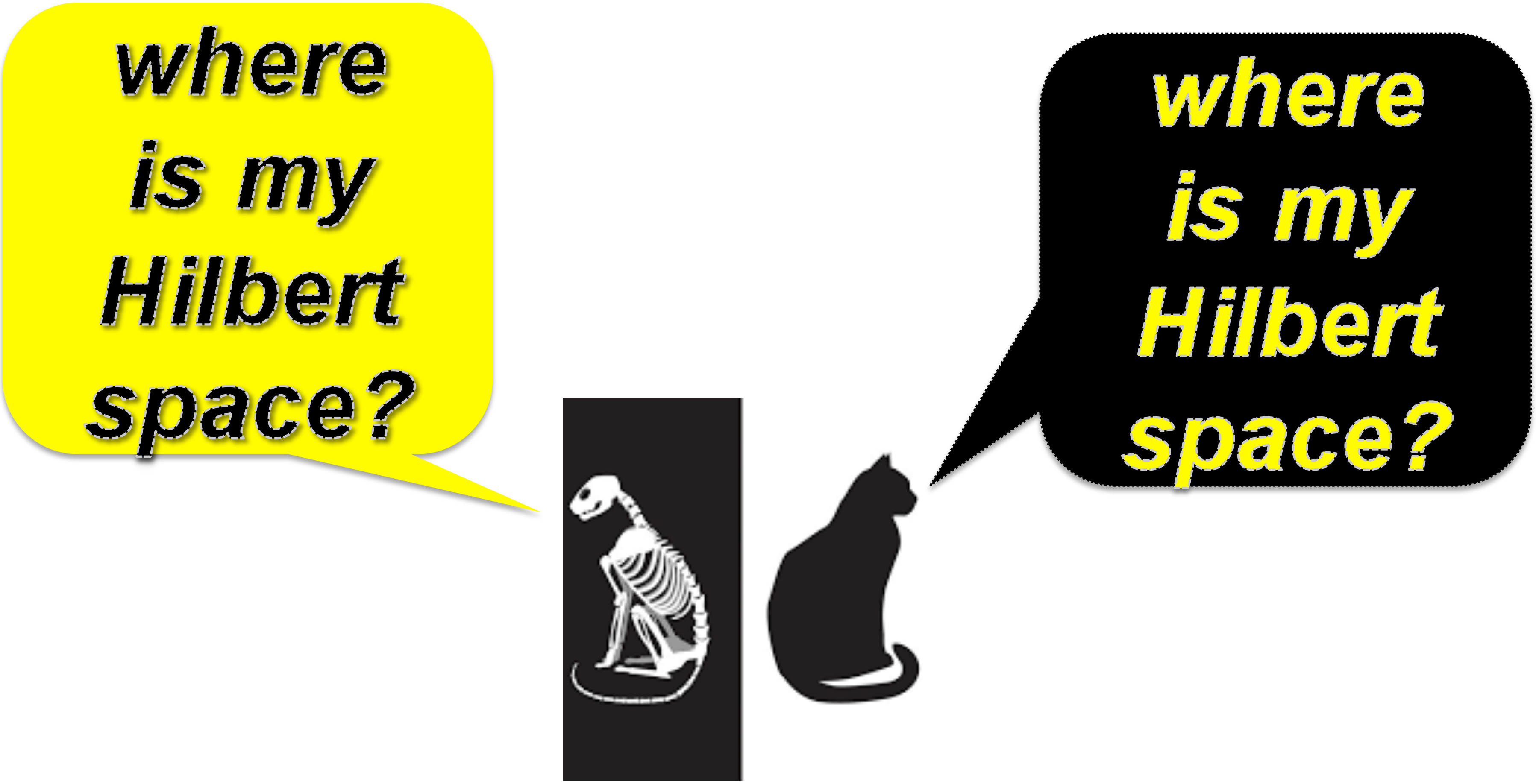

no Birkhoff-von Neumann quantum logic

AF-algebras

(REMEMBER) *A* ***C*-algebra*** *is* a *-algebra B with a norm making B into a **Banach** space such that $\|1_A\| = 1$, $\|yz\| \leq \|y\|\,\|z\|$, and $\|zz^*\| = \|z\|^2$.

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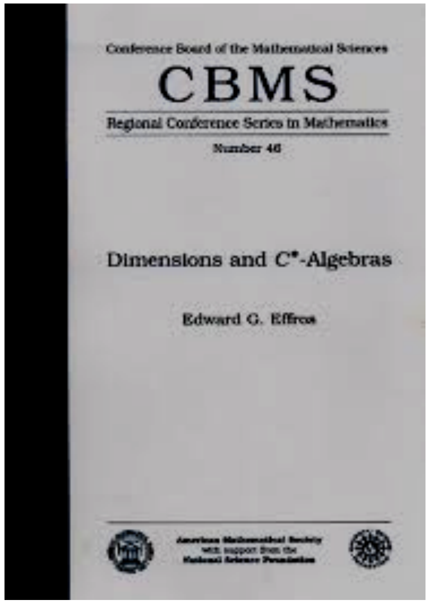
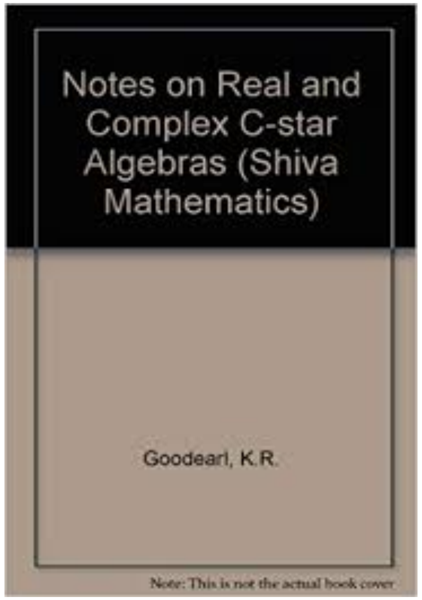
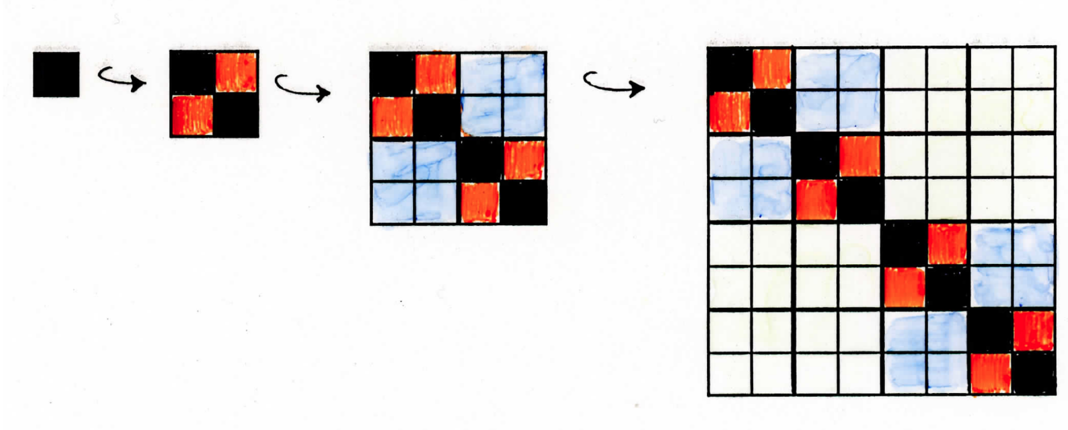




example: the CAR algebra of the Fermi gas

$M_1(\mathbf{C}) \rightarrow M_2(\mathbf{C}) \rightarrow M_4(\mathbf{C}) \rightarrow M_8(\mathbf{C})$ …



…

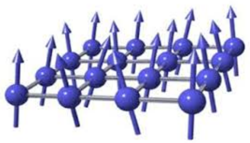
the limit of this sequence of finite-dimensional C*-algebras is known as the **Canonical Anticommutation Relation** (CAR) algebra

Why AF-algebras

In a macroscopic assembly S of particles, S is an infinite volume limit of finite systems $S_1 \subset S_2 \subset S_3 \subset \ldots$ with constant density.

The observables of S are the self-adjoint elements of a C*-algebra A, which is the limit of a sequence of finite-dimensional C*-algebras,

i.e., A is an **AF-algebra,** the C*-algebra of S



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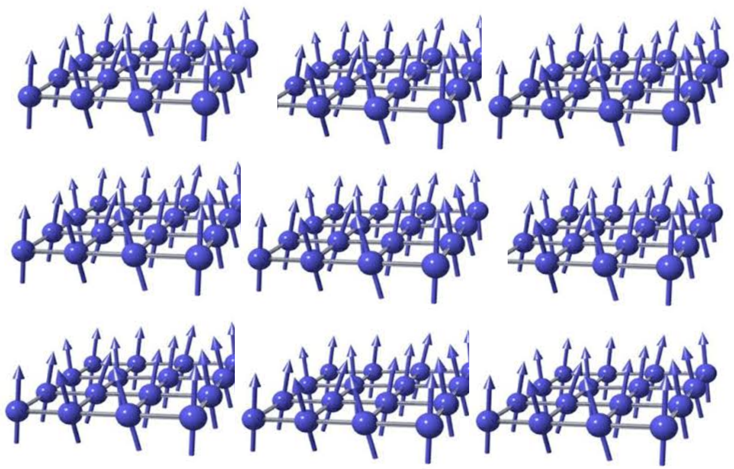

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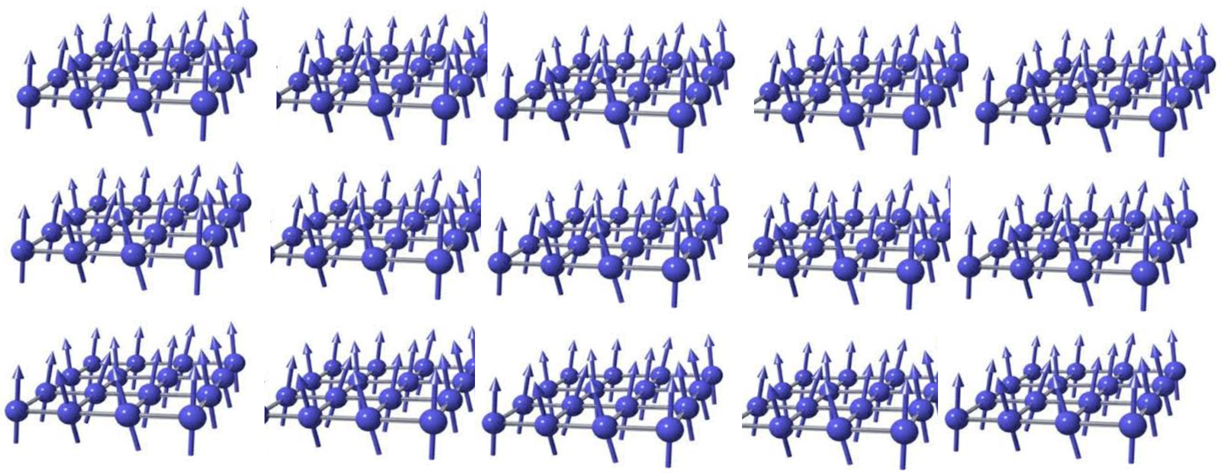

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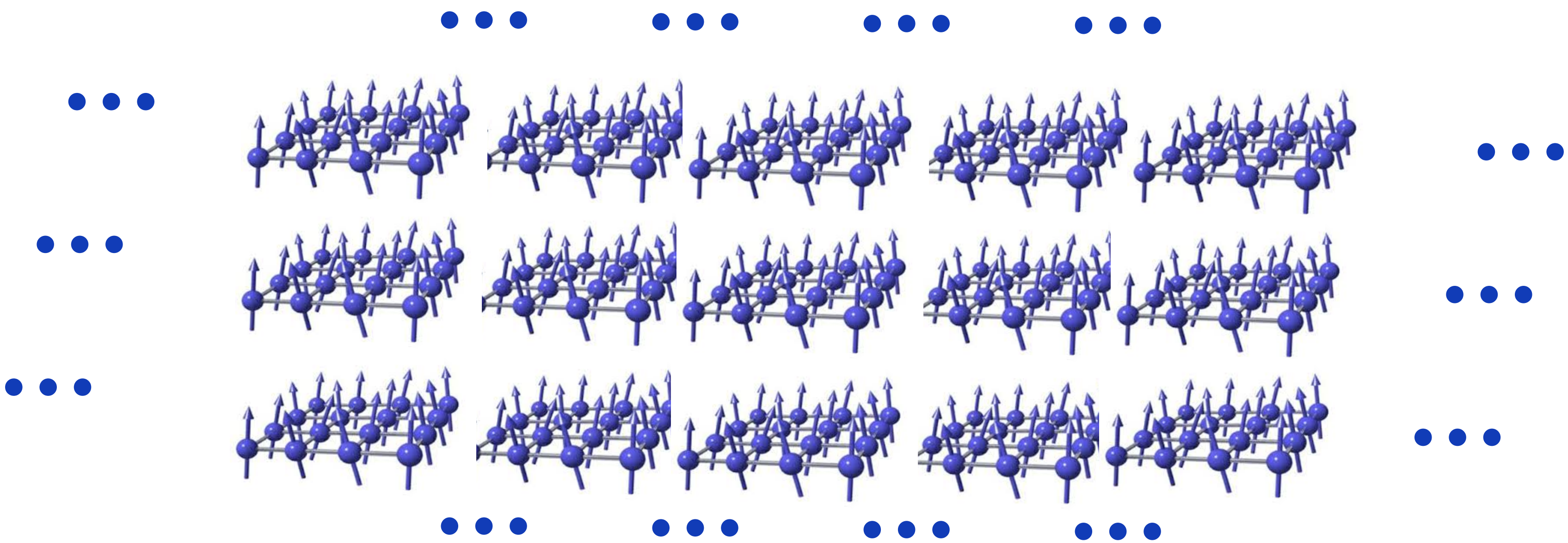

Elliott's **partial** disjunction

Projections p and q in an AF-algebra A are *Murray-von Neumann equivalent* if $p = x^*x$ and $q = xx^*$ for some $x \in A$.

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Elliott's ***partial*** *disjunction* $[p]+[q]$ in $L(A)$ is obtained by summing orthogonal projections.

Elliott's **partial** disjunction

Projections p and q in an AF-algebra A are *Murray-von Neumann equivalent* if $p = x^*x$ and $q = xx^*$ for some $x \in A$.

The *Murray-von Neumann order* on the *set* $L(A)$ *of equivalence classes* is given by: $[p] \le [q]$ iff p is equivalent to a sub-projection of q.

Elliott's ***partial*** *disjunction* $[p]+[q]$ in $L(A)$ is obtained by summing orthogonal projections.

George A. Elliott

THEOREM (**Elliott classification**, J. of Algebra 38 (1976) 29-44) *Two AF-algebras are isomorphic iff so are their* ***partial*** *semigroups*.

A problem in Elliott classification is that the classifier for AFalgebras is a partial algebraic structure.

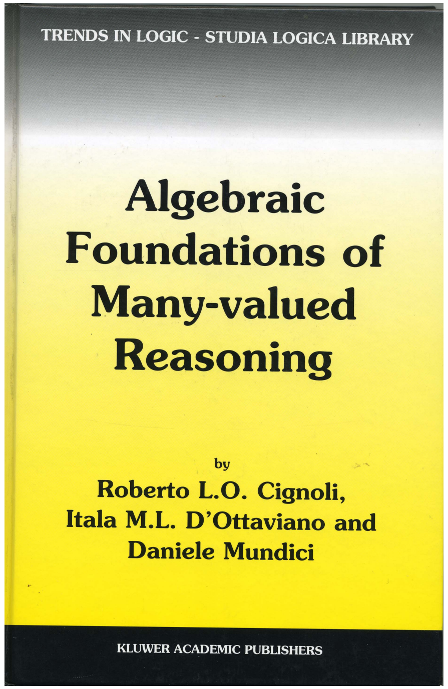
Can we make it total for some interesting and suitably large class of AF-algebras?

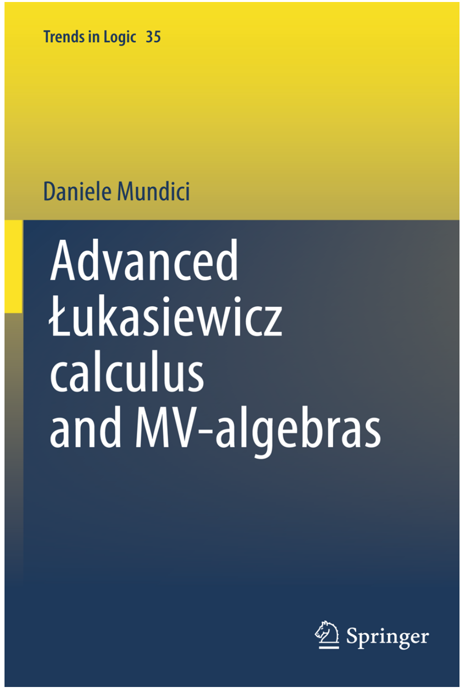
(dimension groups classify AF-algebras)

Like Elliott's disjunction, Boole's disjunction *x*+*y* was only defined for incompatible *x* and *y*. So we may ask whether Elliott's partial operation can be canonically extended to a relevant algebraic structure.

MV-algebraic background, 1986

THEOREM (D.M., J.Functional Analysis, 65, 1986) *There is a categorical equivalence Γ between MV-algebras and lattice-ordered abelian groups with a distinguished strong order unit.*





Elliott's disjunction made **total, 1993**

THEOREM (D.M., G. Panti, J. Functional Analysis, 117 (1993) 461-471)

(i) Elliott's ***partial*** *addition* $+$ *has* ***at most one*** *extension to an associative, commutative, monotone* ***total*** *operation* $\oplus$ *over* $L(A)$ *such that* $[1-p] = \neg[p]$ *is the smallest class* $[q]$ *with* $[q] \oplus [p] = 1$.

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(ii) Such extension $\oplus$ *exists if and only if* $L(A)$ *is lattice-ordered, i.e., iff* A *is an **AFl-algebra**. The resulting involutive monoid* $(L(A), 0, 1, \neg, \oplus) = \Gamma(K_0(A))$ ***is a countable MV-algebra. Every such MV-algebra arises in this way.***

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REMARK (expanded) Like Elliott's disjunction, Boole's disjunction $x+y$ was only defined for incompatible events x and y, so that their probabilities could be added. Jevons and others noted that this **partial** disjunction can be uniquely extended to a **total** operation as in (i) above.

Boolean algebras were born.

AF/-algebras are preeminent in the AF-algebraic literature:

MV-algebras and AF*l*-algebras

COUNTABLE MV-ALGEBRA	AF*l* ALGEBRA
$\{0,1\}$	$\mathbf{C}$ = the complex numbers
finite	finite-dimensional
boolean	commutative
Łukasiewicz chain $\{0, 1/n, \ldots, (n-1)/n, 1\}$	$M_n(\mathbf{C})$, the $n \times n$ complex matrices
totally ordered	with Murray-von Neumann comparability
atomless boolean	$C(2^\omega)$, with 2^ω the Cantor cube
subalgebra of $\mathbf{Q} \cap [0,1]$	UHF, Glimm uniformly hyperfinite
$\mathbf{Q} \cap [0,1]$, $\mathbf{Q}$ = the rationals	Glimm universal UHF algebra
dyadic rationals in the unit interval	CAR algebra of the ideal Fermi gas
algebra generated by $\rho \in [0,1] \setminus \mathbf{Q}$	Effros-Shen algebra F_ρ
Chang algebra C	Behncke-Leptin algebra $A_{0,1}$
real algebraic numbers in $[0,1]$	Blackadar algebra B
free with one generator	the Farey algebra
free with $\aleph_0$ generators	the universal AF C*-algebra $\mathfrak{M}$
Logicians	**Functional analysts and physicists**

as we have seen, MV-algebras are the algebras of Łukasiewicz logic, a logic with an NP-complete deductive algorithmic structure

Formulas in Łukasiewicz logic Ł code equivalence classes of projections of AF*I*-algebras.

The deductive-algorithmic machinery of Ł can be applied to **compute** on these projections

examples of decision problems for any AFI-algebra A and arbitrary input formulas φ, ψ

Do φ and ψ code the same equivalence class of projections of A?

This is the classical *word problem* for A

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Does φ code a *full* projection of A? The *full projection problem*

(*p* is a *full* projection in *A* if the ideal generated by *p* in A coincides with *A*)

THEOREM
(D.M., Advances in Math. 335, 2018)

*The **word problem** is solvable in polytime for the following AFl-algebras:*

All Behnke-Leptin algebras A_{mn}.

The Effros-Shen F_θ *, for* θ *a real algebraic number, or* $\theta = 1/e$, e = *Euler constant*.

Glimm's universal UHF, and the CAR algebra.

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real algebraic numbers in [0,1]	Blackadar algebra *B*
totally ordered	with Murray-von Neumann comparability
Łukasiewicz chain $\{0, 1/n, ..., (n-1)/n, 1\}$	$M_n(\mathbf{C})$, the $n \times n$ complex matrices

THEOREM (D.M., Ann. Pure Appl. Logic, August 2022)

https://doi.org/10.1016/j.apal.2022.103182

*The **order, central, full projection problems** are solvable in polytime for the following AFl-algebras:*

All Behnke-Leptin algebras A_{mn}.

The Effros-Shen F_θ *, for* θ *a real algebraic number, or* $\theta = 1/e$*, e = Euler constant.*

Glimm's universal UHF, and the CAR algebra.

There exist AFl-algebras where all these problems are Gödel incomplete.

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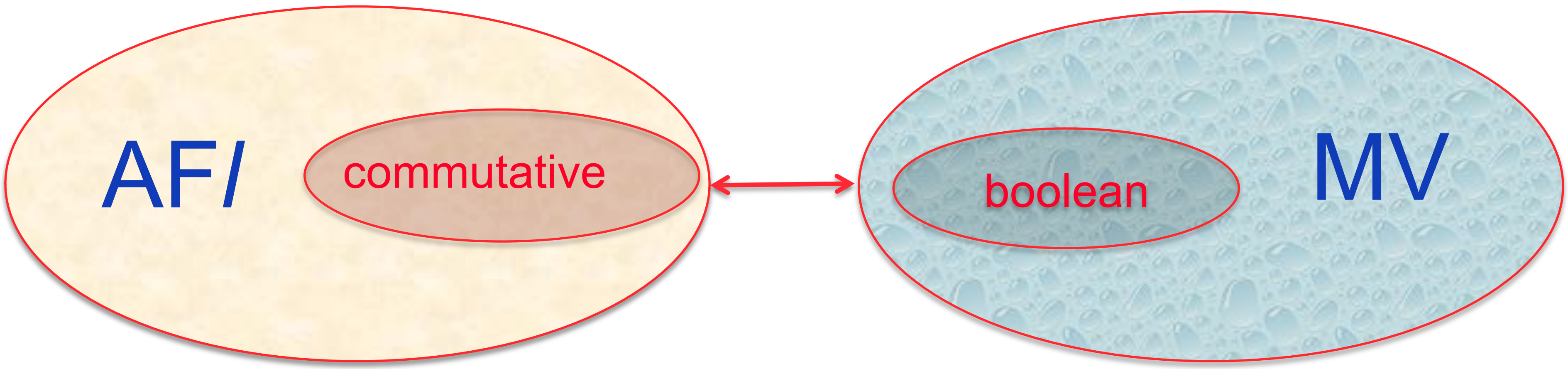
Summing up, Łukasiewicz logic deals with Murray-von Neumann equivalence classes of idempotent observables of quantum statistical systems described by AF*l* algebras.

In particular, boolean logic deals with projections in the fragment given by commutative AF algebras.

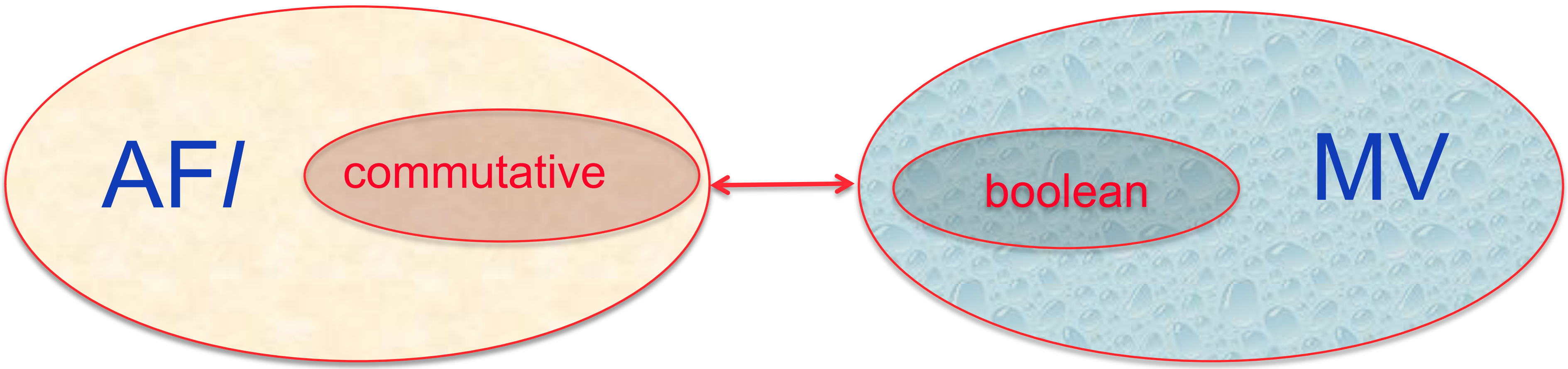
Probability



MV-algebras and their states ≈ Kolmogorov probability to all compact Hausdorff spaces.

Boolean algebras and their states ≈ Kolmogorov probability on boolean spaces.

Computation



Łukasiewicz logic computes on projections of AF*I*-algebras.

Boolean logic computes on projections of commutative AF*I*-algebras.


thank you