First-order fuzzy logics and their model theory

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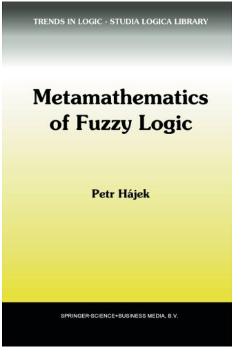
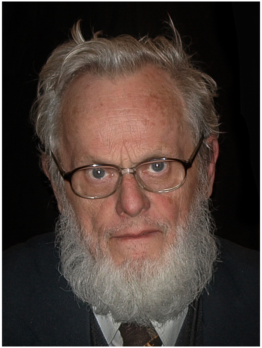
Main goals of the talk

- First-order fuzzy logics as studied in Mathematical Fuzzy Logic
- Two motivations:
  1. Study reasoning with graded predicates
  2. Study weighted (fuzzy) structures that appear in computer science
- Graded Model Theory as a part of MFL, inspired by Classical Model Theory
- Survey of known results
- Lines for future research

Mathematical Fuzzy Logic (MFL) – 1

- Started in the 90s by Petr Hájek, Vilém Novák, Siegfried Gottwald, Francesc Esteva, Lluís Godo, Daniele Mundici, Franco Montagna, Antonio Di Nola, and many others
- Logical foundations of fuzzy set theory
- MFL has become a genuine subdiscipline of Mathematical Logic, specializing in the study of certain many-valued logics
- Hájek's monograph (1998): G, Ł, Π, BL





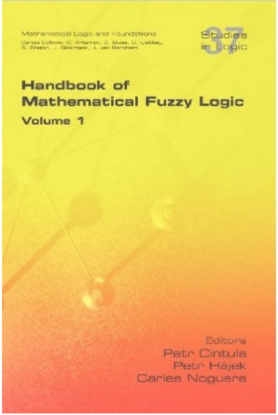
Mathematical Fuzzy Logic (MFL) – 2

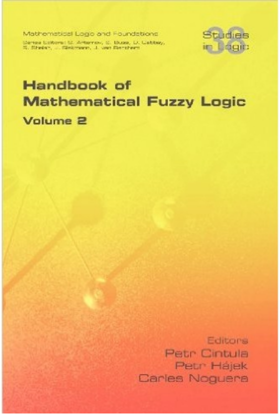
- Standard semantics over the unit interval $[0, 1]$
- Order-based connectives $\vee = \max$ and $\wedge = \min$
- Constants for total truth ($\overline{1}$) and total falsity ($\overline{0}$)
- Another conjunction $\&$ interpreted by a (left-continuous) t-norm: binary commutative, associative, monotone function on $[0, 1]$
- An implication given by the residuum of the t-norm:

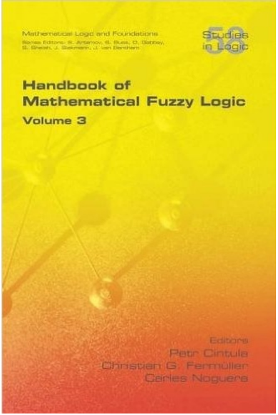
$$a \,\&\, b \leq c \text{ if, and only if, } a \leq b \rightarrow c$$

Mathematical Fuzzy Logic (MFL) – 3

The state of the art (as of 2015) is summarized in:







P. Cintula, C. Fermüller, P. Hájek, C. Noguera (editors). Vol. 37, 38, and 58 of *Studies in Logic: Math. Logic and Foundations*. College Publications, 2011, 2015.

First-order fuzzy logics

- 1961: Mostowski proposes to study first-order many-valued logics (interpreting $\forall$ as infimum, and $\exists$ as supremum).
- 1962: Scarpellini proves that first-order Łukasiewicz logic is not recursively axiomatizable.
- 1963: Rasiowa and Sikorski interpret similarly first-order intuitionistic logic over Heyting algebras.
- 1963: Hay axiomatizes first-order Łukasiewicz logic (with an infinitary rule).
- 1969: Horn axiomatizes first-order Gödel–Dummett logic.
- 1986: Di Nola and G. Gerla, *Fuzzy models of first-order languages*.
- 1990: Novák studies first-order Pavelka logic (Łukasiewicz logic expanded with constants for each real number in $[0, 1]$).
- 1998: Based on these previous works, Hájek gives his general approach to first-order fuzzy logics. He axiomatizes the semantics based on all corresponding linearly algebras.

First motivation: Logic, reasoning, and gradedness

- Logic is concerned with reasoning.
- Everyday language and reasoning successfully handle graded properties all the time, i.e. properties that are a matter of more-or-less such as *red*, *old*, *tall*, or *rich*.
- Fuzzy logics have been proposed as useful mathematical apparatus to formalize (parts of) reasoning with graded properties.
- A logical analysis based only on propositional (or modal) fuzzy logics is clearly insufficient to account for most of instances of everyday reasoning with graded properties.
- We need to develop first-order fuzzy logics!

Second motivation: Study of graded structures

1. Fuzzy structures in fuzzy set theory:
   - fuzzy subalgebras
   - fuzzy orders
   - fuzzy preference relations
   - fuzzy topologies
   - ...
2. Weighted structures in computer science:
   - weighted graphs
   - valued constraint satisfaction problems
   - ...

Going first-order – 1

- Usual classical syntax with a signature $\tau = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$
- Semantics as in Mostowski, Rasiowa, Hájek tradition $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where:
  - $\boldsymbol{A}$ is an algebra of truth-values (for the propositional language)
  - $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where
    - M is a set
    - $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$.
    - $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$
  - An $\mathfrak{M}$-*evaluation* of the object variables is a mapping $\mathrm{v} \colon V \to M$
  - $$\begin{aligned} \|x\|^{\mathfrak{M}}_{\mathrm{v}} &= \mathrm{v}(x), \\ \|F(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\ \|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\ \|\circ(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\ \|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \inf_{\leq_{\boldsymbol{A}}} \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}, \\ \|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \sup_{\leq_{\boldsymbol{A}}} \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}. \end{aligned}$$

Going first-order – 2

- Notion of safe structure, where truth values of all formulas are defined.
- Notion of model: For each v, $\|\sigma\|^{\mathfrak{M}}_{\mathrm{v}} \in \mathcal{F}^{\boldsymbol{A}} = \{a \mid \overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} a\}$.

As in classical logic, we have:

- axiomatic Hilbert-style presentation
- completeness theorem

P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

P. Cintula and C. Noguera. A Henkin-Style Proof of Completeness for First-Order Algebraizable Logics, *Journal of Symbolic Logic*, 80(1):341–358, 2015.

Where do we start?

Classical model theory can be taken as a tentative map of an uncharted territory.

Model theory is the branch of mathematical logic that studies (classical) first-order structures.

- 1915, 1920: Löwenheim–Skolem.
- 1929, 1930, Gödel: completeness, compactness.
- 1950, Trakhtenbrot: undecidability of finite-model tautologies.
- 1953, Fraïssé: back-and-forth for elementary equivalence.
- 1955: Łoś: ultraproducts.
- 1961, Ehrenfeucht, Keisler: games and ultraprowers for elementary equivalence.
- 1961, Vaught: cardinal of models.
- 1965, Morley: categoricity.
- 1966, Robinson: non-standard analysis.
- 1969, Lindström: abstract model theory.
- 1973, Chang and Keisler: *Model Theory*.
- 1993, Hodges: *Model Theory*.
- ...

Wait! Isn't just two-sorted classical first-order logic?

Given a structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$, we build a 2-sorted structure $\boldsymbol{A}_{\mathbf{M}}$:

- The universe of sort 1 is A and the universe of sort 2 is M.
- The symbols $\approx_i$ are interpreted as crisp equality in the corresponding sorts.
- For each propositional n-ary connective λ, define $\lambda^{\boldsymbol{A}_{\mathbf{M}}}$ as $\lambda^{\boldsymbol{A}}$.
- For each n-ary functional symbol $F \in \mathbf{Func}$, define $F^{\boldsymbol{A}_{\mathbf{M}}}$ as $F_{\mathbf{M}}$.
- For each n-ary relational symbol $R \in \mathbf{Pred}$, define $R^{\boldsymbol{A}_{\mathbf{M}}}$ as $R_{\mathbf{M}}$.

P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C. Noguera, Distinguished Algebraic Semantics For T-Norm Based Fuzzy Logics: Methods and Algebraic Equivalencies, *Annals of Pure and Applied Logic* 160(1):53–81, 2009.

Translation to two-sorted structures

Lemma 1

For each formula $\varphi(v_1, \ldots, v_n)$, there is a 2-sorted formula $E_\varphi(v_1, \ldots, v_n, x)$ such that, for every $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$, and each $d_1, \ldots, d_n \in M$,

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = b \quad \textit{if and only if} \quad \boldsymbol{A}_{\mathbf{M}} \models E_\varphi(d_1, \ldots, d_n, b).$$

Corollary 2

A structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is safe if and only if, for every $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$,

$$\boldsymbol{A}_{\mathbf{M}} \models (\forall v_1, \ldots, v_n)(\exists! x) E_\varphi(v_1, \ldots, v_n, x).$$

But it is not exactly the same

- The 2-sorted approach will yield results (taken for free from classical model theory) whenever we regard as $\langle \boldsymbol{A}, \mathbf{M} \rangle$ as two-sorted structures where $\boldsymbol{A}$ and $\mathbf{M}$ "play a symmetrical role".
- However, whenever we want need to stress the "different nature" of $\boldsymbol{A}$ (truth values) and $\mathbf{M}$ (domain of discourse), for instance by keeping a fix intended algebra, the 2-sorted approach will not help.
- Moreover, the translation changes the complexity of formulas, so it is not useful for syntax-sensitive issues.
- Finally, there are cases in which the non-classical result is indeed the translation of a classical result, but not a famous one, rather an awkward never-formulated particular result.

Building up the theory

- Hájek–Cintula (2006): elementary equivalence and embeddings, conservative extensions, diagrams, exhaustive models, witnessed models.
- Dellunde (2011): Notions of homomorphisms and diagrams, reduced models.
- Dellunde (2012): Ultraproducts, Łoś theorem.
- Dellunde (2014): Compactness property for first-order languages with semantics given over a fixed finite MTL-chain: every finitely satisfiable set of sentences is satisfiable.

Immediate challenges for a graded model theory

Compactness fails in general when dealing with infinite MTL-chains: Hájek showed that product predicate logic with the standard semantics on the interval $[0, 1]$ is not compact.

- Can we prove some version of Löwenheim–Skolem theorems?
- What notion of elementary equivalence should we use?
- Can we characterize elementary equivalence?

(Elementary) substructure

$\langle \boldsymbol{B}, \mathbf{N} \rangle$ is a substructure of $\langle \boldsymbol{A}, \mathbf{M} \rangle$ if:

1. $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$.
2. $N \subseteq M$.
3. For each n-ary function symbol $F \in \mathcal{P}$, and elements $d_1, \ldots, d_n \in N$,

$$F_{\mathbf{N}}(d_1, \ldots, d_n) = F_{\mathbf{M}}(d_1, \ldots, d_n).$$

4. For each n-ary predicate $P \in \mathcal{P}$, $P_{\mathbf{N}}$ is the restriction of $P_{\mathbf{M}}$ to N.

Moreover, if both structures are safe, $\langle \boldsymbol{B}, \mathbf{N} \rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{M} \rangle$ if for every $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$, and elements $d_1, \ldots, d_n \in N$,

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{B}}_{\mathbf{N}} = \|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$$

Homomorphisms

The pair $\langle f, g \rangle$ is a homomorphism from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ into $\langle \boldsymbol{B}, \mathbf{N} \rangle$ (safe $\mathcal{P}$-structures) if

1) $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism of L-algebras
2) $g \colon M \to N$ is a mapping from M to N
3) for every n-ary $F \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

4) For every n-ary predicate symbol $P \in \mathcal{P}$, and $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \ \Rightarrow\ P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

It is a σ-homomorphism if f preserves all the existing infima and suprema.
It is a strong homomorphism if for every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$, $f(P_{\mathbf{M}}(d_1, \ldots, d_n)) = P_{\mathbf{N}}(g(d_1), \ldots, g(d_n))$.

Elementary homomorphisms

A homomorphism from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ into $\langle \boldsymbol{B}, \mathbf{N} \rangle$ (safe $\mathcal{P}$-structures) $\langle f, g \rangle$ is elementary if for each $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n \in M$,

$$f(\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|\varphi(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}}$$

Three notions of elementary equivalence

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. We say that they are:

1 Elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence φ, $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\varphi\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.

Assume now that $\boldsymbol{A} = \boldsymbol{B}$.

2 Filter-strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{fs} \langle \boldsymbol{A}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence φ,
$\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\varphi\|^{\boldsymbol{A}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{A}}$
and, moreover, $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{A}}_{\mathbf{N}}$ whenever $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}}$.

3 Strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{s} \langle \boldsymbol{A}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence φ, $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{A}}_{\mathbf{N}}$.

- Clearly, $\equiv$ and $\equiv^{fs}$ are the same notion for logics with weakening, because then $\mathcal{F}^{\boldsymbol{A}} = \{\overline{1}^{\boldsymbol{A}}\}$.

A useful lemma

Lemma 3

$\langle f, g\rangle : \langle \boldsymbol{A}, \mathbf{M}\rangle \to \langle \boldsymbol{B}, \mathbf{N}\rangle$ *strong homomorphism.*
If f is a σ-mapping and g is onto, then $\langle f, g\rangle$ is an elementary homomorphism.
Moreover, if f is one-to-one, we have $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{B}, \mathbf{N}\rangle$.

An example ($\equiv^{fs} \neq \equiv^{s}$)

- Consider a predicate language with only one monadic predicate P and take $\langle [0,1]_G, \mathbf{M} \rangle$ and $\langle [0,1]_G, \mathbf{N} \rangle$, both with the set of natural numbers as domain.

$$P_{\mathbf{M}}(n) = \begin{cases} \frac{3}{4} - \frac{1}{n} & \text{if } n \geq 2, \\ 0 & 0 \leq n \leq 1. \end{cases}$$

$$P_{\mathbf{N}}(n) = \begin{cases} \frac{1}{2} - \frac{1}{n} & \text{if } n \geq 2, \\ 0 & 0 \leq n \leq 1. \end{cases}$$

  $\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{3}{4}$ and $\|(\exists x)P(x)\|_{\mathbf{N}} = \frac{1}{2}$. Taking f as any non-decreasing bijection such that $f(\frac{3}{4}) = \frac{1}{2}, f(1) = 1, f(0) = 0$, and for every $n \in$ N, $f(\frac{3}{4} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}$, and applying the lemma we obtain $\langle [0,1]_G, \mathbf{M} \rangle \equiv \langle [0,1]_G, \mathbf{N} \rangle$.

Another example ($\equiv \neq \equiv^{fs}$)

- Consider a predicate language with only one monadic predicate P and $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A},\mathbf{N}\rangle$ with both domains the set of all natural numbers, and $\boldsymbol{A}$ the standard uninorm given by:

$$x \,\&^{\boldsymbol{A}}\, y = \begin{cases} \min\{x, y\}, & \text{if } x \leq 1 - y, \\ \max\{x, y\}, & \text{if } x > 1 - y. \end{cases}$$

$\mathcal{F}^{\boldsymbol{A}} = [\frac{1}{2}, 1]$

$$P_{\mathbf{M}}(n) = \begin{cases} \frac{4}{5} - \frac{1}{n^4}, & \text{if } n \geq 2, \\ 0, & \text{if } 0 \leq n \leq 1. \end{cases}$$

$$P_{\mathbf{N}}(n) = \begin{cases} \frac{3}{5} - \frac{1}{n^4}, & \text{if } n \geq 2, \\ 0, & \text{if } 0 \leq n \leq 1. \end{cases}$$

$\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{4}{5}$ and $\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{3}{5}$, but taking an appropriate strong σ-homomorphism $\langle f, Id\rangle$ and applying again the lemma, we obtain $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{A}, \mathbf{N}\rangle$.

Downward Löwenheim–Skolem theorem

Theorem 4

Take a safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{M} \rangle$ and assume that every subset of A definable with parameters in $\langle \mathbf{A}, \mathbf{M} \rangle$ has infimum and supremum. Then, for every $Z \subseteq M$ and every cardinal κ such that

$$\max\{\omega, |\mathcal{P}|, |Z|, p(\boldsymbol{A})\} \leq \kappa \leq |\boldsymbol{M}|,$$

there is a safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{N} \rangle$ which is an elementary substructure of $\langle \mathbf{A}, \mathbf{M} \rangle$ such that $|N| = \kappa$ and $Z \subseteq N$.

P. Dellunde, À. García-Cerdaña, and C. Noguera. Löwenheim–Skolem theorems for non-classical first-order algebraizable logics. *Logic Journal of the IGPL* 24(3):321–345, 2016.

Upward Löwenheim–Skolem theorem

Theorem 5

Let $\mathcal{P}$ be an equality-free language. For every infinite safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{M} \rangle$ and every cardinal κ with $\max\{|\mathcal{P}|, |M|\} \leq \kappa$, there is a safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{N} \rangle$ of cardinality κ and an elementary embedding from $\langle \mathbf{A}, \mathbf{M} \rangle$ to $\langle \mathbf{A}, \mathbf{N} \rangle$.

P. Dellunde, À. García-Cerdaña, and C. Noguera. Löwenheim–Skolem theorems for non-classical first-order algebraizable logics. *Logic Journal of the IGPL* 24(3):321–345, 2016.

Failure of the Upward L–S Th. for logics with equality

- Take $\mathrm{G}_\triangle$ and assume that the language contains a unary predicate P and an equality symbol $\approx$.
- Take a semantics of models $\langle [0,1]_{\mathrm{G}_\triangle}, \mathbf{M}\rangle$, where $\approx$ is interpreted as classical equality.
- Consider $\chi = (\forall x)(\forall y)(\neg\triangle(x \approx y) \to \neg\triangle(P(x) \leftrightarrow P(y)))$ that codifies the fact that P is interpreted as an injective mapping from the domain to the algebra of truth-values.
- Therefore, $\langle [0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ is a model of χ if and only if $|M| \leq 2^{\aleph_0}$, and hence the upward theorem does not hold.

Löwenheim–Skolem Theorems (via 2-sorted structures)

Theorem 6

Let $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *be a safe* $\mathcal{P}$*-structure. Then, for every* $Z \subseteq M$, *every* $X \subseteq B$ *and every cardinal* κ *such that* $\max\{|\mathcal{P}|, \omega, |Z|, |X|\} \leq \kappa \leq \max\{|B|, |M|\}$, *there is a safe* $\mathcal{P}$*-structure* $\langle \boldsymbol{A}, \mathbf{O}\rangle$ *which is an elementary substructure of* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *such that* $|A| + |O| = \kappa$, $Z \subseteq O$, *and* $X \subseteq A$.

Theorem 7

Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *be a safe infinite* $\mathcal{P}$*-structure and* κ *a cardinal such that* $\max\{|\mathcal{P}|, |A|, |M|\} \leq \kappa$. *Then there is a safe* $\mathcal{P}$*-structure* $\langle \boldsymbol{B}, \mathbf{N}\rangle$ *such that* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *is an elementary substructure of* $\langle \boldsymbol{B}, \mathbf{N}\rangle$ *and* $|B| + |N| = \kappa$.

Back-and-forth is a sufficient condition for elementary equivalence...

Theorem 8

Let $\mathcal{P}$ be a finite predicate language. Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$, $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. The following holds:

$$\langle \boldsymbol{A}, \mathbf{M} \rangle \cong_f \langle \boldsymbol{B}, \mathbf{N} \rangle \quad \Rightarrow \quad \langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle.$$

Furthermore, if there is $\langle I_n \mid n \in \mathsf{N} \rangle : \langle \boldsymbol{A}, \mathbf{M} \rangle \cong_f \langle \boldsymbol{A}, \mathbf{N} \rangle$ such that for every n, and every $\langle p, r \rangle \in I_n$, $p \restriction A \subseteq \mathrm{Id}_A$, then,

$$\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^s \langle \boldsymbol{A}, \mathbf{N} \rangle.$$

P. Dellunde, À. García-Cerdaña, and C. Noguera. Back-and-forth systems for fuzzy first-order models. *Fuzzy Sets and Systems* 345(1):83–98, 2018.

...but it is not necessary!

Let $\mathcal{P}$ be a finite predicate language.

Let $\langle \boldsymbol{B}_2, \mathbf{M} \rangle$ be a classical first-order $\mathcal{P}$-structure over a finite domain M.

Now take an infinite L-algebra $\boldsymbol{A}$.

Since $\boldsymbol{B}_2 \subseteq \boldsymbol{A}$, we can also see $\langle \boldsymbol{B}_2, \mathbf{M} \rangle$ as a structure over $\boldsymbol{A}$.

Clearly $\langle \boldsymbol{B}_2, \mathbf{M} \rangle \equiv^s \langle \boldsymbol{A}, \mathbf{M} \rangle$.

But it is not true that $\langle \boldsymbol{B}_2, \mathbf{M} \rangle \cong_f \langle \boldsymbol{A}, \mathbf{M} \rangle$.

Unions of elementary chains

Elementary chain: $\{\langle \boldsymbol{A}, \mathbf{M}_i\rangle \mid i < \gamma\}$ where, for all $i < j < \gamma$, $\langle \boldsymbol{A}, \mathbf{M}_i\rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{M}_j\rangle$.

Theorem 9

Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be the union of an elementary chain $\{\langle \boldsymbol{A}, \mathbf{M}_i\rangle \mid i < \gamma\}$. Then, for each sequence $\overline{a}$ of elements of $\mathbf{M}_i$ and each formula $\varphi(\overline{x})$,
$\|\varphi(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}_i}$.
Moreover, $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a safe structure.

G. Badia and C. Noguera. Fraïssé classes of graded relational structures. *Theoretical Computer Science* 737(1):81–90, 2018.

Fraïssé limits

Age$(\mathbf{A}, \mathbf{M})$: all finitely generated substructures of $\langle \mathbf{A}, \mathbf{M} \rangle$ and their isomorphic copies.

Homogeneous structure: if every isomorphism between two finitely generated substructures extends to an automorphism of the structure.

Theorem 10

$\mathbb{K}$: countable class of finitely generated $\mathbf{A}$-structures (same language).

- *$\mathbb{K} = Age(\mathbf{A}, \mathbf{N})$ for some $\langle \mathbf{A}, \mathbf{N} \rangle$ iff $\mathbb{K}$ satisfies HP and JEP.*
- *If $\mathbb{K}$ has HP, JEP, and AP, then there is a unique countable homogeneous structure $\langle \mathbf{A}, \mathbf{M} \rangle$ such that $\mathbb{K} = Age(\mathbf{A}, \mathbf{M})$.*
- *If a structure is a homogeneous, then its age has AP.*

G. Badia and C. Noguera. Fraïssé classes of graded relational structures. *Theoretical Computer Science* 737(1):81–90, 2018.

Saturated models

Consistent pair: $\langle T, U \rangle$ such that $T \models \bigvee U_0$ for no finite $U_0 \subseteq U$.

Type: $\langle p, p' \rangle$ pair of sets of formulas in x and parameters in $D \subseteq M$ such that $\langle \mathit{Th}_D(\mathfrak{M}) \cup p, \overline{\mathit{Th}}_D(\mathfrak{M}) \cup p' \rangle$ is consistent.

Set of all types: $S^{\mathfrak{M}}(D)$.

κ-saturated model: for any $D \subseteq M$ such that $|D| < \kappa$, any type in $S^{\mathfrak{M}}(D)$ is satisfiable in $\mathfrak{M}$.

Theorem 11

For each cardinal κ, each model can be elementarily extended to a κ^+-saturated model.

G. Badia and C. Noguera. Saturated models of first-order many-valued logics. *Logic Journal of the IGPL* 30:1–20, 2022.

Omitting types theorem

Theorem 12

Let $\langle T, U \rangle$ be a tableau, such that at most finitely many of its elements involve object constants, realized by some model, and $\langle p, p' \rangle$ an unsupported n-type of $\langle T, U \rangle$.
Then, there is a model satisfying $\langle T, U \rangle$ which omits $\langle p, p' \rangle$.

G. Badia, C. Noguera. A general omitting types theorem in mathematical fuzzy logic, *IEEE Transactions on Fuzzy Systems* 29:1386–1394, 2021.

An abstract model theory

Lindström theorem (1969)

Classical first-order logic is the *strongest logic* enjoying Löwenheim–Skolem and Compactness.

Abstract Model Theory: Jon Barwise, Solomon Feferman (eds). *Model-theoretic logics*, Springer–Verlag, 1985.

Do first-order fuzzy logics have Lindström-style characterizations? Would an abstract graded model theory be viable?

Hájek (2002): The analogues of Lindström theorem, in a certain sense, fail for some of the main first-order fuzzy logics (BL$\forall$, Ł$\forall$, $\Pi\forall$, and G$\forall$).

Maximality of first-order fuzzy logics – 1

We consider logics based on a finite MTL-chain $\boldsymbol{A}$.

$\mathscr{L}_1^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_2^{\boldsymbol{A}}$ means that for every formula of the first logic there is a 1-equivalent formula in the second. It is a preorder.

$\mathscr{L}_1^{\boldsymbol{A}} \simeq \mathscr{L}_2^{\boldsymbol{A}}$ means that $\mathscr{L}_1^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_2^{\boldsymbol{A}}$ and $\mathscr{L}_2^{\boldsymbol{A}} \trianglelefteq \mathscr{L}_1^{\boldsymbol{A}}$ (expressively equivalent abstract logics).

$\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ is the abstract logic obtained from considering our first-order languages with crisp equality and constants for each element of $\boldsymbol{A}$.

Maximality of first-order fuzzy logics – 2

Theorem 13

Let $\mathscr{L}^{\mathbf{A}}$ be an abstract logic such that $\mathscr{L}^{\mathbf{A}}_{\omega\omega} \trianglelefteq \mathscr{L}^{\mathbf{A}}$. If $\mathscr{L}^{\mathbf{A}}$ has the Löwenheim–Skolem property and the Compactness property for countable sets of formulas, then $\mathscr{L}^{\mathbf{A}} \simeq \mathscr{L}^{\mathbf{A}}_{\omega\omega}$.

G. Badia, C. Noguera. Lindström theorems in graded model theory, *Annals of Pure and Applied Logic*, 172(3):102916, 2021.

Asymptotic probabilities

For any τ-sentence φ, $a \in A$, and $n \geqslant 1$,
$l_n^a(\varphi)$: cardinality of the (finite) set K_τ^a consisting of each model $\mathfrak{M}$ for the signature τ with domain $\{1, 2, \ldots, n\}$ such that $\|\varphi\|^{\mathfrak{M}} = a$.
$l_n(\tau)$: cardinality of the (finite) set containing all model for the signature τ with domain $\{1, 2, \ldots, n\}$.
Now, let

$$\mu_n^a(\varphi) = \frac{l_n^a(\varphi)}{l_n(\tau)}.$$

The asymptotic probability of φ getting value a is defined as follows:

$$\mu^a(\varphi) = \lim_{n \to \infty} \mu_n^a(\varphi).$$

0-1 laws

Theorem 14

If φ is a sentence in the finite relational signature τ, then there is $a \in A$ such that $\mu^a(\varphi) = 1$ and for any other truth-value a', $\mu^{a'}(\varphi) = 0$.

Theorem 15

If φ is a sentence of $\mathscr{L}_{\infty\omega}^{k\mathbf{A}}$ in the finite relational signature τ, then there is $a \in A$ such that $\mu^a(\varphi) = 1$ and for any other truth-value a', $\mu^{a'}(\varphi) = 0$.

G. Badia, C. Noguera. A 0-1 law in mathematical fuzzy logic, *IEEE Transactions on Fuzzy Systems*, 2022.

Future research

- Characterization of elementary equivalence.
- Levels of generality.
- An approach based on useful graded structures.
- Finite graded model theory.
- Extensions of the language: infinitary connectives, generalized quantifiers.