A Journey in Intuitionistic Modal Logic: normal and non-normal modalities

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Based on joint works with Tiziano Dalmonte and Charles Grellois

Outline

- A short tale of Intuitionistic and constructive modal logic
- Constructive non-mormal modalities
- Proof theory: back to Constructive modal logic

Intuitionistic modal logics: two traditions

Intuitionistic modal logics (Fischer Servi 1977,1980, Plotkin & Stirling 1986, Ewald 1986, Simpson 1994)

- Theoretical interest of combining these two forms of logics.
- Define the intuitionistic analogue(s) of some classical modal logics.
- Justified by intuitionistic meta-theory: translation into first-order IL

Constructive modal logics (Prawitz 1965, Goldblatt 1981, Wijesekera 1990, Masini 1993, Fairtlough & Mendler 1997, Bellin, de Paiva, Ritter 2001,)

- Designed for specific applications of logic to computer science.
- Verification and Knowledge representation.
- Natural deduction systems and type-theoretic interpretations.

Some principles (Simpson 94)

- IML should be a conservative extensions of IL
- $\Box$ and $\Diamond$ should be independent
- Disjunction Property: $\vdash A \vee B$ implies either : $\vdash A$ or $\vdash B$
- adding $A \vee \neg A$ to IML we get Classical ML: controversial!

How to build an intuitionistic modal logic?

Semantics

- Possible-world semantics (among others)
- Extend Kripke models of Intuitionistic Logic: bi-relational models
- Hereditary Property (refinement)

How to build an intuitionistic modal logic?

Models

$M = (\mathcal{W}, \preceq, R, V)$ where:
$\preceq$ is a pre-order on W , $R \subseteq W \times W$, and $V : W \to \mathcal{P}(Atom)$ and satisfies:

- $x \preceq y$ implies $V(x) \subseteq V(y)$

Hereditary Property

- We want Hereditary Property: for any formula A
  If $M, x \Vdash A$ and $x \preceq y$ then $M, y \Vdash A$
- How to define the truth conditions of $\Box$ and $\Diamond$ in order to ensure the Hereditary Property?

Simplest solution

Truth conditions for $\Box$ and $\Diamond$

Build the Hereditary property into the forcing relation:

- $M, x \Vdash \Box A$ if $\forall x'.x \preceq x'\ \forall y.Rx'y$ implies $M, y \Vdash A$
- $M, x \Vdash \Diamond A$ if $\forall x'.x \preceq x'\ \exists y.Rx'y\ \&\ M, y \Vdash A$

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Other truth conditions

The same as in IL

- $M, x \Vdash P$ (Atom) if $P \in V(x)$
- $M, x \nVdash \bot$
- $M, x \Vdash A \wedge B$ iff $M, x \Vdash A$ and $M, x \Vdash B$
- $M, x \Vdash A \vee B$ iff $M, x \Vdash A$ or $M, x \Vdash B$
- $M, x \Vdash A \supset B$ iff $\forall x'.x \preceq x'$: if $M, x' \Vdash A$ then $M, x' \Vdash B$

Notation: we write just $x \Vdash A$ instead of $M, x \Vdash A$ when no confusion arise

Wijesekera **WK**

This corresoponds to the propositional part of Wijesekera's logic CCDL (Wijesekera 1990), that we call **WK**

- IL+MP
- $\frac{A}{\Box A}$ (Nec)
- $\Box(A \supset B) \supset \Box A \supset \Box B$
- $\Box(A \supset B) \supset \Diamond A \supset \Diamond B$
- $\neg\Diamond\bot$

Features of **WK**

- Non Normal: it does not prove

$$C_\diamond : \quad \diamond(A \vee B) \supset \diamond A \vee \diamond B$$

  (although it proves $C_\Box$: $\Box A \wedge \Box B \supset \Box(A \wedge B)$)
- It satisfies Disjuction Property
- **WK** $+A \vee \neg A \neq$ Classical K (Simpson 94)

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$$\mathbf{WK} + A \vee \neg A + \Box A \vee \Diamond\neg A \;\; = \text{Classical } K$$

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$$\textbf{WK} + A \vee \neg A + \Box A \vee \diamond\neg A \; = \text{Classical } K$$

- Criticism (Bellin, De Paiva, Mendler etc.):
  there are reason to reject also the nullary version of $C_\diamond$: $\diamond\bot \supset \bot$

CK: Constructive K

The system **CK** (Bellin, de Paiva, Ritter 2001, Mendler & de Paiva 2000),

The system **CK** is just **WK** - $\{\neg\Diamond\bot\}$.

- computer science applications (types)
- categorical semantics

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Relational models for **CK** (Mendler & de Paiva 2005)

Models for **WK** + fallible worlds $\mathcal{F} \subseteq \mathcal{W}$:

- $w \Vdash \bot \ \forall w \in \mathcal{F}$
- $Atom \subseteq \mathcal{V}(w) \ \forall w \in \mathcal{F}$
- $\mathcal{F}$ is R-closed and $\preceq$-closed

In these models $\Diamond\bot$ is satisfiable.

System **IK** (Fischer Servi 1977, Simpson 94)

- Motivated by intuitionistic meta-theory
- Normal Modal Logic

Models for **IK**

The same as for **WK**, but the definition of $\diamond$ is local:

$x \Vdash \diamond A \quad \textit{if}\ \exists y.Rxy\ \&\ y \Vdash A$

But there are additional frame conditions

Additional frame conditions for **IK**

- (F1) if $x \preceq x'$ and Rxy then $\exists y'.y \preceq y'$ and $Rx'y'$

$$
\begin{tikzcd}
x' \arrow[r, dashed, "R"] & y' \\
x \arrow[u, "\preceq"] \arrow[r, "R"'] & y \arrow[u, dashed, "\preceq"']
\end{tikzcd}
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- (F2) if Rxy and $y \preceq y'$ and then $\exists x'.x \preceq x'$ and $Rx'y'$

$$
\begin{tikzcd}
x' \arrow[r, dashed, "R"] & y' \\
x \arrow[u, dashed, "\preceq"] \arrow[r, "R"'] & y \arrow[u, "\preceq"']
\end{tikzcd}
$$

Additional frame conditions for **IK**

- (F1) is needed to get the heredidary property for $\Diamond$
- (F2) is needed for completeness wrt. intuitionistic meta-theory

Axiomatisation

- Axioms and rules of **WK**
- $\Diamond(A \lor B) \supset \Diamond A \lor \Diamond B \quad (C_\Diamond)$
- $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

Properties of **IK**

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Translaton in FOIL

- $p^t = p(x)$
- $\bot^t = \bot$
- $(A \# B)^t = A^t \# B^t$ for $\# \in \{\vee, \wedge, \supset\}$
- $(\Box A)^t = \forall y(Rxy \supset A(y)^t)$
- $(\Diamond A)^t = \exists y(Rxy \wedge A(y)^t)$

Theorem (Simpson 94)

$\vdash_{IK} A \textit{ iff } \vdash_{FOIL} \forall x A^t$

More Systems!

- Make the $\Box$ condition local:
  $M, x \Vdash \Box A$ if $\forall y. Rxy$ implies $M, y \Vdash A$
- in order to get Herediclary Property add (F3):
  if $x \preceq x'$ and $Rx'y'$ then $\exists y. y \preceq y'$ and Rxy

$$\begin{tikzcd} x' \arrow[rr, "R"] & & y' \\ & & \\ x \arrow[uu, "\preceq"] \arrow[rr, dashed, "R"'] & & y \arrow[uu, dashed, "\preceq"'] \end{tikzcd}$$

- (F3) validates

$$\Box(A \vee B) \supset \Box A \vee \Diamond B$$

Considered in (Božić and Došen 1984, D'Agostino et als. 97, Balbiani et als. 2021)

Three intuitionistic versions of classical **K**

Intuitionistic K (**IK**)

(Fischer Servi 1977, Simpson 1994)

$$\frac{A}{\Box A}$$

$\Box(A \supset B) \supset (\Box A \supset \Box B)$

$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

$\neg\Diamond\bot$

$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$

$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

"Wijesekera's K" (**WK**)

(Wijesekera 1990)

$$\frac{A}{\Box A}$$

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Constructive K (**CK**)

(Bellin et al. 2001, Mendler & de Paiva 2005)

$$\frac{A}{\Box A}$$

$\Box(A \supset B) \supset (\Box A \supset \Box B)$

$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

What are the relations?

Obvious remarks

- The $\Box$-fragment of **CK** and **WK** is the same.
- $\Diamond$ is non-normal in **CK** and **WK** as $C_\Diamond$ fails:

$$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$$

- Different intuitionistic counterparts of the same classical modal logic.

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Less obvious remark!

See: `https://prooftheory.blog/2022/08/19/brouwer-meets-kripke-constructivising-modal-logic/`
Even $\Box$-fragment of **IK** is different from the one of **CK** and **WK**

- Counterexample by Das and Marin: $\neg\neg\Box p \supset \Box\neg p$
- Counterexample by Grefe in 96? (communicated by A. Simpson):

$$(\neg\Box\bot \supset \Box\bot) \supset \Box\bot$$

- The $\Box$-fragment of **IK** is not finitely axiomatisable (by C. Grefe reported by Alex Simpson)
- The $\Box$-fragment of **WK** $+ C_\Diamond$ is stronger than the $\Box$-fragment of **WK** alone

Sequent calculi

- Constructive modal logic **WK** and **CK** have simple Gentzen calculi
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Sequent calculus for Classical K

Modal Rules (with weakening)

$$\mathrm{R}\Box \frac{\Sigma \Rightarrow A, \Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Diamond\Pi, \Delta} \qquad \mathrm{R}\Diamond \frac{\Sigma, A \Rightarrow \Pi}{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Diamond\Pi, \Delta}$$

$\Box\Sigma = \{C \mid \Box C \in \Sigma\}$
$\Diamond\Pi = \{D \mid \Diamond D \in \Pi\}$

Σ and Π may be empty

Sequent Calculus for **WK** and **CK**

Modal Rules to add to a standard Gentzen calculus for IL: just make the rules single succedent.

$$\mathrm{R}\Box\ \frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A}$$

$$\mathrm{R}\Diamond\ \frac{\Sigma, A \Rightarrow B}{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathrm{R}\Diamond_{\perp}\ \frac{\Sigma, A \Rightarrow}{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Delta}$$

- In rule $\mathrm{R}\Diamond_{\perp}$ $|\Delta| \leq 1$.
- For **CK**: remove $\mathrm{R}\Diamond_{\perp}$

Constructive vs. Intuitionistic modal logic from a proof-theorical viewpoint

To sum up:

- Natural internal Gentzen calculi for Constructive modal logic
- No (internal) cut-free Gentzen calculus is available for **IK**, not even for its $\Box$-fragment.
  - Sonia's talk :
    Nested sequents (Galmiche and Salhi 2010, Strassburger 2013), labelled sequents (Simpson 1994), (Marin, Morales, Straßburger 2021)

A good reason to distinguish Construtive from Intuitionistic Modal Logic

Reject $C_\Diamond$

There are reasons to reject $C_\Diamond$ (Wijesekera 1990, Mendler & Scheele 2011)

- Lack of control/choice:
  1. Agent/process: $\Diamond A$ means the agent/process can ensure A
  2. Contextual : $\Diamond A$ means A holds in some context
  3. Types: $\Diamond A$ means the program can produce a value of type A
- Normality : $\Diamond A$ means A holds in normal circumstances

Same reasons lead to reject $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

From normal to non-normal intuitionistic modal logic

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Do we keep $C_\Box$?

What about the dual

$$C_\Box : \Box A \land \Box B \supset \Box(A \land B) \ ?$$

Why do not study Intuitionistic Non Normal Modal Logic?

- Classical non-normal modal logics are well studied (Chellas 1980, Pacuit 2017)
- No general investigation of intuitionistic logics with non-normal modalities.

Question

What are the intuitionistic counterparts of classical non-normal modal logics?

An axiomatisation of **K**

- Monotonicity

$$M \quad \Box(A \wedge B) \rightarrow \Box A \qquad \text{or} \quad RM\ \frac{A \rightarrow B}{\Box A \rightarrow \Box B}$$

- Aggregation

$$C \quad \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$$

- Necessitation

$$N \quad \Box\top \qquad \text{or} \quad RN\ \frac{A}{\Box A}$$

Objections

- Monotonicity: Deontic paradoxes, Omniscience
- Aggregation: Deontic paradoxes, Agent' ability, "High probablity"
- Necessitation: Omniscience, deontic interpretation

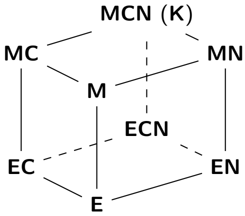
Non-normal modal logics: Classical cube

Basic system **E**

$$\textbf{CPL} \quad + \quad \Diamond A \leftrightarrow \neg\Box\neg A \quad + \quad RE\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

Extensions of **E** add any combination of:

$M \quad \Box(A \wedge B) \rightarrow \Box A$	or $RM\ \dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$
$C \quad \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$	
$N \quad \Box\top$	or $RN\ \dfrac{A}{\Box A}$



- 8 non-equivalent systems
- $M/C/N$ derivable only if they explicitly belong to the axiomatisation
- Top system coincides with **K**

An attempt

A landcape of constructive non-normal modal logics that

- can be seen as intuitionistic counterparts of classical non-normal modal logics;
- hope to accommodate also **CK** and **WK**.

Intuitionistic non-normal modal logics must contain

- Characteristic modal axioms and rules of the systems of the classical cube.

$RE_{\Box}$	$\dfrac{A \supset\subset B}{\Box A \supset\subset \Box B}$	$RE_{\Diamond}$	$\dfrac{A \supset\subset B}{\Diamond A \supset\subset \Diamond B}$
$M_{\Box}$	$\Box(A \wedge B) \supset \Box A$	$M_{\Diamond}$	$\Diamond A \supset \Diamond(A \vee B)$
$C_{\Box}$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$C_{\Diamond}$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
$N_{\Box}$	$\Box\top$	$N_{\Diamond}$	$\neg\Diamond\bot$

- But what interactions between $\Box$ and $\Diamond$?

Weak duality principles

- Interactions between $\Box$ and $\Diamond$ that can be seen as weak duality principles
- State when $\Box A$ and $\Diamond B$ are jointly incompatible:
  - when one of the two A or B is $\top$ and the other is $\bot$
  - when one is the negation of the other
  - when they are incompatible $\vdash \neg(A \wedge B)$

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$weak_a$	$\neg(\Box\top \wedge \Diamond\bot)$	$weak_b$	$\neg(\Box\bot \wedge \Diamond\top)$	str	$\dfrac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$
neg_a	$\neg(\Box\neg A \wedge \Diamond A)$	neg_b	$\neg(\Box A \wedge \Diamond\neg A)$		

Further requirements (cf. Simpson 1994)

1. Conservativity over **IPL**.
2. Disjunction property.
3. Homogeneus treatment of $\Box$ and $\Diamond$
4. But constructive : Not contain $C_\Diamond \quad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$.

How to proceed?

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How to proceed?

Proof-theoretic approach

- Extend Gentzen calculus with modal rules corresponding to principles
- Accept only the combinations of principles which give a cut-free calculus.

Sequent rules for non-normal modalities (G3i-style)

$$\mathsf{E}_\Box \ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{M}_\Box \ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{N}_\Box \ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A}$$

$$\mathsf{C}_\Box \ \frac{A_1, \dots, A_n \Rightarrow B \quad B \Rightarrow A_1 \ \dots \ B \Rightarrow A_n}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B} \qquad \mathsf{MC}_\Box \ \frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B}$$

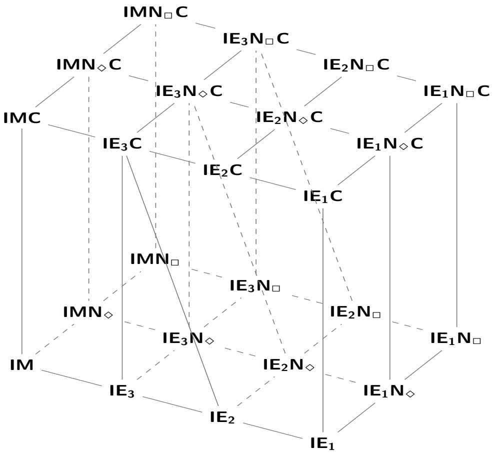
$$\mathsf{E}_\Diamond \ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \mathsf{M}_\Diamond \ \frac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \mathsf{N}_\Diamond \ \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta}$$

Interaction rules “weak duality principles”

$$\mathsf{weak}_\mathsf{a} \ \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad \mathsf{weak}_\mathsf{b} \ \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{neg}_\mathsf{a} \ \frac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad \mathsf{neg}_\mathsf{b} \ \frac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{str} \ \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$



- 24 distinct cut-free calculi
- Monotonicity compatible only with the strongest interaction
- $N_\Box$ only in presence of $N_\Diamond$
- All of them weaker than **WK**

Trivial intuitionistic counterparts of classical non-normal modal logics?

Example: the system $\mathbf{IE_3}$

$$\mathbf{IPL} + \quad RE_{\Box}\ \frac{A \supset\subset B}{\Box A \supset\subset \Box B} \quad RE_{\Diamond}\ \frac{A \supset\subset B}{\Diamond A \supset\subset \Diamond B} \quad str\ \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$$

- The rule *str* is classically equivalent to axiom M $\quad\Rightarrow\quad$ $\mathbf{IE_3} \not\subseteq \mathbf{E}$.
- $\mathbf{IE_3}$ doesn't contain $M_{\Box}$ or $M_{\Diamond}$.
- $\mathbf{IE_3}$ doesn't trivially correspond neither to $\mathbf{E}$ nor to $\mathbf{M}$.

Richer picture than in the classical setting

- Finer distinctions among principles that are equivalent in classical logics.
- Systems that do not correspond to any classical logic.

Further properties

- Disjunction property
- All of them decidable
- Craig interpolation for some of them
  - Uniform interpolation for the $\Box$-fragment of **IM** [Tabatabai & als. 2022]

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Further systems

- Deontic Logic with an intuitionistic basis [DGO WOLLIC 22], namely $\mathbf{IE_1}$ extended with:
  $(C_\diamond)$ $\mathcal{P}(A \vee B) \supset (\mathcal{P}A \vee \mathcal{P}B)$
  (D) $\mathcal{O}A \supset \mathcal{P}A$
  $(N_\diamond)$ $\neg\mathcal{P}\bot$
- Extension with Axioms of the standard cube of normal modal logic (T,D,4,5,B): to be studied.

Possible worlds semantics for intuitionistic non-normal modal logics

Neighbourhood models for non-normal modal logics

$\langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$

Kripke models for intuitionistic logic

$\langle \mathcal{W}, \preceq, \mathcal{V} \rangle$

Models for intuitionistic non-normal modal logics

$\langle \mathcal{W}, \preceq, \mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}, \mathcal{V} \rangle$

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Models for intuitionistic non-normal modal logics
$\langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$

- Two distinct neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ handling the modalities separately
- Different connections between $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ corresponding to the interaction axioms
- The combination must preserve the hereditary property:

$$\text{If } w \Vdash A \text{ and } w \preceq v \text{, then } v \Vdash A$$

Coupled intuitionistic neighbourhood models

CINMs: $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, where:

- $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ as usual in Kripke models of IL
- $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{PP}(\mathcal{W})$ s.t.

$$w \preceq v \text{ implies } \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v) \text{ and } \mathcal{N}_\Diamond(w) \subseteq \mathcal{N}_\Diamond(v).$$

Forcing conditions

Standard for p, $\bot$, $\top$, $B \wedge C$, $B \vee C$, $B \supset C$

$$w \Vdash \Box B \quad \text{iff} \quad [\![B]\!] \in \mathcal{N}_\Box(w);$$

$$w \Vdash \Diamond B \quad \text{iff} \quad [\![B]\!] \in \mathcal{N}_\Diamond(w).$$

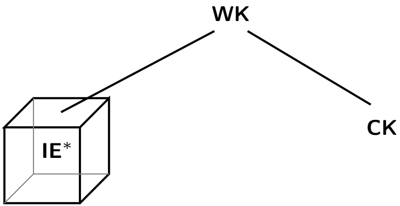
Conditions corresponding to the axioms

- Conditions on $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ ($\circ = \Box, \Diamond$)

$$
\begin{array}{ll}
\text{If } \alpha \in \mathcal{N}_\circ(w) \text{ and } \alpha \subseteq \beta, \text{ then } \beta \in \mathcal{N}_\circ(w) & (M_\circ) \\
\text{If } \alpha, \beta \in \mathcal{N}_\Box(w), \text{ then } \alpha \cap \beta \in \mathcal{N}_\Box(w) & (C_\Box) \\
\mathcal{W} \in \mathcal{N}_\circ(w) & (N_\circ)
\end{array}
$$

- Relations between $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ ($-\alpha = \{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}$)

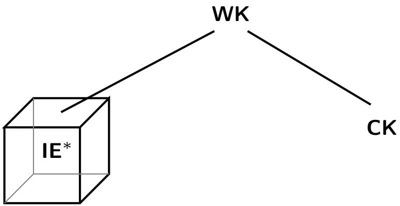
$$
\begin{array}{ll}
\text{If } \alpha \in \mathcal{N}_\Box(w), \text{ then } \mathcal{W} \setminus \alpha \notin \mathcal{N}_\Diamond(w) & (\mathsf{weakInt}) \\
\text{If } -\alpha \in \mathcal{N}_\Box(w), \text{ then } \alpha \notin \mathcal{N}_\Diamond(w) & (\mathsf{negInt_a}) \\
\text{If } \alpha \in \mathcal{N}_\Box(w), \text{ then } -\alpha \notin \mathcal{N}_\Diamond(w) & (\mathsf{negInt_b}) \\
\text{If } \alpha \in \mathcal{N}_\Box(w) \text{ and } \alpha \subseteq \beta, \text{ then } \mathcal{W} \setminus \beta \notin \mathcal{N}_\Diamond(w) & (\mathsf{strInt})
\end{array}
$$



Neighbourhood semantics for **CK** and **WK**

Common conditions

- Supplementation for $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$
- $N_\Box$: $W \in \mathcal{N}_\Box(w)$
- $C_\Box$: $\alpha, \beta \in \mathcal{N}_\Box(w)$, then $\alpha \cap \beta \in \mathcal{N}_\Box(w)$



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- $C_\Box$: $\alpha, \beta \in \mathcal{N}_\Box(w)$, then $\alpha \cap \beta \in \mathcal{N}_\Box(w)$

Additional conditions:

- WInt$_\cap$: If $\alpha \in \mathcal{N}_\Box(w)$ and $\beta \in \mathcal{N}_\Diamond(w)$, then $\alpha \cap \beta \in \mathcal{N}_\Diamond(w)$
- weakInt for **WK** : If $\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus \alpha \notin \mathcal{N}_\Diamond(w)$ (equivalent condition $\emptyset \notin \mathcal{N}_\Diamond(w)$)

Neighbourhood semantics for **CK** and **WK**

- Models for **CK** without "fallible worlds"
- Direct transformations with their original relational models
  - But relational models are much larger...
- Alternative Neighbourhood semantics for **WK** (Kojima 2012, Dalmonte 2022)

From Relational models to Neighbourhood models

For WK

Given $M = (\mathcal{W}, \preceq, R, V)$. For $x \in W$ let

$$\mathcal{R}(x) = \{y \in W : Rxy\}$$

we define $M_n = (\mathcal{W}, \preceq, R, V, \mathcal{N}_\Box, \mathcal{N}_\Diamond)$ where

$$\mathcal{N}_\Box(x) = \{\alpha \subseteq \mathcal{W} \mid \forall y \succeq x. \mathcal{R}(y) \subseteq \alpha\}$$
$$\mathcal{N}_\Diamond(x) = \{\alpha \subseteq \mathcal{W} \mid \forall y \succeq x. \mathcal{R}(y) \cap \alpha \neq \emptyset\}$$

For CK

$M = (\mathcal{W}, \mathcal{F}, \preceq, R, V)$, where $\mathcal{F} \subseteq \mathcal{W}$ and for all $x \in \mathcal{F}$ $x \Vdash \bot$. For let

$$\alpha^+ = \alpha - \mathcal{F} \textit{ for } \alpha \subseteq \mathcal{W},$$

we define $M_n = (\mathcal{W}, \preceq, \mathcal{V}, \mathcal{N}_\Box, \mathcal{N}_\Diamond)$ where

$$\mathcal{N}_\Box(x) = \{\alpha^+ \subseteq \mathcal{W} \mid \forall y \succeq x. \mathcal{R}(y) \subseteq \alpha\}$$
$$\mathcal{N}_\Diamond(x) = \{\alpha^+ \subseteq \mathcal{W} \mid \forall y \succeq x. \mathcal{R}(y) \cap \alpha^+ \neq \emptyset\}$$

From Neighbourhood models to Relational models

For **WK**

Given $M = (\mathcal{W}, \preceq, \mathcal{V}, \mathcal{N}_\Box, \mathcal{N}_\Diamond)$. We define $M^* = (\mathcal{W}, \preceq^*, \mathcal{R}^*, \mathcal{V}^*)$ as follows:

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in W, W - \alpha \notin \mathcal{N}_\Diamond(w), and\ \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$
- $(w, \alpha) \preceq^* (u, \beta)$ *iff* $w \preceq u$
- $(w, \alpha)\mathcal{R}^*(u, \beta)$ *iff* $u \in \alpha$
- $\mathcal{V}^*((w, \alpha)) = \mathcal{V}(p)$

From Neighbourhood models to Relational models

For **CK**

Given $M = (\mathcal{W}, \preceq, \mathcal{V}, \mathcal{N}_\Box, \mathcal{N}_\Diamond)$. Let $\mathbf{f} \notin W$ we define We define $M^* = (\mathcal{W}, \preceq^*, \mathcal{R}^*, \mathcal{V}^*)$ as follows:

$$\begin{aligned} \mathcal{W}^* \quad = \quad & \{(w, \alpha) \mid w \in W, \emptyset \notin \mathcal{N}_\Diamond(w), W - \alpha \notin \mathcal{N}_\Diamond(w), \text{and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\} \\ & \cup \{(w, \bigcap \mathcal{N}_\Box(w) \cup \{\mathbf{f}\}) \mid w \in W, \emptyset \in \mathcal{N}_\Diamond(w))\} \\ & \cup \{(\mathbf{f}, \{\mathbf{f}\})\} \end{aligned}$$

and :

- $(w, \alpha) \preceq^* (u, \beta)$ *iff* $w \preceq u$ *or* $w = v = \mathbf{f}$
- $\mathcal{R}^*$ *as for* ***WK***
- $\mathcal{V}^*((w, \alpha)) = \mathcal{V}(w)$
- $\mathcal{V}^*((\mathbf{f}, \{\mathbf{f}\})) = Atom \cup \{\bot\}$

Back to proof-theory for CMLs

- A proof system must provide a decision procedure, whenever the logic is decidable
- Proofs and countermodels are equally important:
  - A proof is a witness of the validity of a formula
  - A countermodel is a witness of its non-validity

State of the art

- Gentzen calculi for **WK** and **CK** (Wijesekera 1990; Bellin, de Paiva & Ritter 2001) (not possible for IMLs)
- 2-sequent calculus for **CK** (Mendler & Scheele 2011)
- Nested calculi for **CK** and some extensions (Arisaka, Das & Straßburger 2015)

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What we want

Proof systems for CMLs that

- are simple: Gentzen calculi without additional structure, as allowed by CMLs
- Are strictly terminating (provide direct decision procedure)
- Allow for direct countermodel extraction from failed proofs

G3-calculi for intuitionistic logic are not strictly terminating

$$\mathsf{L}\supset \frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

- Left premiss of L$\supset$ can be more complex than the conclusion
- Need of loop-checking mechanism

Remark

- Explicit contraction or loop-checking $\Rightarrow$ no strict termination
- No study of countermodel extraction

Starting point: base calculus for **IPL**

- Dyckhoff's G4ip (Dyckhoff 1992): terminating, simple, famous.
- Pinto & Dyckhoff's refutation calculus (Pinto & Dyckhoff 1995): calculus for "non-derivability" in G4ip: directly constructs countermodels.

L⊃ of G3-calculus

$$\mathrm{L}\supset\ \frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

replaced with four rules, one for every possible connective in the antecedent of ⊃:

$$\mathrm{L0}\supset\ \frac{\Gamma, p, B \Rightarrow \Delta}{\Gamma, p, p \supset B \Rightarrow \Delta} \qquad \mathrm{L}\wedge\supset\ \frac{\Gamma, C \supset (D \supset B) \Rightarrow \Delta}{\Gamma, (C \wedge D) \supset B \Rightarrow \Delta}$$

$$\mathrm{L}\vee\supset\ \frac{\Gamma, C \supset B, D \supset B \Rightarrow \Delta}{\Gamma, (C \vee D) \supset B \Rightarrow \Delta} \qquad \mathrm{L}\supset\supset\ \frac{\Gamma, C, D \supset B \Rightarrow D \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, (C \supset D) \supset B \Rightarrow \Delta}$$

- All premisses have a smaller complexity than the conclusion (according to a suitable notion of complexity).
- Bottom-up proof search is terminating.

Starting point II (countermodels)

Pinto & Dyckhoff 1995: refutation calculus for "non-derivability" in G4ip

- Anti-sequents $\Gamma \not\Rightarrow \Delta$ $\rightsquigarrow$ "$\bigvee \Delta$ does not follow from $\bigwedge \Gamma$"

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- Anti-sequents $\Gamma \not\Rightarrow \Delta \;\rightsquigarrow\;$ "$\bigvee \Delta$ does not follow from $\bigwedge \Gamma$"
- Convert G4-rules into refutation rules: if the premisses are not derivable, then the conclusion is not derivable

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- Anti-sequents $\Gamma \not\Rightarrow \Delta$ $\rightsquigarrow$ "$\bigvee \Delta$ does not follow from $\bigwedge \Gamma$"
- Non-derivable initial Sequents $\Gamma \not\Rightarrow \Delta$
- Convert G4-rules into refutation rules

One rule for every invertible premiss of a rule of G4ip:
If the premiss is not derivable, then the conclusion is not derivable

Examples:

$$\mathrm{L}\vee \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \longrightarrow \mathrm{L}\vee_1 \frac{\Gamma, A \not\Rightarrow \Delta}{\Gamma, A \vee B \not\Rightarrow \Delta} \qquad \longrightarrow \mathrm{L}\vee_2 \frac{\Gamma, B \not\Rightarrow \Delta}{\Gamma, A \vee B \not\Rightarrow \Delta}$$

$$\mathrm{L}{\supset}{\supset} \frac{\Gamma, C, D \supset B \Rightarrow D \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, (C \supset D) \supset B \Rightarrow \Delta} \longrightarrow \mathrm{L}{\supset}{\supset} \frac{\Gamma, B \not\Rightarrow \Delta}{\Gamma, (C \supset D) \supset B \not\Rightarrow \Delta}$$

(Only the right premiss of L⊃⊃ is invertible).

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- Anti-sequents $\Gamma \not\Rightarrow \Delta \;\rightsquigarrow\;$ "$\bigvee \Delta$ does not follow from $\bigwedge \Gamma$"
- Non-derivable initial Sequents $\Gamma \not\Rightarrow \Delta$
- Convert G4-rules into refutation rules

A single rule dealing with all non-invertible premisses of G4ip:
Excludes all possible derivations of the conclusion

$$\mathrm{L}{\supset}{\supset}\ \frac{\Gamma, C_i, D_i \supset B_i \Rightarrow D_i \qquad \Gamma, B_i \Rightarrow \Delta}{\Gamma, (C_i \supset D_i) \supset B_i \Rightarrow \Delta} \qquad\qquad \mathrm{R}{\supset}\ \frac{\Gamma, E_j \Rightarrow F_j}{\Gamma \Rightarrow E_j \supset F_j, \Delta}$$

$$\mathrm{nip}\ \frac{\Gamma_1, D_1 \supset B_1, C_1 \not\Rightarrow D_1 \quad \dots \quad \Gamma_n, D_n \supset B_n, C_n \not\Rightarrow D_n \quad \Gamma', E_1 \not\Rightarrow F_1 \quad \dots \quad \Gamma', E_m \not\Rightarrow F_m}{\Gamma, (C_1 \supset D_1) \supset B_1, \dots, (C_n \supset D_n) \supset B_n \not\Rightarrow E_1 \supset F_1, \dots, E_m \supset F_m, \Delta}$$

- $\Gamma' = \Gamma, (C_1 \supset D_1) \supset B_1, \dots, (C_n \supset D_n) \supset B_n$. $\quad \Gamma_i = \Gamma' \setminus \{(C_i \supset D_i) \supset B_i\}$.

+ Suitable application conditions (= no other rule is applicable)

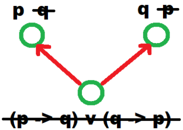
Pinto & Dyckhoff 1995: refutation calculus for “non-derivability” in G4ip

- A REFUTATION IS A COUNTERMODEL

Invertible rules $\Rightarrow$ Local rules $\Rightarrow$ Same world
Non-invertible rules $\Rightarrow$ Create new worlds reachable through $\preceq$

$$\dfrac{\dfrac{p \not\Rightarrow q \qquad q \not\Rightarrow p}{\not\Rightarrow p \supset q, q \supset p}\ \text{nip}}{\not\Rightarrow (p \supset q) \vee (q \supset p)}\ \text{R}\vee$$

$$\dfrac{\dfrac{p \not\Rightarrow q \qquad q \not\Rightarrow p}{\not\Rightarrow p \supset q, q \supset p}\ \text{nip}}{\not\Rightarrow (p \supset q) \vee (q \supset p)}\ \text{R}\vee$$



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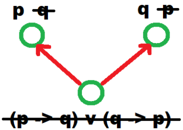


Remark: 1-1 correspondence between premisses of non-invertible rules and worlds of the model

"Positive" G4-calculus for **WK** and **CK**

- Rules of G4ip
- Standard modal rules:

$$\mathrm{K}_\Box\ \frac{\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta} \qquad \mathrm{K}_\Diamond\ \frac{\Sigma, B \Rightarrow C}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Diamond C, \Delta}$$

$$\mathrm{N}_\Diamond\ \frac{\Sigma, B \Rightarrow}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Delta}\ \text{(for } \mathbf{WK}\text{)}$$

- Special rule $\mathrm{K}_\Box$

$$\mathrm{L}\Box\supset\ \frac{\Sigma \Rightarrow C \qquad \Gamma, \Box\Sigma, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Box C \supset B \Rightarrow \Delta}$$

- Special rule $\mathrm{L}\Box\supset$

$$\mathrm{L}\Diamond\supset\ \frac{\Sigma, D \Rightarrow C \qquad \Gamma, \Box\Sigma, \Diamond D, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B \Rightarrow \Delta}$$

Note: G4$+\mathrm{K}_\Box+$ $\mathrm{L}\Box\supset$ $=$ calculus for the $\Box$-fragment of **CK** by [Iemhoff 2018]

Remarks

- The calculus is terminating
- Only the right premiss of $\mathrm{L}\Box\supset$ and $\mathrm{L}\Diamond\supset$ are invertible.

Initial anti-sequents

Initial sequents: are not axioms, nor a conclusion of any rule:

$$(\text{init})\ \Gamma, \Box\Gamma' \not\Rightarrow \Diamond\Delta', \Delta \qquad (\text{init})_{CK}\ \Gamma, \Box\Gamma', \Diamond\Gamma'' \not\Rightarrow \Delta$$

- $\Gamma \cap \Delta = \emptyset$.
- Γ contains only propositional variables, atomic implications, and implications of the form $\Diamond A \supset B$;
- Δ contains only atomic formulas;
- if $p \supset A \in \Gamma$, then $p \notin \Gamma$;
- if Γ contains an implication $\Diamond A \supset B$, then $\Diamond\Gamma'' = \emptyset$;

One rule for every invertible premiss

$$\text{L}\Box\supset \frac{\Sigma \Rightarrow C \qquad \Gamma, \Box\Sigma, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Box C \supset B \Rightarrow \Delta} \longrightarrow \text{L}\Box\supset \frac{\Gamma, B \not\Rightarrow \Delta}{\Gamma, \Box C \supset B \not\Rightarrow \Delta}$$

$$\text{L}\Diamond\supset \frac{\Sigma, D \Rightarrow C \qquad \Gamma, \Box\Sigma, \Diamond D, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B \Rightarrow \Delta} \longrightarrow \text{L}\Diamond\supset \frac{\Gamma, \Diamond D, B \not\Rightarrow \Delta}{\Gamma, \Diamond D, \Diamond C \supset B \not\Rightarrow \Delta}$$

A single rule (nip) dealing with all non-invertible premisses invertible premiss

Non-invertible premisses of modal or $\supset$-rules

$$\frac{\begin{array}{ll} \{\Gamma^{\Box} \not\Rightarrow A \mid \Box A \supset B \in \Gamma\} & \{\Gamma^{\Box} \not\Rightarrow A \mid \Box A \in \Delta\} \\ \{\Gamma^{\Box}, C \not\Rightarrow A \mid \Diamond A \supset B, \Diamond C \in \Gamma\} & \{\Gamma^{\Box}, A \not\Rightarrow B \mid \Diamond A \in \Gamma, \Diamond B \in \Delta\} \\ \{\Gamma', D \supset B, C \not\Rightarrow D \mid (C \supset D) \supset B \in \Gamma\} & \{\Gamma, A \not\Rightarrow B \mid A \supset B \in \Delta\} \end{array}}{\Gamma \not\Rightarrow \Delta}$$

- $\Gamma' = \Gamma \setminus \{(C \supset D) \supset B\}$.
  If $\Box A_1, ..., \Box A_n$ are *all* the $\Box$-formulas of Γ, then $\Gamma^{\Box} = A_1, ..., A_n$.
- Application conditions similar to G4 (= no other rule applicable)
- for **WK**: an additional rule: similar but includes non-derivability with $N_{\Diamond}$.

What about countermodels?

What about countermodels?

A refutation is a neighbourhood countermodel

What about countermodels?

A refutation is a neighbourhood countermodel

Rules and worlds

- Invertible rules $\Rightarrow$ Local rules $\Rightarrow$ Same world
- Non-invertible rules $\Rightarrow$ Create new worlds
  - Premisses determined by $\supset$-formulas $\Rightarrow$ worlds reachable through $\preceq$
  - Premisses determined by modal formulas $\Rightarrow$ worlds belonging to the neighbourhood

Extracting the neighbourhood (idea)

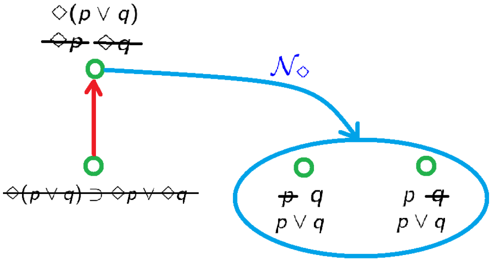
- $A^+ = \{\Gamma \not\Rightarrow \Delta \mid A \in \Gamma\}$.
- $\Box A, \Gamma \not\Rightarrow \Delta \rightsquigarrow w \quad \longrightarrow \quad A^+ \in \mathcal{N}_\Box(w)$.
- $\Diamond A, \Gamma \not\Rightarrow \Delta \rightsquigarrow w \quad \longrightarrow \quad A^+ \in \mathcal{N}_\Diamond(w)$.

A refutation is a neighbourhood countermodel

- Local rules ⇒ same world
- Premisses determined by ⊃-formulas ⇒ worlds reachable through $\preceq$
- Premisses determined by modal formulas ⇒ worlds belonging to the neighbourhood

$$\dfrac{\dfrac{\dfrac{\dfrac{q \not\Rightarrow^{2} p}{p \lor q \not\Rightarrow^{2} p}\,\mathrm{L}\lor \qquad \dfrac{p \not\Rightarrow^{3} q}{p \lor q \not\Rightarrow^{3} q}\,\mathrm{L}\lor}{\Diamond(p \lor q) \not\Rightarrow^{1.1} \Diamond p, \Diamond q}\,\mathrm{nip}}{\Diamond(p \lor q) \not\Rightarrow^{1.1} \Diamond p \lor \Diamond q}\,\mathrm{R}\lor}{\not\Rightarrow^{1} \Diamond(p \lor q) \supset \Diamond p \lor \Diamond q}\,\mathrm{nip}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{q \not\Rightarrow^{2} p}{p \lor q \not\Rightarrow^{2} p}\,\mathrm{L}\lor \qquad \dfrac{p \not\Rightarrow^{3} q}{p \lor q \not\Rightarrow^{3} q}\,\mathrm{L}\lor}{\Diamond(p \lor q) \not\Rightarrow^{1.1} \Diamond p, \Diamond q}\,\mathrm{nip}}{\Diamond(p \lor q) \not\Rightarrow^{1.1} \Diamond p \lor \Diamond q}\,\mathrm{R}\lor}{\not\Rightarrow^{1} \Diamond(p \lor q) \supset \Diamond p \lor \Diamond q}\,\mathrm{nip}$$



Annotation

annotation $\sigma = n_1.n_2.\ \dots\ .n_k$.

Anti-sequents are annotated $\Gamma \not\Rightarrow^{\sigma} \Delta$ as follows:

- The root anti-sequent $\Gamma \not\Rightarrow \Delta$ is annotated with 1.
- (Rule) $\neq$ different from nip: the premiss is annotated with the same annotation then its conclusion
- (nip): if the conclusion is annotated with σ, then its premisses are annotated as follows:
  - Premisses from $(C \supset D) \supset B$ on the left of the conclusion, or $A \supset B$ on the right, are annotated each with a different $\sigma.n$, with n new
  - Other Premisses each with a different new k

A refutation is a neighbourhood countermodel

For an annotated refutation $\mathcal{R}$ of $\Gamma \not\Rightarrow^1 \Delta$:

$$\Gamma^\sigma = \bigcup\{\Gamma \mid \Gamma \not\Rightarrow^\sigma \Delta \in \mathcal{R}\} \text{ and } \Delta^\sigma = \bigcup\{\Delta \mid \Gamma \not\Rightarrow^\sigma \Delta \in \mathcal{R}\}.$$

- $\mathcal{W}$ = the set of annotations occurring in $\mathcal{R}$
- $\sigma \preceq \rho$ iff $\rho = \sigma.\pi$ for some possibly empty annotation π.
- $\mathcal{V}(p) = \{\sigma \in \mathcal{W} \mid p \in \Gamma^\sigma\}$.
- For every $\Box A, \Diamond A$ occurring in $\mathcal{R}$ $A^+ = \{\sigma \in \mathcal{W} \mid A \in \Gamma^\sigma\}$.
- For every $\sigma \in \mathcal{W}$, $\mathcal{N}_\Box(\sigma)$ and $\mathcal{N}_\Diamond(\sigma)$ are defined as follows:
  - If there are no $\Box$-formulas in Γ^σ, then:
    - $\mathcal{N}_\Box(\sigma) = \{\mathcal{W}\}$.
    - $\mathcal{N}_\Diamond(\sigma) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Diamond B \in \Gamma^\sigma \text{ s.t. } B^+ \subseteq \alpha\}$.
  - Otherwise, if $\Box A_1, ..., \Box A_n$ are *all* the $\Box$-formulas in Γ^σ, then:
    - $\mathcal{N}_\Box(\sigma) = \{\alpha \subseteq \mathcal{W} \mid A_1^+ \cap \ldots \cap A_n^+ \subseteq \alpha\}$.
    - $\mathcal{N}_\Diamond(\sigma) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Diamond B \in \Gamma^\sigma \text{ s.t. } A_1^+ \cap \ldots \cap A_n^+ \cap B^+ \subseteq \alpha\}$.

A refutation is a neighbourhood countermodel

Countermodel for $\Diamond(p \lor q) \supset \Diamond p \lor \Diamond q$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \nRightarrow^{2} p}\,\text{init}}{p \lor q \nRightarrow^{2} p}\,\text{L}\lor \qquad \dfrac{\dfrac{}{p \nRightarrow^{3} q}\,\text{init}}{p \lor q \nRightarrow^{3} q}\,\text{L}\lor}{\Diamond(p \lor q) \nRightarrow^{1.1} \Diamond p, \Diamond q}\,\text{nip}}{\Diamond(p \lor q) \nRightarrow^{1.1} \Diamond p \lor \Diamond q}\,\text{R}\lor}{\nRightarrow^{1} \Diamond(p \lor q) \supset \Diamond p \lor \Diamond q}\,\text{nip}$$

$\mathcal{W} = \{1, 1.1, 2, 3\}$
$1 \preceq 1.1$
$\mathcal{V}(3) = \{p\} \quad \mathcal{V}(2) = \{q\}$
$\mathcal{N}_{\Box}(w) = \{\mathcal{W}\}$ for every $w \in \mathcal{W}$
$\mathcal{N}_{\Diamond}(w) = \emptyset$ for every $w \in \mathcal{W}$, $w \neq 1.1$
$\mathcal{N}_{\Diamond}(1.1) = \{\alpha \mid (p \lor q)^{+} \subseteq \alpha\} =$
$\{\{2,3\}, \{2,3,1\}, \{2,3,1.1\}, \{2,3,1,1.1\}\}$

A refutation is a neighbourhood countermodel

Countermodel for $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \not\Rightarrow^{2.1} q}\,\text{init}}{\not\Rightarrow^{2} p \supset q}\,\text{nip}}{\Diamond p \supset \Box q \not\Rightarrow^{1.1} \Box(p \supset q)}\,\text{nip}}{\not\Rightarrow^{1} (\Diamond p \supset \Box q) \supset \Box(p \supset q)}\,\text{nip}$$

$\mathcal{W} = \{1, 1.1, 2, 2.1\}$
$1 \preceq 1.1 \quad 2 \preceq 2.1$
$\mathcal{V}(2.1) = \{p\} \quad \mathcal{V}(w) = \emptyset$ for $w \neq 2.1$
$\mathcal{N}_{\Box}(w) = \{\mathcal{W}\}$ for every $w \in \mathcal{W}$
$\mathcal{N}_{\Diamond}(w) = \emptyset$ for every $w \in \mathcal{W}$

Countermodel for $\Diamond\bot \supset \bot$

$$\dfrac{\dfrac{}{\Diamond\bot \not\Rightarrow^{1.1} \bot}\ \mathrm{init}_{CK}}{\not\Rightarrow^{1} \Diamond\bot \supset \bot}\ \mathrm{nip}$$

$\mathcal{W} = \{1, 1.1\}$
$1 \preceq 1.1.$
$\mathcal{N}_\Box(1) = \{\mathcal{W}\} \quad \mathcal{N}_\Box(1.1) = \{\mathcal{W}\}$
$\mathcal{N}_\Diamond(1) = \emptyset \quad \mathcal{N}_\Diamond(1.1) = \mathcal{P}(\mathcal{W})$

How can we get relational countermodels?

- One can apply the transformation from neighbourhood to relational models
- *But*: The resulting relational model can be exponentially larger than the original neighbourhood one
  - Special case: what about the fragment with only $\Box$?
- Loss of 1-1 correspondence between premisses of non-invertible rules of the refutation and worlds of the countermodels

Neighbourhood models are the natural semantics of the refutation calculus.

Conclusion

What we know

- Constructive modal logics lead naturally to study Non-Normal modalities with an intuitionistic base
- We have a framework for intuitionistic non-normal modal logic:
  - $\Rightarrow$ Simple proof theory: Sequent calculi
  - $\Rightarrow$ modular semantic by Neighbourhood models
- **CK** and **WK** have their place
- The simple proof-theory of **CK** and **WK** allows us to define good terminating calculi for provability and refutation
- The refutation calculus justifies the Neighbourhood semantics as the natural one

Conclusion

What we do not know yet

- Other systems to study: Extensions by the classical cube (T,D,B,4,5)
- (Uniform) interpolation: recent results for some systems
- terminating (refutation) calculi for other Non-normal CMLs
- Extract relational countermodels for **CK** and **WK** : of the same size as neighbourhood ones, directly from the calculus or by transformation Is it possible?
- Complexity: we conjecture that all these logics, including CK and WK are in PSPACE, but we are not aware of any proof
  - If so, find optimal calculi taking as a base an optimal calculus for **IPL** (G4ip is not)
- Type-theoretic interpretation of Non-Normal CMLs in the style of (Bellin, De Paiva, Ritter 2001)

Thank you!

Some References

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