From unified correspondence to parametric correspondence

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Unified correspondence

Duality-theoretic approach to nonclassical logics

- canonical extensions;
- ALBA + translation;
- uniform definition of Sahlqvist/inductive formulas/inequalities;
- applications to structural proof theory (ALBA- driven generation of analytic rules).

Methodologically unified mathematical theory of LE-logics (LE: lattice expansions)

- duality-induced relational semantics (polarity-based frames, graph-based frames...);
- generalized Sahlqvist correspondence and canonicity;
- syntactic and semantic cut elimination, finite model property;
- Goldblatt-Thomason theorem.

Main tool: the algorithm ALBA

- computes the first-order correspondent of LE-terms/inequalities.
- reduction steps sound on complex algebras of relational structures (perfect LEs)

Examples: reflexivity and transitivity

	$\forall p[\Box p \leq p]$	
iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \Box p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]$	(generators)
iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\blacklozenge \mathbf{j} \leq p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]$	(adjunction)
iff	$\forall \mathbf{j} \forall \mathbf{m}[\blacklozenge \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]$	(Ackermann)
iff	$\forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}]$	(Ackermann)

	$\forall p[\Diamond\Diamond p \leq \Diamond p]$	
iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq p \ \& \ \Diamond p \leq \mathbf{m}) \Rightarrow \Diamond\Diamond \mathbf{j} \leq \mathbf{m}]$	(generators)
iff	$\forall \mathbf{j} \forall \mathbf{m}[\Diamond \mathbf{j} \leq \mathbf{m} \Rightarrow \Diamond\Diamond \mathbf{j} \leq \mathbf{m}]$	(Ackermann)
iff	$\forall \mathbf{j}[\Diamond\Diamond \mathbf{j} \leq \Diamond \mathbf{j}]$	(Ackermann)

Modularity: One reduction, many translations!

On Kripke frames (W, R):

$$\forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \quad \rightsquigarrow \quad \forall w\,(\Delta[w] \subseteq R[w]) \qquad \text{i.e.} \quad \Delta \subseteq R$$

$$\forall \mathbf{j}[\Diamond\Diamond \mathbf{j} \leq \Diamond \mathbf{j}] \quad \rightsquigarrow \quad \forall w\left(R^{-1}[R^{-1}[w]] \subseteq R^{-1}[w]\right) \qquad \text{i.e.} \quad R \circ R \subseteq R$$

On polarity-based frames $(A, X, I, R_{\Box}, R_{\Diamond})$:

$$\forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \quad \rightsquigarrow \quad \forall a\left(R_{\Box}^{(1)}[a] \subseteq I^{(1)}[a]\right) \qquad \text{i.e. } R_{\Box} \subseteq I$$

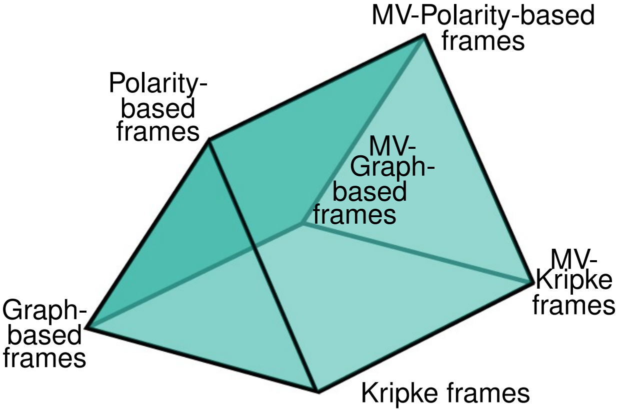
$$\forall \mathbf{j}[\Diamond\Diamond \mathbf{j} \leq \Diamond \mathbf{j}] \quad \rightsquigarrow \quad \forall a((R_{\Diamond} ;_I R_{\Diamond})^{(0)}[a] \subseteq R_{\Diamond}^{(0)}[a]) \qquad \text{i.e. } R_{\Diamond} \subseteq R_{\Diamond} ;_I R_{\Diamond}$$

On graph-based frames $(Z, E, R_{\Box}, R_{\Diamond})$:

$$\forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \quad \rightsquigarrow \quad \forall z\left(E^{[1]}[z] \subseteq R_{\Box}^{[1]}[z]\right) \qquad \text{i.e. } E \subseteq R_{\Box}$$

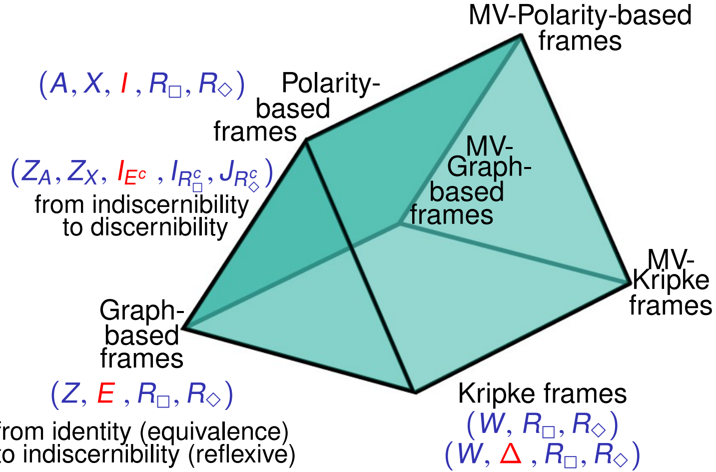
$$\forall \mathbf{j}[\Diamond\Diamond \mathbf{j} \leq \Diamond \mathbf{j}] \quad \rightsquigarrow \quad \forall z((R_{\Diamond} \star_E R_{\Diamond})^{[0]}[z] \subseteq R_{\Diamond}^{[0]}[z]) \qquad \text{i.e. } R_{\Diamond} \star_E R_{\Diamond} \subseteq R_{\Diamond}$$

A commutative diagram of semantic contexts 1/2



- These semantic contexts relate to each other via embeddings;
- Can we systematically relate the f.o.-correspondents of (modal) axioms to each other along these embeddings?
- Can we retrieve the intuitive meaning of these axioms in each context?

A commutative diagram of semantic contexts 2/2



- Complex algebras preserved under embeddings.
- $z_A I_R z'_X$ iff zRz' $\qquad z_X J_R z'_A$ iff zRz';
- Lifting preserves composition! E.g. $I_{(R\circ S)^c} = I_{R^c} ;_I I_{S^c}$.

Epistemic interpretation of modal axioms

Axiom	Kripke frames	Polarity-based frames	Graph-based frames
$\Box p \rightarrow p$ Factivity: if agent knows p then p true	$\Delta \subseteq R$ states that agent tells apart are non-identical	$R \subseteq I$ agent's attributions factually correct	$E \subseteq R$ states that agent tells apart are not inher. indist.
$\Box p \rightarrow \Box\Box p$ Positive introspection: if agent knows p then agent knows of knowing p	$R \circ R \subseteq R$ if agent tells apart x, y then agent can distinguish y from any z agent cannot tell apart from x	$R \subseteq R\,;R$ If agent thinks object a is an x-object, then agent must also attribute to a all features shared by x-objects according to i	$R \circ_E R \subseteq R$ positive introspection + inherent indistinguishab.

Skimming through the technicalities

In each semantic context:

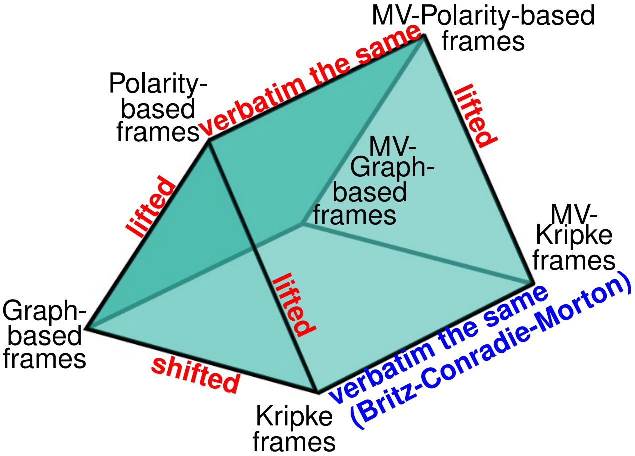
- Various **compositions of binary relations** have been defined;
- these compositions used as interpretations of **term-constructors in an algebraic language of binary relations**;
- **f.o.-correspondents** of Sahlqvist MRPs (modal reduction principles) translated **as term-inequalities** in this language.

Thanks to compositions being preserved under embeddings, **term-inequalities "lifted" and "shifted" along the embeddings** among contexts.

Preliminary results: the f.o.-correspondents of...

blue = all Sahlqvist formulas

red = all modal reduction principles



Conclusions: towards parametric correspondence

- Groundwork for a framework for systematically comparing f.o.-corr's of inductive formulas/inequalities across different relational semantics;
- correspondence theories for different logics and semantic contexts both methodologically *unified* by the same algebraic and algorithmic mechanisms, and *parametrically* related in terms of their outputs.
- Question: can other results be parametrically transferred in analogous ways?

How far can these results be extended?

- From Sahlqvist to inductive?
  - Yes, this should be no problem (but see below);
- to all LE-signatures?
  - Yes, this should be no problem (but see below);
- from modal reduction principles to **all** Sahlqvist/inductive inequalities?
  - No. Consider $\Diamond(p \vee q) \leq \Diamond(p \wedge q)$, which is Sahlqvist. Its f.o.-corr. on Kripke frames is $R \subseteq \varnothing$, which lifts to $X \times A \subseteq R_\Diamond$, which is NOT equivalent to its f.o.-corr. on polarity-based frames.
- What goes wrong?
  - Conjecture: we believe the failure is due to the loss of order-theoretic properties of the interpretation of $\wedge$ in moving from the classical to the lattice-based environment. Notice that no such loss occurs for the connectives occurring in modal reduction principles.