Łukasiewicz logic reasons about probability: encoding de Finetti coherence in MV-algebras

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George Boole, *An investigation of the laws of thought on which are founded the mathematical theories of logic and probabilities, 1854*:

"The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the **symbolical language of a Calculus***, and upon this foundation to establish the science of* **Logic** *and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of* **Probabilities***."*

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This talk is about the connection between: logic, probability, and algebra, via geometrical methods, in the many-valued setting.

The uncertainty of many-valued events

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Academical sorites paradox: losing one participant does not turn the attendance from high to not high...

Classical events	vs	**Many-valued events**
Either true or false		May be *partially* true
Classical logic		Many-valued logic
Probability maps		State maps

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Real-valued events: the truth value lies in the real unit interval $[0,1]$.

Suitable logical framework: Mathematical Fuzzy Logic in general, Łukasiewicz logic in particular.

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with $x \oplus y = \min\{x+y, 1\}$, $\neg x = 1 - x$.
Equivalently, as a residuated lattice (signature $\{\cdot, \to, \wedge, \vee, 0, 1\}$):

$$[0,1]_{\mathsf{MV}} = ([0,1], \cdot_{Ł}, \to_{Ł}, \min, \max, 0, 1)$$

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- $[0,1]_{\text{Ł}}$ generates the variety of MV-algebras, the equivalent algebraic semantics of Łukasiewicz logic.
- We will see how the quasiequational theory of MV-algebras is expressive enough to encode probabilistic reasoning.

(McNaughton's theorem): $\mathbf{F}_{\mathsf{MV}}(n)$ is the algebra of McNaughton functions (piecewise linear with integer coefficients) from $[0,1]^n$ to $[0,1]$ with operations defined pointwise from $[0,1]_{\text{Ł}}$.

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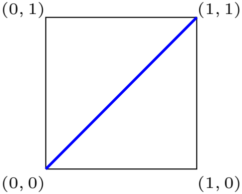
Formula $\varphi(x_1 \dots x_n) \longrightarrow$ McNaughton function $f_\varphi : [0,1]^n \to [0,1]$



$\varphi(x) = x$

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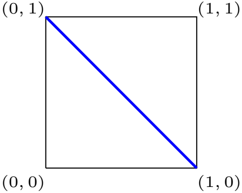

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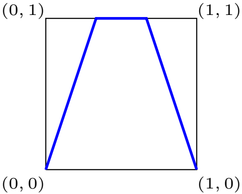

$$\varphi(x) = \neg(\neg x \oplus x^2)^3$$

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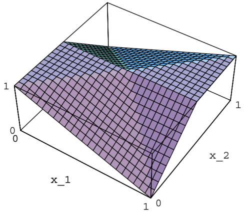
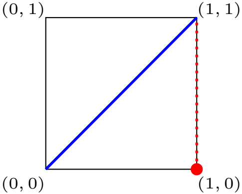


[1]

[1] Figure from S. Aguzzoli, S. Bova, *The free n-generated BL-algebra*, APAL, 2010.

Consider $\varphi(x_1 \dots x_n)$. Let $\mathcal{O}(\varphi) = \{\mathbf{x} \in [0,1]^n : f_\varphi(\mathbf{x}) = 1\}$.



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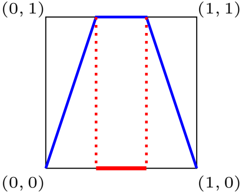

$\mathcal{O}(\neg(\neg x \oplus x^2)^3) = [1/3, 2/3]$

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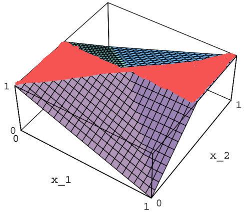
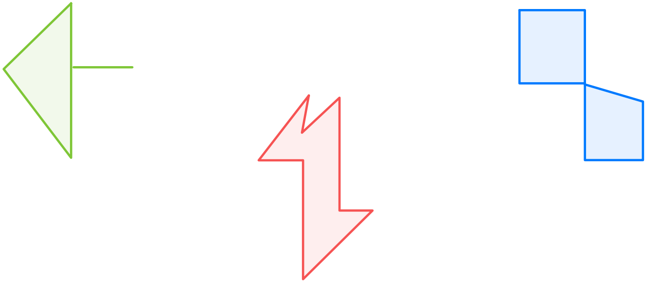


1-sets of Łukasiewicz formulas = rational polyhedra (Mundici)

Polytope: convex hull of finitely many rational points of $\mathbb{R}^n$
Rational polyhedron: finite unions of polytopes.



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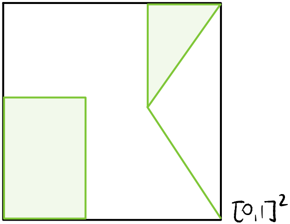

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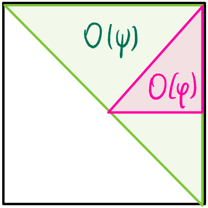
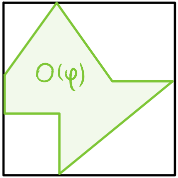
- $\varphi \vdash_{Ł} \psi$ iff $\mathcal{O}(\varphi) \subseteq O(\psi)$.



To each polyhedron $\mathscr{P} \subseteq [0,1]^n$ we associate a formula $\varphi_{\mathscr{P}}(x_1, \ldots, x_n)$.

Rational polyhedra in $[0,1]^n \Leftrightarrow$ Principal congruence filters of $\mathbf{F}_{MV}(n)$



$$\Longleftrightarrow \quad \mathbf{F}_{MV}(n)/\theta_{f_\varphi} = \mathbf{F}_{MV}(n)/\mathcal{O}_\varphi$$

The correspondence lifts to a duality between finitely presented MV-algebras (with homomorphisms) and rational polyhedra (with $\mathbb{Z}$-maps: componentwise McNaughton's functions)(Marra, Spada).

State theory and de Finetti coherence

Grounding on the work of Goodearl on lattice ordered abelian groups, Mundici introduces **states** on MV-algebras.

Let $\mathbf{A}$ be an MV-algebra. A state of **A** is a map $s : A \to [0, 1]$ such that $s(1) = 1$ and for every $a, b \in A$:

$$\text{if } a \cdot b = 0, \text{ then } s(a \oplus b) = s(a) + s(b).$$

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Examples:

- If $\mathbf{A}$ is a Boolean algebra, states = probability maps;
- homomorphisms of an MV-algebra $\mathbf{A}$ to $[0,1]_{\text{Ł}}$.
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States give a proper generalization of both classical and subjective probability theory to the many-valued setting.

- Kroupa-Panti integral-representation theorem: given any s of $\mathbf{F}_{\mathsf{MV}}(n)$

$$s(f) = \int_{[0,1]^n} f \mathrm{d}\mu.$$

One-one correspondence between states of $\mathbf{F}_{\mathsf{MV}}(n)$ and Borel probability measures on $[0,1]^n$ (extends to all MV-algebras).

- Formulas of Łukasiewicz logic = random variables.
States = expected values of bounded random variables.

- **Algebraic finite additivity** corresponds to measure theoretic σ-additivity.

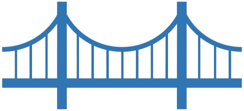
de Finetti, 1930s: foundation of subjective probability, in alternative to, for instance, the frequentist approach.

Example (F. Montagna):

Someone wants to build a bridge between Sicily and Calabria. What is the probability that the bridge is still up after 100 years?

Frequentist answer: build many bridges, wait for 100 years, compute the ratio between the number of bridges which are still up and the total number of bridges.

de Finetti's answer is based on a betting game.



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The book is **coherent** if there are no bets ensuring a sure loss.

$$\beta_H = 1/2,\ \beta_T = 1/2 \quad ✓$$

$$\beta_H = 1/3,\ \beta_T = 2/3 \quad ✓$$

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Let $\alpha_H = \alpha_T = -1$. Gambler pays 1, receives $4/3$.
Sure loss for the bookmaker!

Let $\mathcal{E} = \{\varphi_1 \dots \varphi_k\}$ be elements of a Boolean algebra.
Book $\beta : \mathcal{E} \to \{0, 1\}$, bets $\alpha_1, \dots, \alpha_k$, truth evaluation $v : \mathcal{E} \to \{0, 1\}$.
Balance of bookmaker: $\sum_{i=1}^{k} \alpha_i(\alpha_i - v(\varphi_i))$.

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This approach translates *mutatis-mutandis* to the MV-algebraic setting.

Mundici's Theorem: take a finite set of events $\mathcal{E}$ to be in an MV-algebra $\mathbf{A}$, then a book on $\mathcal{E}$ is coherent iff it can be extended to a state of $\mathbf{A}$.

Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ a finite set of events (in n variables).

The set of coherent books can be characterized **geometrically** (Mundici, Paris).

Fact: coherent books are in the convex closure of homomorphisms from $\mathbf{F}_{\mathsf{MV}}(n)$ to $[0,1]_{\text{Ł}}$ ($\leftrightarrow$ points of $[0,1]^n$).

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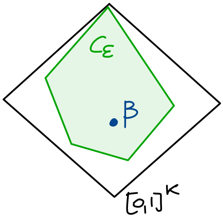
Take the McNaughton functions $f_{\varphi_1}, \ldots, f_{\varphi_k} : [0,1]^n \to [0,1]$.

Then a book $\beta : \mathcal{E} \to [0,1]$ is **coherent** iff

$$\beta = (\beta(\varphi_1), \ldots, \beta(\varphi_k)) \in \mathscr{C}_{\mathcal{E}} \subseteq [0,1]^k$$

where $\mathscr{C}_{\mathcal{E}} = \overline{\mathrm{co}}\{(f_{\varphi_1}(\mathbf{x}), \ldots, f_{\varphi_k}(\mathbf{x})) \mid \mathbf{x} \in [0,1]^n\}$.



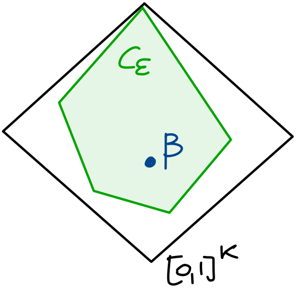
Given $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ its coherence set $\mathscr{C}_\mathcal{E}$ is a **convex rational polyhedron** of $[0,1]^k$.

It corresponds to:

- a finitely presented MV-algebra, $\mathbf{F}_{\mathsf{MV}}(k)/\mathscr{C}_\mathcal{E}$.
- a Łukasiewicz formula $\chi_\mathcal{E}$.

One-one correspondence:

- Coherent books on $\mathcal{E}$
- Points of $\mathscr{C}_\mathcal{E}$
- homomorphisms of $\mathbf{F}_{\mathsf{MV}}(k)/\mathscr{C}_\mathcal{E}$ to $[0,1]_Ł$



States have been introduced in other substructural logics.
Integral representation theorems:

- Gödel logic (Aguzzoli, Gerla, Marra 2008)
- Nilpotent minimum algebras (Aguzzoli, Gerla 2010)
- Product logic (Flaminio, Godo, U. 2018)

Other classes of residuated lattices:

- Representable CIRL (Flaminio, U. 2020)
- Pseudo MV-algebras (Dvurečenskij, 2001)
- Finite GBL-algebras (Flaminio, Gerla, Marigo 2017)
- Bosbach states, Riečan states (Ciungu, Georgescu, ...)
- ...

Plus the work of many other authors (Diaconescu, Di Nola, Lapenta, Leustean, Montagna ...)

Reasoning about probabilities

We use:

- Probabilistic logic FP(C, Ł) (Hájek, Godo, Esteva 1995)
- Probabilistic logic FP(Ł, Ł) (Flaminio, Godo 2007)

Key ideas:

- Probability as a partial operator on formulas representing events.
  $P(\varphi)$: φ is probable.
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Different approaches in the literature:

- 1970s: Keisler, Hoover
- Ognjanović, Rašković, Marković
- Fagin, Halpern, and Megiddo (1990). Shown to be equivalent to FP(C, Ł) with Δ by Baldi, Cintula, Noguera (2020).

Formulas of FP(C,Ł):

- Classical formulas $\varphi_1, \ldots, \varphi_n \ldots$
- Łukasiewicz formulas over variables of the form $P(\varphi_1), \ldots, P(\varphi_n) \ldots$

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$$P(\varphi \vee \psi) \oplus \neg P(\varphi \to \neg\psi)$$ ✓

$$(\varphi \vee \psi) \oplus \neg P(\varphi \to \neg\psi)$$ ✗

$$P(P(\varphi \vee \psi) \oplus \neg P(\varphi \to \neg\psi))$$ ✗

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Axioms and rules

- Classical logic axioms and rules for the inner logic
- Łukasiewicz axioms and rules for modal formulas,

(A1) $P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$,

(A2) $P(\neg\varphi) \equiv \neg P(\varphi)$,

(A3) $P(\varphi \vee \psi) \equiv ((P(\varphi) \to P(\varphi \wedge \psi)) \to P(\psi))$,

(N) The necessitation rule: from φ, deduce $P(\varphi)$.

FP(C,Ł) enjoys completeness with respect to finitely additive probabilities.

Let Φ be a formula of FP(C,Ł), then

$$\Phi = t[P(\varphi_1) \dots P(\varphi_k)]$$

t Łukasiewicz term; $\varphi_1 \dots \varphi_k$ classical formulas in n variables.

Theorem (Esteva, Godo, Hàjek)

Φ is a theorem of FP(C,Ł) if and only if given any probability map μ defined on the free Boolean algebra over n generators,

$$t^{[0,1]_Ł}[\mu(\varphi_1), \dots, \mu(\varphi_k)] = 1$$

Theorems and deductions of FP(C,Ł) can be translated to Łukasiewicz logic.

- Formulas of FP(C,Ł) can be translated to formulas of Łukasiewicz logic by a map $^\bullet$:
  - $(P(\psi))^\bullet = p_\psi$ (fresh propositional variable);
  - $(\top)^\bullet = \top$;
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- $\Phi \vdash_{\mathrm{FP}(\mathrm{C},\mathrm{Ł})} \Psi$ iff $\Phi^\bullet, FP^\bullet \vdash_{\mathrm{Ł}} \Psi^\bullet$.

  $FP^\bullet$: translation of all the axioms and rule of FP(C,Ł) on the **finite** free Boolean algebra over the variables in Φ, Ψ.

(Flaminio, Godo 2007) replace the inner logic with Łukasiewicz logic.

Formulas of FP(Ł,Ł):

$$t[P(\psi_1), \ldots, P(\psi_k)]$$

$\psi_1, \ldots \psi_k$ Łukasiewicz formulas; t Łukasiewicz term over "variables" of the form $P(\psi_1), \ldots, P(\psi_k)$.

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Axioms and rules:

- Łukasiewicz axioms and rules for the inner logic
- Łukasiewicz axioms and rules for modal formulas,

(P1) $P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$,

(P2) $P(\neg\varphi) \equiv \neg P(\varphi)$,

(P3) $P(\varphi \oplus \psi) \equiv P(\varphi) \oplus (P(\psi) \ominus P(\varphi \cdot \psi))$,

(N) Necessitation rule: from φ, deduce $P(\varphi)$.

FP(Ł,Ł) is a two-layered logic. An equivalent algebraic semantics would need two-sorted algebras (work in this direction by Kroupa and Marra).

The inner logic can be replaced by other logics: for instance we use Product logic (Flaminio, Godo, U. 2018).

General work on two-layered systems by Cintula and Noguera (2014).

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We will now see:

1. FP(Ł,Ł) reasons about states or coherent books;
2. theorems and deductions of FP(Ł,Ł) are actually contained in the quasiequational theory of MV-algebras.

Improving a completeness result in (Flaminio, 2021) we show **standard completeness** of $\mathrm{FP}(Ł,Ł)$ with respect to coherent books.

Let $\Phi = t[P(\varphi_1), \ldots, P(\varphi_k)], \Psi = u[P(\psi_1), \ldots, P(\psi_j)]$. Then:

Theorem (Flaminio, U.)

$\Phi \vdash_{\mathrm{FP}(Ł,Ł)} \Psi$ *iff given any coherent book* β *on* $\{\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_j\}$,

if $t^{[0,1]_Ł}[\beta(\varphi_1), \ldots, \beta(\varphi_k)] = 1$ *then* $u^{[0,1]_Ł}[\beta(\psi_1), \ldots, \beta(\psi_j)] = 1$

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So $\mathrm{FP}(Ł,Ł)$ reasons about coherent books (and states).

Key ideas:

- Formulas of FP(Ł,Ł) can be translated to formulas of Łukasiewicz logic by a map $^\bullet$ as follows:
  - $(P(\psi))^\bullet = x_\psi$ (Łukasiewicz propositional variable);
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- CANNOT use previous strategy (not locally finite)

- The variables $x_{\psi_1}, \ldots, x_{\psi_k}$ have to be evaluated as coherent books on the formulas $\psi_1 \ldots \psi_k$, so with values in the proper $\mathscr{C}_{\mathcal{E}}$.

Consider a set of events $\mathcal{E} = \{\varphi_1 \dots \varphi_k\}$.

Let $\chi_{\mathcal{E}}$ denote the Łukasiewicz formula whose 1-set is $\mathscr{C}_{\mathcal{E}}$.

Theorem (Flaminio, U.)

For all formulas Φ, Ψ over the events $\mathcal{E}$, the following are equivalent:

1. $\Phi \vdash_{FP} \Psi$;
2. $\chi_{\mathcal{E}}, \Phi^{\bullet} \vdash_{Ł} \Psi^{\bullet}$;
3. $\models_{MV} [(\chi_{\mathcal{E}} \approx 1) \& (\Psi^{\bullet} \approx 1)] \Rightarrow (\Psi^{\bullet} \approx 1)$
4. $\mathscr{C}_{\mathcal{E}} \cap \mathcal{O}_{\Phi^{\bullet}} \subseteq \mathcal{O}_{\Psi^{\bullet}}$.

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1. $\Phi \vdash_{FP} \Psi$; Probability
2. $\chi_{\mathcal{E}}, \Phi^{\bullet} \vdash_{Ł} \Psi^{\bullet}$; Logic
3. $\models_{MV} [(\chi_{\mathcal{E}} \approx 1) \& (\Psi^{\bullet} \approx 1)] \Rightarrow (\Psi^{\bullet} \approx 1)$ Algebra
4. $\mathscr{C}_{\mathcal{E}} \cap \mathcal{O}_{\Phi^{\bullet}} \subseteq \mathcal{O}_{\Psi^{\bullet}}$. Geometry

As direct consequences:

Corollary

$\mathrm{FP}(Ł,Ł)$ has a local deduction theorem:
$\Phi \vdash_{\mathrm{FP}(Ł,Ł)} \Psi$ *iff* $\exists n \in \mathbb{N}$ *such that* $\vdash_{\mathrm{FP}(Ł,Ł)} \Phi^n \to \Psi$

Corollary

The deducibility relation of $\mathrm{FP}(Ł,Ł)$ is decidable.

Coherent MV-algebras

Let $\mathcal{E}$ be a set of events.

Fixing $\mathcal{E}$ fixes $\mathscr{C}_{\mathcal{E}}$ and the finitely presented MV-algebra $\mathbf{F}_{\mathsf{MV}}(|\mathcal{E}|)/\mathscr{C}_{\mathcal{E}}$.

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Using work by Cabrer and Mundici we can show:

Theorem (Flaminio, U.)

Let $\mathscr{C} \subseteq [0,1]^k$ be convex polyhedron. The following are equivalent:

1. *$\mathscr{C}$ is the coherence set of some set of events $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$, i.e. $\mathscr{C} = \mathscr{C}_{\mathcal{E}}$;*
2. *$\mathbf{F}_{\mathsf{MV}}(k)/\mathscr{C}$ is a projective MV-algebra.*

P projective in a variety V if it is a retract of a free algebra $\mathbf{F}_{\mathsf{V}}(\kappa)$.

That is, there are homomorphisms i, j s.t. $\mathbf{A} \underset{j}{\overset{i}{\rightleftarrows}} \mathbf{F}_{\mathsf{V}}(\kappa)$, with $j \circ i = id$.

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Projective algebras are connected to: splitting algebras, unification problems...

We call coherent the MV-algebras isomorphic to some $\mathbf{F}_{\mathsf{MV}}(k)/\mathscr{C}$.

Coherent MV-algebras are finitely presented: we can restrict Marra-Spada duality.

Coherence set - Coherent MV-algebra - Coherent book

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$\mathbb{Z}$-maps - Homomorphism - Probabilistic substitution

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Probabilistic substitution: $\sigma : \{P(\varphi_i) : i = 1 \ldots k\} \to Form(\mathrm{FP}(\text{Ł},\text{Ł}))$ satisfying substitution invariance:

$$\text{if } \vdash_{FP} \Phi \leftrightarrow \Psi \text{ then } \vdash_{FP} \sigma(\Phi) \leftrightarrow \sigma(\Psi)$$

With this notion, we can study probabilistic unification problems.

Unification problem: finite set of identities $\Sigma = \{s_i = t_i : i = 1 \ldots n\}$ over variables in X.
Solution or unifier: substitution making the identities provable in the logic.

For an algebraizable logic $\mathcal{L}$ with equivalent algebraic semantics V, equivalently by (Ghilardi, 1997):

Algebraic unification problem: finitely presented algebra $\mathbf{F}_{\mathsf{V}}(X)/\theta_\Sigma$
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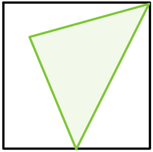
For Łukasiewicz logic, geometrically (Marra, Spada):

Geometric unification problem: Rational polyhedron $\mathcal{O}_\Sigma$
Geometric solution or unifier: $\mathbb{Z}$-map from $\mathbb{Z}$-retract to $\mathcal{O}_\Sigma$.



Observe:

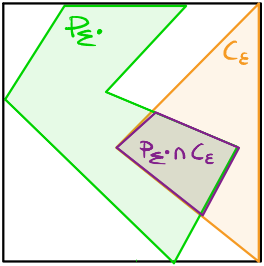
- Two layers: cannot apply usual notions.
- Not an equivalent algebraic semantics.
- One cannot reduce a probabilistic unification problem to a usual Łukasiewicz unification problem.

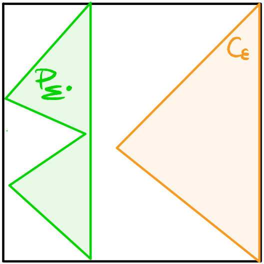
Nonetheless, we can define an analogue of Ghilardi's approach to unification for FP(Ł,Ł) using: coherent MV-algebras, probabilistic variables and probabilistic substitutions.

Idea: solve Łukasiewicz unification problems with probabilistic variables.

$\Sigma = \{s_i[P(\varphi_1)\dots P(\varphi_k)] = t_i[P(\varphi_1)\dots P(\varphi_k)] : i = 1\dots n\}$
$\Sigma^\bullet = \{s_i^\bullet = t_i^\bullet : i = 1\dots n\}$, $\mathcal{E} = \{\varphi_1, \dots \varphi_k\}$.

Algebraic unification problem: $\mathbf{F}_{MV}(k)/(\mathscr{P}_{\Sigma^\bullet} \cap \mathscr{C}_{\mathcal{E}})$.





- **Probabilistic Unification problem**: finite set of identities $\Sigma = \{s_i[P(\varphi_1)\dots P(\varphi_k)] = t_i[P(\varphi_1)\dots P(\varphi_k)] : i = 1\dots n\}$.

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  A **unifier** is a probabilistic substitution σ: $\vdash_{FP} \sigma(s_i) = \sigma(t_i)$

- **Algebraic unification problem:** finitely presented algebra $\mathbf{F}_{\mathsf{V}}(k)/(\mathscr{P}_{\Sigma^\bullet} \cap \mathscr{C}_{\mathcal{E}})$, where $\mathcal{E} = \{\varphi_1, \dots, \varphi_k\}$.

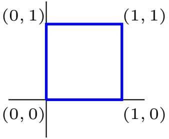
  Algebraic unifier: homomorphism $u : \mathbf{F}_{\mathsf{V}}(k)/(\mathscr{P}_{\Sigma^\bullet} \cap \mathscr{C}_{\mathcal{E}}) \to$ **C**, with **C** coherent MV-algebra.

Theorem (Flaminio, U.)

Given a (symbolic) probabilistic unification problem Σ for $\mathrm{FP}(Ł,Ł)$, its corresponding algebraic probabilistic unification problem has a solution or unifier iff Σ does.

Moreover:

- Probabilistic unification type (cardinality of best solutions = maximal unifiers ordered by generality) equivalent in the symbolic and algebraic case.
- Analogously to Łukasiewicz logic, there are probabilistic unification problems with no best solution (nullary type).



$$\Sigma = \{P(x) \vee \neg P(x) \vee P(y) \vee \neg P(y) = P(1)\}$$

- The probability logic $\mathrm{FP}($Ł,Ł$)$ is the logic of states and coherent books over Łukasiewicz events.

- Theorems and deductions of $\mathrm{FP}($Ł,Ł$)$ can be translated to Łukasiewicz logic, MV-algebras, and rational polyhedra.

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- Algebraizable extension of $\mathrm{FP}($Ł,Ł$)$: logic of MV-algebras with internal states (Flaminio, Montagna). Open problems:
  - Standard semantics? (Ongoing joint work with T. Flaminio and S. Lapenta).
  - Can we apply the ideas presented here?

Thank you!