Direttore Responsabile: Ruggero Ferro
Iscrizione al Registro Stampa del Tribunale di Padova n. 1235 del 26.9.1990

Pubblicato con il contributo di:
CARIPLO
CASSA DI RISPARMIO DELLE PROVINCIE LOMBARDE S.p.A.

Stampa Veronese (Padova)
Laser Fotocomposizioni (Padova)

NON STANDARD REGULAR FINITE SET THEORY

STEFANO BARATELLA AND RUGGERO FERRO

Abstract. We propose a set theory, called **NRFST**, in which the cantorian axiom of infinity is negated, and a new notion of infinity is introduced via non standard methods, i.e. via adequate notions of *standard* and *internal*, two unary predicates added to the language of **ZF**. After some initial results on **NRFST**, we investigate its relative consistency with respect to **ZF** and Kawai's **WNST**.

CONTENTS

Introduction

As we have stated in our previous paper [BF93], our goal is to introduce a set theory negating the cantorian axiom of infinity, but, at the same time, still able to support modern mathematics. A hint that this is possible comes from

- the results that mathematics reached even before the end of the past century with the use of infinitesimals and without the cantorian axiom of infinity;
- the possibility of exhibiting good foundations for the infinitesimal calculus via Robinson's non standard analysis (even though non standard analysis accepts the cantorian axiom of infinity);
- the hope of being able to introduce a non standard set theory including the negation of the axiom of infinity relativized to the standard and internal elements: such a hope is based on the impossibility of characterizing the notion of finiteness (and hence of infinity) in a first order language.

Lying underneath this entire approach there is a notion of infinity, different from the cantorian one, whose heuristic motivations will be discussed in detail in the appendix at the end of the paper.

We will admit the existence of natural numbers of which we do not recall the entire process of construction. These numbers may be regarded as infinite. So we get another notion of infinity that takes into account the role of the observer: he regards a set as infinite if he is not able to recall its entire construction.

In the set theory that we propose there is, in addition to the usual membership predicate, an undefined unary predicate, *standard*, that will be denoted by st and whose intended interpretation is that it holds on the elements that are finite. All the elements of the observed world form the *internal universe* and, among them, those that are too large to be grasped by the observer will be called *non standard*.

We will assume that the observer has a vague notion of the non standard elements of the observed world, and, from the point of view of the set theory that we propose, we want to be able to consider these vague notions as well. Thus we need a further undefined unary predicate, *internal*, denoted by in, that will distinguish between the objects of the observed world and the extensions of vague notions associated to them.

We will call *external* all the objects that we will deal with, whether they are obtained from the observed world or from a vague perception of the observed world or by an operation on external sets previously obtained.

To state something about the observed world we will use formulas relativized to the internal elements, i.e. formulas of the language of **ZF** in which we substitute "$\forall x\ in(x) \rightarrow$" and "$\exists x\ in(x) \wedge$" for "$\forall x$" and "$\exists x$", respectively. If ϕ is a formula in the language of **ZF**, we will denote by ${}^I\phi$ the relativization of ϕ to the internal elements.

Similarly, ${}^S\phi$ (${}^E\phi$) will denote the relativization of ϕ to the standard (external) elements.

We will denote by **NRFST** (*Non standard Regular Finite Set Theory*) the theory whose axioms state the main features of the concepts of standard, internal and membership.

We accept that the definable properties of the internal universe retain their truth value when relativized to the standard universe. This assumption is formalized as follows:

if ϕ is a formula in the language of **ZF** whose free variables are among $x_1, \dots, x_n$, then the following sentence is an axiom of our theory:

(*Transfer*)
$$\forall x_1 \forall x_2 \dots \forall x_n((st(x_1) \wedge st(x_2) \wedge \dots \wedge st(x_n)) \rightarrow ({}^S\phi \leftrightarrow {}^I\phi)).$$

This axiom schema will be called the *Transfer Principle*.

Furthermore, since we want the observed world to be a universe of sets in which the negation of the cantorian axiom of infinity holds, we describe the membership relation via the axioms of the theory **RFST** introduced in [BF93]. We recall that the axioms of **RFST** are the usual axioms of extensionality, pair, union, replacement, emptyset and the following axioms of finiteness and strong regularity:

(*Fin*)
$$\forall x Fin^f(x)$$

where $Fin^f(x)$ is the formula

$$\exists f \exists n(Fun(f) \wedge Nat(n) \wedge Inj(f) \wedge Dom(f) = n \wedge Ran(f) = x),$$

$Fun(f)$ and $Nat(n)$ are the formulas asserting that f is a function and n is a natural number, respectively.

(*SReg*)
$$\forall x \exists \alpha(On(\alpha) \wedge x \subseteq R(\alpha))$$

where the hierarchy of $R(\alpha)$'s is the usual one. Notice that the formulation of *SReg* is legitimate since, as proved in [BF93], *Pow* (the *Power Set Axiom*) follows from the axioms listed before *SReg*.

Now we are ready to state the axioms of **NRFST** concerning the membership relation.

The axioms of **RFST** are assumed to hold among the internal sets and can be relativized to the standard universe, in view of the transfer principle.

Thus the following will be an axiom schema for **NRFST**:

(${}^S RFST$) ${}^S\phi$ for each axiom ϕ of **RFST**

Since the elements of the the observed world are viewed as collections, we add the following *transitivity of the collection of internal sets* axiom:

(*Intrans*)
$$\forall x \forall y((x \in y \wedge\ in(y)) \rightarrow in(x))$$

We require the intersection of a standard set with any set, even with a non internal one, to be a standard set. This attitude is formalized by introducing in **NRFST** the following *axiom of standardization*:

(*Stan*)
$$\forall x \forall y[st(y) \rightarrow \exists z((st(z) \wedge \forall u(u \in z \leftrightarrow (u \in x \wedge u \in y))]$$

In the usual non standard set theory (see [Kaw83]) the name "axiom of standardization" is reserved for the following statement:

(*UStan*)
$$\forall x(\exists y(st(y) \wedge \forall z((z \in x \wedge st(z)) \rightarrow z \in y)) \rightarrow \exists z(st(z) \wedge \forall u(st(u) \rightarrow (u \in x \leftrightarrow u \in z))))$$

Nevertheless we stick to our terminology because the two formalizations can be proved equivalent using the other axioms that will be introduced in **NRFST** in the sequel.

The axioms introduced so far do not imply the existence of either non standard internal elements or non internal elements. To ensure the existence of such elements, as it is customary in nonstandard theories, we assume the *Axiom Schema of Enlargement*:

Let $\phi(x, y, y_1, ..., y_k)$ be a formula in the language of **ZF** whose free variables are among those displayed (where $y_1, ..., y_k$ are standard parameters). Then the following statement is an axiom of **NRFST**.

(*Enl*)

$$\forall y_1 \dots \forall y_n[(st(y_1) \wedge \dots \wedge st(y_n)) \rightarrow [\forall z(st(z) \rightarrow \exists y(st(y) \wedge \forall x(x \in z \rightarrow {}^S\phi(x, y, y_1, ..., y_k))))] \rightarrow \exists \mathbf{y}[in(\mathbf{y}) \wedge \forall x(st(x) \rightarrow {}^I\phi(x, \mathbf{y}, y_1, ..., y_k))]]$$

This formulation of *Enl* is limited to relations acting among standard elements and definable through a **ZF** formula relativized to the standard universe. We will overcome these restrictions when strengthening **NRFST** in Section 3.

Regarding the external elements, we require them to have the basic features of the notion of set, so we include in **NRFST** the axioms *Ext*, *Pair* and *Union* without any relativization.

For the moment, we propose to accept in **NRFST** the *Power Set Axiom* (*Pow*), the *Axiom of Choice*:

(*AC*)

$$\forall x \exists y(Fun(y) \wedge Dom(y) = x \wedge Ran(y) = \cup x \wedge \forall z((z \in x \wedge z \neq \emptyset) \rightarrow f(z) \in z))$$

and the *Axiom Schema of Separation*:

(*Sep*) $$\forall y_1, ..., y_k \forall u \exists z \forall y(y \in z \leftrightarrow (y \in u \wedge \phi(y_1, ..., y_k, y)))$$

for any formula $\phi(y_1, ..., y_k, y)$ in the language of **NRFST** whose free variables are among those displayed.

In Section 3, we will consider a strengthening of **NRFST** obtained by replacing *Sep* with the schema of replacement for external sets and the axiom of enlargement with the axiom schema of saturation.

The study of the consequences of possible restrictions on the use of the predicates *st* and *in* in the axioms for the external universe, as well as a deeper investigation of which properties have to hold in such a universe, are goals that will be pursued in future researches.

In the sequel we will use ideas from [Kaw83] and compare **NRFST** with the theories **WNST** and **NST**introduced by Kawai in [Kaw83]. In Kawai's formulation, the symbols **S** and **I** are two constant symbols of the language whose intended interpretations are the collections of standard and internal sets respectively. With respect to the notation introduced so far, we could say that

$$\mathbf{S} = \{x : st(x)\},\ \mathbf{I} = \{x : in(x)\},\ x \in \mathbf{S} \text{ iff } st(x),\ x \in \mathbf{I} \text{ iff } in(x)$$

We prefer to have the predicates *st* and *in* in the language rather then the constants **S** and **I**, because we do not want to assume a priori that the collections of the standard sets and of the internal sets be themselves sets; but in what follows we will use either notation when this will cause no problem.

We will stick to Kawai's notation: lower-case variables in a formula are intended to range on *standard* sets, upper-case variables on *external* sets, and boldface variables on *internal* sets. We will speak of *standard, internal, external* variables when referring to the tree kinds of variables mentioned above.

If ϕ is any formula in the language of **ZF**, ${}^S\phi$ (${}^I\phi$, ${}^E\phi$ respectively) is the formula obtained by replacing all bound variables of ϕ by standard (internal, external) variables. The free variables of ϕ are intended to be replaced by external variables in all ${}^S\phi$, ${}^I\phi$, ${}^E\phi$, if not otherwise specified.

Remark that the *Weak Extension Principle* (*WEP*) and the *Axiom of Infinity* for external sets are consequences of the axioms of **NRFST**.

Notice also that (even in Kawai's **WNST**) it is not necessary to assume that every standard set is internal. For, an application of the Transfer Principle yields

$$\forall x(\exists y(x \in y) \leftrightarrow \exists \mathbf{y}(x \in \mathbf{y}))$$

as a theorem in both **WNST** and **NRFST**. Since every standard set is member of some standard set, the consequence that every standard set is internal follows from the transitivity of the collection of internal sets.

The differences between the Axiom Schema of Enlarging in [Kaw83] and *Enl* are due to the presence of the axiom ${}^SFin$ in **NRFST**.

1. Basic Development of NRFST

We start by remarking that any subset of an internal (standard) set defined by a **ZF**-formula possibly with internal (standard) parameters is internal (standard).

Notice also that the notion of empty set is absolute.

In **RFST** the property of being a natural number can be expresses by means of a formula $Nat(x)$ (see [BF93]), thus in **NRFST** one can consider X-natural numbers, where X can be S, I or E. X-natural numbers are defined by the formula ${}^XNat(x)$. Notice that

$$\mathbf{NRFST} \vdash \forall x({}^SNat(x) \leftrightarrow {}^INat(x)),$$

as a consequence of Transfer Principle.

Also, the E-natural numbers are well-ordered by the membership relation and therefore, for any formula $\phi(x)$ in the language of **NRFST** (possibly with parameters), we have

$$\mathbf{NRFST} \vdash \phi(\mathbf{0}) \wedge \forall N({}^ENat(N) \wedge \phi(N) \rightarrow \phi(N \cup \{N\})) \rightarrow \forall N({}^ENat(N) \rightarrow \phi(N)).$$

We will refer to the above as to the *Principle of Induction on External Natural Numbers.*

Proposition 1.1. **NRFST** $\models \forall X({}^ENat(X) \leftrightarrow {}^SNat(X) \wedge st(X))$.

Proof. ($\rightarrow$) By induction on the external well-ordering of E-natural numbers.

($\leftarrow$) Let n be a standard element such that ${}^SNat(n) \wedge (\neg {}^ENat(n))$. Let

$$K = \{X \leq n :\ st(X) \wedge (\neg {}^ENat(X))\}.$$

It is not difficult to prove that K is well-defined, nonempty and that it is indeed a standard set. The existence of a least element in K leads to a contradiction. □

The above result, combined with *Transfer* and *Intrans* yields the following:

Proposition 1.2. **NRFST** $\vdash \forall x\ {}^E(Fin^f(x))$.

As a consequence of the previous proposition, we have that

$$\mathbf{NRFST} \vdash \forall x\ {}^X Fin^f(x), \qquad \text{with } X = S, I, E.$$

So, in the sequel, we will feel free to omit the relativization on the formula $Fin^f(X)$, when the variable X is instantiated on a standard argument.

As another application of the principle of external natural numbers, we notice that every standard finite external set of internal elements is indeed internal. In a sense this is the internal analogue of the axiom of standardisation.

The next theorem (proved by external induction) shows that every E-finite set is standard *iff* all its elements are standard.

Theorem 1.3.

$$\mathbf{NRFST} \vdash \forall X({}^E Fin^f(X) \to (st(X) \leftrightarrow \forall Y(Y \in X \to st(Y)))).$$

The transitivity of the collection of standard sets now follows immediately:

Corollary 1.4. **NRFST** $\vdash \forall X(st(X) \to \forall Y(Y \in X \to st(Y)))$.

Remark 1.1. Let $\phi(v, w)$ be the formula $\forall z(z \in v \to z \in w)$. For $X = S, I, E$, we abbreviate ${}^X\phi(V, W)$ by $V\ {}^X\subseteq W$. Then

$$\mathbf{NRFST} \vdash \forall x \forall y(x\ {}^S\subseteq\ y \leftrightarrow x\ {}^I\subseteq\ y \leftrightarrow x\ {}^E\subseteq\ y),$$

by *Transfer* and by transitivity of the collection of internal sets.

If x and y are standard variables, we can thus write $x \subseteq y$ without fear of ambiguity.

Now we show that **NRFST** proves the existence of an internal (and hence I-finite) set that contains all the standard sets.

Proposition 1.5. **NRFST** $\vdash \exists \mathrm{x} \forall y(y \in \mathrm{x})$.

Proof. Apply the Axiom Schema of Enlarging to the formula $\phi(x, y) \equiv x \in y$. □

Thus **NRFST** proves that the collection of standard sets is a set: apply separation by means of the formula $st(x)$ to an internal set containing all the standard sets. It follows that **NRFST** is powerful enough to prove the existence of non-standard internal I-natural numbers and the existence of external non-internal sets.

Theorem 1.6. **NRFST** $\vdash \exists \mathrm{x}({}^I Nat(\mathrm{x}) \wedge (\neg st(\mathrm{x})))$.

Proof. Let b be an internal set that includes all the standard sets (see the previous Proposition). By *Separation* on internal sets we define $\mathrm{c} = \{\mathrm{x} \in \mathrm{b} : {}^I Nat(\mathrm{x})\}$. The set c contains all the standard sets x such that ${}^S Nat(x)$.

It is easy to prove that $\mathrm{d} = \cup \mathrm{c}$ satisfies ${}^I On(\mathrm{x})$. Moreover, since d is I-finite, it satisfies also ${}^I Nat(\mathrm{x})$. Eventually, d is not a standard set. For, if it were not, the Axiom of Strong Regularity relativized to the standard sets would be contradicted. □

So, from the proof of Theorem 1.6, we have that, in **NRFST**, every internal set satisfying ${}^I On(\mathrm{x})$ satisfies also ${}^I Nat(\mathrm{x})$.

We have already remarked that the collection of standard sets is a set. Let us call such a set $\mathbf{S}$. It is easy to show that **NRFST** $\vdash \neg\ in(\mathbf{S})$. We point out that, as expected, the standard sets satisfying ${}^S Nat(x)$ form an initial segment of the internal sets satisfying ${}^I Nat(\mathrm{x})$ with respect to the order given by the membership relation.

We prove now that the axioms ${}^E Inf$ and WEP of the theory **WNST** (see [Kaw83]) are theorems of **NRFST**.

Theorem 1.7. **NRFST** $\vdash {}^E Inf$.

Proof. Apply separation for external sets to an internal set containing all the standard sets with respect to the formula ${}^E Nat(X)$. □

Theorem 1.8. **NRFST** $\vdash WEP$.

Proof. First of all, by transitivity of the collection of standard sets, we can reformulate Kawai's WEP as follows:

$$\forall x \forall \mathrm{y} \forall F((F : x \longrightarrow \mathrm{y}\ (\text{map}))$$
$$\to \exists \mathrm{f}((\mathrm{f} : x \longrightarrow \mathrm{y}\ (\text{map})) \wedge \forall z \in x(F(z) = \mathrm{f}(z)))).$$

The proof is by induction on the external natural numbers applied to the formula $\phi(N)$ so defined:

$$\forall x \forall \mathrm{y} \forall F((F : x \longrightarrow \mathrm{y}\ (\text{map})) \wedge (\text{"}x \text{ is bijective with } N\text{"}) \to$$
$$\exists \mathrm{g}((\mathrm{g} : x \longrightarrow \mathrm{y}\ (\text{map})) \wedge (\forall z \in x(F(z) = \mathrm{g}(z)))).$$

□

Remark 1.2. *Stan* and *UStan* are provably equivalent in

$$\mathbf{NRFST}^- = \mathbf{NRFST} \setminus \{Stan\}.$$

For clarity we rewrite *Stan* and *UStan* using the convention on variables:

$$(Stan) \qquad \forall X \forall y \exists z \forall U(U \in z \leftrightarrow U \in X \wedge U \in y).$$
$$(UStan) \qquad \forall X(\exists y({}^\circ X \subseteq y) \to \exists z \forall x(x \in X \leftrightarrow x \in z)),$$

where for every external set X, we let ${}^\circ X = X \cap \mathbf{S}$.

In order to prove that $\mathbf{NRFST}^- \vdash Stan \to UStan$, let X be such that there exists a standard y with ${}^\circ X \subseteq y$. By *Stan* applied to X and y there is a standard set z such that, for all U, $U \in z$ iff $U \in X \cap y$. *A fortiori* the *iff* holds for the U's that are standard and the conclusion follows.

In order to prove that $\mathbf{NRFST}^- \vdash UStan \to Stan$, we notice first that the statements 1.1 - 1.3 and Corollary 1.4 hold by replacing **NRFST** with $\mathbf{NRFST}^- + UStan$ in the assumptions.

The key point is to show $(\leftarrow)$ of Proposition 1.1 under the new assumptions (the rest is immediate). One can follow the proof of Proposition 1.1 until the remark that K has only standard elements. Then one can apply *UStan* to K (recall that $K \subseteq n \cup \{n\}$) so getting a (nonempty) standard set ${}^*K$. The existence of a least element among the standard members of ${}^*K$ leads to a contradiction.

Now, in order to show that $\mathbf{NRFST}^- \vdash$ *UStan* $\to$ *Stan*, choose X and y and consider $Z = X \cap y$. By *UStan* there is a standard z such that, for all standard x, $x \in Z$ iff $x \in z$ and, by the corresponding of Corollary 1.4 proved for $\mathbf{NRFST}^-+$ *UStan*, $U \in z$ *iff* $U \in X$ and $U \in y$, for all U.

2. Relative Consistency of NRFST with respect to WNST

In this section we prove that **NRFST** is consistent relative to Kawai's **WNST**.

Without further notice we will stick to Kawai's convention on the use of variables.

What we will show is indeed that the theory $\mathbf{NRFST}^+$ obtained by reformulating **NRFST** in the language of Kawai with the constant symbols **S** and **I** (so assuming that the collections of standard and internal sets are themselves sets) is consistent relative to **WNST**. Clearly, that implies the consistency of **NRFST** with respect to **WNST**.

In [Kaw83], the author provides a model for **WNST**, under the assumption of consistency of **ZFC**. The model is obtained as a direct limit of a sequence of enlargements beginning from a Zermelo universe.

Incidentally, notice that Kawai's construction performed in his Lemma 2 ([Kaw83]) not always leads to a structure that is a model of the axiom of infinity, in contrast to what stated by the author. For, let us define inductively $R_0 = \emptyset$ and $R_{i+1} = P(R_i)$ (power set). Let $\mathbf{M}_{i+1}$ be $[R_{i+1}, \in_{i+1}]$, where $\in_{i+1}$ is $\in$ restricted to R_{i+1}. It is easy to verify that if we apply repeatedly the Proposition preceding Lemma 2 in [Kaw83], starting from $\mathbf{M}_1$, we get the chain of structures

$$(\mathbf{M}_{i+1}, \iota_{i+1} : R_{i+1} \to R_{i+2})_{i\in\omega},$$

where ι_{i+1} is the inclusion map for all $i \in \omega$.

The direct limit of the chain $(\mathbf{M}_{i+1}, \iota_{i+1} : R_{i+1} \to R_{i+2})_{i\in\omega}$ is the structure

$$[R_\omega, \in] = [\bigcup_{i\in\omega} R_i, \bigcup_{i\in\omega} \in_i],$$

that clearly does not satisfy the axiom of infinity. In the sequel we show how to modify a given model $\mathbf{V} = (V, \in, S, I)$ of **WNST** in order to get a model of $\mathbf{NRFST}^+$. We commit the abuse of using the symbol "$\in$" for the membership relation in $\mathbf{V}$ since from now on $\mathbf{V}$ will be our universe of discourse. We let

$$I^* = \{\mathrm{x} \in I : \forall \mathrm{y}(\mathrm{y} \in (Trcl(\mathrm{x}) \cup \{\mathrm{x}\}) \to^I (Fin^f(\mathrm{y})))\} \quad \text{and} \quad S^* = S \cap I^*.$$

Notice that the definitions of I^* and S^* are legitimate by the *Separation Schema* for external sets of **WNST**.

Remark 2.1. By the Transfer Principle of **WNST**, it holds also that

$$S^* = \{x \in S : \forall y(y \in (Trcl(x) \cup \{x\}) \to^S (Fin^f(y)))\}.$$

If ϕ is a formula in the language of **ZF**, we shall use the abbreviations $(\forall x \in S^*)\phi$ and $(\exists x \in S^*)\phi$ for $\forall x(\forall y(y \in (Trcl(x) \cup \{x\}) \to^S (Fin^f(y)) \to \phi)$ and for $\exists x(\forall y(y \in Trcl(x) \cup \{x\} \to^S (Fin^f(y))) \wedge \phi)$, respectively.

The abbreviations $(\forall \mathrm{x} \in I^*)\phi$ and $(\exists \mathrm{x} \in I^*)\phi$ are introduced in a similar way.

Given ϕ as above, we shall denote by ${}^{S^*}\phi$ (${}^{I^*}\phi$) the formula obtained by relativizing all the quantifications in ϕ to S^* (I^*).

Before proving that $\mathbf{V}^* = (V, \in, S^*, I^*) \models \mathbf{NRFST}^+$, we need a preliminary result.

Lemma 2.1. *For every formula $\phi(v_1, \ldots, v_n)$ in the language of* **ZF** *whose free variables are among those displayed and for all $a_1, \ldots, a_n \in S^*$, the following are equivalent:*

(i) $\mathbf{V}^* \models {}^S\phi(a_1, \ldots, a_n)$;

(ii) $\mathbf{V} \models {}^{S^*}\phi(a_1, \ldots, a_n)$;

(iii) $\mathbf{V} \models {}^{I^*}\phi(a_1, \ldots, a_n)$;

(iv) $\mathbf{V}^* \models {}^I\phi(a_1, \ldots, a_n)$.

Proof. Let us show that (i) $\Leftrightarrow$ (ii), proceeding by induction on the complexity of ϕ.

The only nontrivial case is when ϕ has the form $\exists v_0 \psi(v_0, v_1, \ldots, v_n)$:

$$\mathbf{V}^* \models {}^S\phi(a_1, \ldots, a_n) \Leftrightarrow$$

$\Leftrightarrow$ there exists $a_0 \in S^*$ such that $\mathbf{V}^* \models {}^S\psi(a_0, a_1, \ldots, a_n)$

$\Leftrightarrow$ (by ind. hyp.) there exists $a_0 \in S^*$ such that $\mathbf{V} \models {}^{S^*}\psi(a_0, a_1, \ldots, a_n)$

$\Leftrightarrow \mathbf{V} \models (\exists v_0 \in S^*)\ {}^{S^*}\psi(a_0, a_1, \ldots, a_n) \Leftrightarrow \mathbf{V} \models {}^{S^*}\phi(a_1, \ldots, a_n)$.

The proof of (iii) $\Leftrightarrow$ (iv) goes in the same way. □

Eventually, the equivalence of (ii) and (iii) is a consequence of the *Transfer Principle* of **WNST**.

Corollary 2.2. *Transfer holds in* $\mathbf{V}^*$.

Lemma 2.3. *Intrans holds in* $\mathbf{V}^*$, *i.e.* $\mathbf{V}^* \models \forall X \forall \mathrm{y}(X \in \mathrm{y} \to X \in \mathbf{I})$.

Proof. Straightforward. □

The next Lemma shows that S^* is transitive in $\mathbf{V}^*$.

Lemma 2.4. $\mathbf{V}^* \models \forall x \forall \mathrm{y}(\mathrm{y} \in x \to \mathrm{y} \in \mathbf{S})$.

Proof. Recall that (see [Kaw83])

$$\mathbf{WNST} \vdash \forall x({}^S(Fin^f(x)) \leftrightarrow {}^E(Fin^f(x)))$$

and $$\mathbf{WNST} \vdash \forall A({}^E(Fin^f(A)) \to (A \in \mathbf{S} \leftrightarrow A \subseteq \mathbf{S})).$$

Thus

$$\mathbf{V} \models \forall x \forall \mathrm{y}(x \in S^* \wedge \mathrm{y} \in x \to \mathrm{y} \in \mathbf{S}),$$

from which the conclusion follows. □

Lemma 2.5. *Enl holds in* $\mathbf{V}^*$.

Proof. Let $\phi(u, v)$ be a formula in the language of **ZF** (possibly with parameters) such that

(1) $$\mathbf{V}^* \models \forall w \exists v \forall u \in w\ {}^S\phi(u, v).$$

Then (by (i) $\Leftrightarrow$ (ii) of Lemma 2.1) we have

$$\mathbf{V} \models \forall w \in S^* \exists v \in S^* \forall u \in (w \cap S^*)\ {}^{S^*}\phi(u, v), \quad \text{i.e.}$$

$$\mathbf{V} \models \forall w \in S^* \exists v \forall u \in w(v \in S^* \wedge (u \in S^* \to {}^{S^*}\phi(u, v))).$$

Let ${}^{S^*}\psi(u,v)$ be $v \in S^* \wedge (u \in S^* \to {}^{S^*}\phi(u,v))$. We claim that

$$\mathbf{V} \models \forall w({}^S(Fin^f(w)) \to \exists v \forall u \in w\ {}^{S^*}\psi(u,v)). \tag{2}$$

Assume not. Then $\mathbf{V} \models \exists w({}^S(Fin^f(w)) \wedge \forall v \exists u \in w(\neg^{S^*}\psi(u,v))$, i.e.

$$\mathbf{V} \models \exists w({}^S(Fin^f(w)) \wedge \forall v \exists u \in w(\neg(v \in S^*) \vee (u \in S^* \wedge \neg^{S^*}\phi(u,v))).$$

So $\mathbf{V} \models \exists w({}^S(Fin^f(w)) \wedge \forall v \in S^* \exists u \in (w \cap S^*) \neg^{S^*}\phi(u,v))$.

Let $b \in V$ be a witness for the above formula and let $C = b \cap S^*$. Since $b \in S$, the characterization of S^* given in Remark 2.1 allows us to infer that $C \in S$ since C is obtained with an application of the *Axiom Schema of Separation* for standard set of **WNST**. Also, $C \in S^*$ by definition and because it can be easily proved in **WNST** that a standard subset of a finite standard set is finite. Therefore

$$\mathbf{V} \models \exists w \in S^* \forall v \in S^* \exists u \in (w \cap S^*)(\neg^{S^*}\phi(u,v))$$

$$\text{and hence} \quad \mathbf{V}^* \models \exists w \forall v \exists u \in w\ \neg^S\phi(u,v),$$

so contradicting (1).

Thus (2) holds and, by the *Schema of Enlarging* of **WNST**, we get

$$\mathbf{V} \models \exists \mathrm{v} \forall u\ {}^{I^*}\psi(u,\mathrm{v}).$$

So

$$\mathbf{V} \models \exists \mathrm{v} \forall u(\mathrm{v} \in I^* \wedge (u \in I^* \to {}^{I^*}\phi(u,\mathrm{v})))$$

$$\text{and hence} \quad \mathbf{V} \models \exists \mathrm{v} \in I^* \forall u \in S^*\ {}^{I^*}\phi(u,\mathrm{v})$$

$$\text{and, finally,} \quad \mathbf{V}^* \models \exists \mathrm{v} \forall u\ {}^{I}\phi(u,\mathrm{v}),$$

as requested. □

Lemma 2.6. *The axiom of standardisation Stan holds in* $\mathbf{V}^*$.

Proof. We show that *UStan* holds in $\mathbf{V}^*$. This, together with all the other axioms, shows that *Stan* holds in $\mathbf{V}^*$ by Remark 1.2.

Let A be an external set such that $\mathbf{V}^* \models \exists x({}^\circ A \subseteq x)$. Therefore there exists $a \in S^*$ such that

$$\mathbf{V} \models A \cap S^* \subseteq a. \tag{3}$$

$A \cap S^*$ is obtained by *Separation* on external sets. By (3), we can apply the *Axiom of Standardisation* (that holds in $\mathbf{V}$) to $A \cap S^*$, so getting $b \in S$ such that

$$\mathbf{V} \models \forall x(x \in b \leftrightarrow x \in (A \cap S^*)).$$

Let $c = a \cap b$. Clearly $c \in S^*$ and $\mathbf{V} \models \forall x \in S^*(x \in c \leftrightarrow x \in A)$.

Hence $\mathbf{V} \models \exists y \in S^* \forall x \in S^*(x \in y \leftrightarrow x \in A)$ and the required result follows by applying (i) $\Leftrightarrow$ (ii) of Lemma 2.1. □

Lemma 2.7. *All the axioms for the external sets hold in* $\mathbf{V}^*$.

Proof. The axioms for the external sets hold in $\mathbf{V}$ and do not contain occurrences of the constant symbols **S** and **I**. Hence, by definition of $\mathbf{V}^*$, they are true also in $\mathbf{V}^*$. □

Let us show now that all the axioms of ${}^S RFST$ hold in $\mathbf{V}^*$. This is easily seen for ^{S}*Emptyset*, ^{S}*Pair*, ^{S}*Ext*, ^{S}*Union*.

Lemma 2.8. $\mathbf{V}^* \models {}^S Repl$.

Proof. We want to show that, for every formula $\phi(y_1,\ldots,y_k,x,y)$ in the language of **ZF** whose free variables are among those displayed,

$$\mathbf{V}^* \models \forall y_1,\ldots,y_k \forall u(\forall x \in u \exists! y\ {}^S\phi(x,y) \to \exists z \forall y(y \in z \leftrightarrow \exists x \in u\, {}^S\phi(x,y))).$$

For a sake of clarity, let us forget about the parameters $y_1,\ldots,y_k$ and let $a \in S^*$ be such that $\mathbf{V}^* \models \forall x \in a \exists! y\ {}^S\phi(x,y)$.

Hence, by (i) $\Leftrightarrow$ (ii) of Lemma 2.1, we have

$$\mathbf{V} \models \forall x \in S^*(x \in a \to \exists! y(y \in S^* \wedge {}^{S^*}\phi(x,y))). \tag{4}$$

By Lemma 2.4, (4) is equivalent to

$$\mathbf{V} \models \forall x \in a \exists! y(y \in S^* \wedge {}^{S^*}\phi(x,y)). \tag{4'}$$

We are now in a position to apply ${}^S Repl$ (that holds in $\mathbf{V}$) to get

$$\mathbf{V} \models \exists z \forall y(y \in z \leftrightarrow \exists x \in a(y \in S^* \wedge {}^{S^*}\phi(x,y))). \tag{5}$$

Let $b \in S$ be such that $\mathbf{V} \models \forall y(y \in b \leftrightarrow \exists x \in a(y \in S^* \wedge {}^{S^*}\phi(x,y)))$. From the finiteness of a and from (4'), one can easily prove that $\mathbf{V} \models {}^S(Fin^f(b))$.

Hence, by (5), $b \in S^*$ and, again by (5),

$$\mathbf{V} \models \forall y \in S^*(y \in b \leftrightarrow \exists x \in (a \cap S^*)\ {}^{S^*}\phi(x,y)).$$

So $\mathbf{V} \models \exists z \in S^* \forall y \in S^*(y \in b \leftrightarrow \exists x \in (a \cap S^*)\ {}^{S^*}\phi(x,y))$, from which the required result follows by applying (1) $\Leftrightarrow$ (2) of Lemma 2.1. □

Lemma 2.9. $\mathbf{V}^* \models \forall x\ {}^S(Fin^f(x))$.

Proof. First of all notice that $\mathbf{V} \models \forall x\ {}^S(Fin^f(x))$. Thus, by induction on the standard natural number giving the finiteness of $a \in S^*$, we can prove that

$$\mathbf{V} \models \exists f \in S^* \exists m \in S^*({}^{S^*}Fun(f) \wedge {}^{S^*}Nat(m) \wedge {}^{S^*}(\text{"}f \text{ is a bijection of } m \text{ onto } a\text{"})).$$

Therefore $\mathbf{V} \models \forall x \in S^*\ {}^{S^*}(Fin^f(x))$, from which the conclusion follows by applying (1) $\Leftrightarrow$ (2) of Lemma 2.1. □

Before proving that $\mathbf{V}^* \models {}^S SReg(x)$, we need some preliminary results.

Lemma 2.10. $\mathbf{V} \models \forall n({}^S Nat(n) \leftrightarrow (n \in S^*) \wedge {}^{S^*}Nat(n))$

Proof. The left-to-right implication can be proved by induction on standard natural numbers. The base step is trivial. For the inductive step,

$$\mathbf{V} \models (n \in S) \wedge {}^S Nat(n \cup \{n\}) \quad \Rightarrow \quad \mathbf{V} \models {}^S Nat(n)$$

$$\Rightarrow \quad (\text{by ind. hyp.})\mathbf{V} \models (n \in S^*) \wedge {}^{S^*}Nat(n)$$

$$\Rightarrow \quad \mathbf{V} \models (n \cup \{n\} \in S^*) \wedge {}^{S^*}Nat(n \cup \{n\}).$$

For the other implication, let $n \in S^*$ be such that $\mathbf{V} \models {}^{S^*}Nat(n) \wedge \neg^S Nat(n)$.

In $\mathbf{V}$ consider $m = (n \cup \{n\}) \cap \{x \in S^* : {}^{S^*}Nat(x) \wedge \neg^S Nat(x)\}$. The set m is in S^* and is nonempty. The existence of a least S^*–element in m leads to a contradiction. □

Notice that since **RFST** $\setminus \{SReg\} \vdash$ *Pow* (see [BF93]), as a consequence of the axioms so far proved to hold in $\mathbf{V}^*$, we have $\mathbf{V}^* \models {}^S Pow$. Hence we can define in $\mathbf{V}^*$ the hierarchy of $R(\alpha)$'s and prove the required result:

Lemma 2.11. $\mathbf{V}^* \models {}^S SReg$.

Proof. We shall prove that

(6) $$\mathbf{V}^* \models \forall x \exists y({}^S Nat(y) \wedge x \subseteq R(y)), \quad \text{i.e.}$$
$$\mathbf{V} \models \forall x \in S^* \exists y \in S^*({}^{S^*} Nat(y) \wedge x \subseteq {}^{S^*} R(y)).$$

Now, induction on standard natural numbers yields that

$$\mathbf{V} \models \forall x({}^S Nat(x) \rightarrow R(x) = {}^{S^*} R(x)),$$

hence, by Lemma 2.10, (5) is equivalent to

(7) $$\mathbf{V} \models \forall x \in S^* \exists y({}^S Nat(y) \wedge x \subseteq R(y)).$$

Since $\mathbf{V} \models {}^S SReg$, (7) can be proved by induction on rank. □

3. Strengthening NRFST

We will strengthen **NRFST** by replacing the axiom schema *Enl* with a stronger form (denoted by *Sat*) and by introducing the *axiom schema of Replacement for external sets* (*ERepl*).

We have proved in Section 1 that, under the axioms of **NRFST**, the collection of standard sets is indeed an external set. The following definition is customary to strengthen *Enl* (see [Kaw83]):

Definition 3.1. (**NRFST**) A set A is *S-size* (notation: $Ss(A)$) if there exists a surjective (external) function from **S** onto A.

We can now state the axiom schema of *Saturation*: let $\phi(x, y, z_1, \ldots, z_n)$ be a formula of **ZF** whose free variables are among those displayed. Then

(*Sat*)
$$\forall X \forall \mathbf{x}_1, \ldots \mathbf{x}_n[[Ss(X) \wedge \forall \mathbf{x}({}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq X \rightarrow \exists \mathbf{u} \forall \mathbf{v} \in \mathbf{x}\ {}^I\phi(\mathbf{v}, \mathbf{u}, \mathbf{x}_1, \ldots, \mathbf{x}_n))]$$
$$\rightarrow \exists \mathbf{w} \forall \mathbf{v} \in X\ {}^I\phi(\mathbf{v}, \mathbf{w}, \mathbf{x}_1, \ldots, \mathbf{x}_n)].$$

Here X is not bound to be included in the internal universe because this restriction would not alter the strength of *Sat*.

It is clear that *Sat* is stronger than the axiom *Enl* introduced in Section 2 for **S** is clearly S-size. However, we have to keep both *Enl* and *Sat* since the notion of S-size does make sense thanks to *Enl*.

We will call **NRFST*** the theory obtained by adding to **NRFST** the axioms *ERepl* and *Sat*.

We claim that **NRFST*** is consistent relative to **ZFC**. Indeed, what we will prove is that **NRFST*** is consistent relative to Kawai's **NST**. Since **NST** is a conservative extension of **ZFC** (see [Kaw83]), the proof of the claim then follows.

We argue as in Section 2, after formulating **NRFST*** in the language of Kawai with the constant symbols **S** and **I**. Under the assumption of consistency of **ZFC**, there exists a model $\mathbf{W} = (W, \in, S, I)$ of **NST**. Again, we use "$\in$" for the membership relation in **W** because **W** will be our universe of discourse. We define $\mathbf{W}^* = (W, \in, S^*, I^*)$, where the definitions of S^* and I^* are those given in Section 2 relative to **W**. A straightforward check shows that all the results of Section 2, but Lemma 2.5, hold relative to **W** and $\mathbf{W}^*$. Also, $\mathbf{W}^* \models ERepl$, because *ERepl* holds in **W**. So, we are left with the proof of the following:

Lemma 3.1. *Sat holds in* $\mathbf{W}^*$.

Proof. Let $\phi(x, y, z_1, \ldots, z_n)$ be a formula of **ZF** whose free variables are among those displayed and let A in W and $b_1, \ldots, b_n$ in I^* be such that

$$\mathbf{W}^* \models Ss(A) \wedge \forall \mathbf{x}[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \rightarrow \exists \mathbf{u} \forall \mathbf{v} \in \mathbf{x}\ {}^I\phi(\mathbf{v}, \mathbf{u}, b_1, \ldots, b_n)].$$

In the sequel, we will omit the parameters $b_1, \ldots, b_n$. Notice that $\mathbf{W}^* \models Ss(A)$ implies $\mathbf{W} \models Ss(A)$. Also, we have

$$\mathbf{W} \models \forall \mathbf{x} \in I^*[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \rightarrow \exists \mathbf{u} \forall \mathbf{v} \in \mathbf{x}(\mathbf{u} \in I^* \wedge (\mathbf{v} \in I^* \rightarrow {}^{I^*}\phi(\mathbf{v}, \mathbf{u})))].$$

Let ${}^{I^*}\psi(\mathbf{v}, \mathbf{u})$ be $\mathbf{u} \in I^* \wedge (\mathbf{v} \in I^* \rightarrow {}^{I^*}\phi(\mathbf{v}, \mathbf{u}))$. We claim that

(8) $$\mathbf{W} \models \forall \mathbf{x}[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \rightarrow \exists \mathbf{u} \forall \mathbf{v} \in \mathbf{x}\ {}^{I^*}\psi(\mathbf{v}, \mathbf{u})].$$

Assume not. Then

$$\mathbf{W} \models \exists \mathbf{x}[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \wedge \forall \mathbf{u} \exists \mathbf{v} \in \mathbf{x}(\neg(\mathbf{u} \in I^*) \vee (\mathbf{v} \in I^* \wedge \neg^{I^*}\phi(\mathbf{v}, \mathbf{u})))].$$

Therefore $\mathbf{W} \models \exists \mathbf{x}[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \wedge \forall \mathbf{u} \in I^* \exists \mathbf{v} \in \mathbf{x} \cap I^* \neg^{I^*}\phi(\mathbf{v}, \mathbf{u})]$. Let c be a witness for the formula in square brackets. Since

$$\mathbf{NST} \vdash \forall \mathbf{x}({}^E Fin^f(\mathbf{x}) \rightarrow {}^I Fin^f(\mathbf{x})),$$

we have that $\mathbf{W} \models {}^I Fin^f(c)$. Let $d = c \cap I^*$. Then $d \in I^*$ and so

$$\mathbf{W} \models \exists \mathbf{x} \in I^*[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \wedge \forall \mathbf{u} \in I^* \exists \mathbf{v} \in \mathbf{x} \neg^{I^*}\phi(\mathbf{v}, \mathbf{u})].$$

Therefore $\mathbf{W}^* \models \exists \mathbf{x}[{}^E Fin^f(\mathbf{x}) \wedge \mathbf{x} \subseteq A \wedge \neg \exists \mathbf{u} \forall \mathbf{v} \in \mathbf{x}\ {}^I\phi(\mathbf{v}, \mathbf{u})]$: a contradiction.

Thus, by the *Axiom of Saturation* of **NST**, we get from 8 that

$$\mathbf{W} \models \exists \mathbf{v} \forall \mathbf{u} \in A\ {}^{I^*}\psi(\mathbf{v}, \mathbf{u})$$

and, by definition of ${}^{I^*}\psi(\mathbf{v}, \mathbf{u})$, $\mathbf{W}^* \models \exists \mathbf{v} \forall \mathbf{u} \in A\ {}^I\phi(\mathbf{v}, \mathbf{u})$, as requested. □

Theorem 3.2. **NRFST*** *is equiconsistent with* **ZFC**.

Proof. The previous Lemma and the discussion preceding it show that **NRFST*** is consistent relative to **ZFC**. For the converse, assume that **NRFST*** is consistent. Then $\mathbf{ZFC}^-$ is consistent (among the external axioms of **NRFST*** there are those of $\mathbf{ZFC}^-$) and, as is well known, the consistency of **ZFC** thus follows. □

One might ask whether it is possible to further enlarge **NRFST*** in order to include the axiom of restricted regularity and at the same time retaining the relative consistency with respect to **ZFC**. We show that it is not possible to proceed as done before.

Under the assumption of consistency of **ZFC**, let $\mathbf{W} = (W, E, S, I)$ be a model of **NST**. If we are looking for a model obtained from **W** in which the axiom of restricted regularity holds, then it is natural to drop from W all the elements not satisfying the restricted regularity (for instance the external set of W that is the collection of all internal non standard natural numbers). Moreover, since we want the negation of the axiom of infinity to hold in the internal universe, then we are forced to define S^* and I^* as it was done in Section 2, whilst the definition of the universe of the new model requires now some preliminary work. We let

$$I^* = \{x \in I : \ \mathbf{W} \models \forall y(y \in (Trcl(x) \cup \{x\}) \to^I (Fin^I(y)))\} \quad \text{and} \quad S^* = S \cap I^*.$$

From the point of view of **ZFC**, E is a set, so it does make sense to consider (in **ZFC**) the transitive closure $\overline{E}$ of E. For X in W, let

$$Tcl_E(X) = \{y \in W : \ \text{there exists } x \in X \text{ such that } (y, x) \in \overline{E}\}.$$

We define

$$W^* = \{X \in W : \forall Y \in Tcl_E(X) \cup \{X\} \\ \mathbf{W} \models (Y \neq \mathbf{0}) \wedge (Y \cap I^* = \mathbf{0}) \to \exists Z \in Y(Z \cap Y = \mathbf{0})\}.$$

Our candidate is now $\mathbf{W}^* = (W^*, E, S^*, I^*)$, where we denote in the same way the relation E on W and its restriction to W^*. $\mathbf{W}^*$ satisfies the axiom of restricted regularity (**N8**) by construction. Notice that $I \in W^*$ and also that $I^* \subseteq W^*$ as a consequence of the well-foundedness of I.

Unfortunately, *ERepl* (the axiom of replacement for external sets) does not hold in $\mathbf{W}^*$, as we are going to prove now.

Let α, β, γ be ranging over the internal ordinals of $\mathbf{W}$. Since $\mathbf{W}$ is a model of **NST**, we have

$$\mathbf{W} \models \forall\alpha\forall\beta\exists!\gamma\exists \mathbf{f} : \ (\alpha \times \{0\} \cup \beta \times \{1\}, \mathbf{R}) \xrightarrow{\cong} (\gamma, E), \tag{9}$$

where $\mathbf{R}$ is the lexicographic order on the pairs of $\alpha \times \{0\} \cup \beta \times \{1\}$ and γ plays the role of the ordinal $\alpha + \beta$ in $\mathbf{W}$.

Let $\omega \in W$ be the standard set containing the natural numbers and let $A = \omega \setminus S$ (read everything inside $\mathbf{W}$). We have that $A \in W^*$ since $Tcl_E(A) = \omega$ and since $\mathbf{W} \models \neg(A \cap I^* = \mathbf{0})$.

Consider now the formula $\mathbf{On}(X)$ stating that X is in I and that it satisfies the definition of ordinal relativized to I. Such a formula containing the parameter I can be interpreted in $\mathbf{W}^*$ since, as we have already noticed, $I \in W^*$. So $\mathbf{On}(X)$ allows us to recover inside $\mathbf{W}^*$ the internal ordinals of $\mathbf{W}$ that are merely external sets in $\mathbf{W}^*$ (i.e. the internal infinite ordinals of $\mathbf{W}$). We claim that

$$\mathbf{W}^* \models \forall N \in A\exists!C(\mathbf{On}(C) \wedge \exists F : \ (\omega \times \{0\} \cup N \times \{1\}, \mathbf{R}) \xrightarrow{\cong} (C, E)).$$

The existence of such a C follows from the well-foundedness of I which implies that α, β, γ, $\mathbf{R}$ and $\mathbf{f}$ in 9 are all in W^*. If C were not unique, we would have two internal distinct ordinals of $\mathbf{W}$ that are isomorphic: a contradiction.

Finally, if *ERepl* hold, then

$$\mathbf{W}^* \models \exists Y\forall Z(Z \in Y \leftrightarrow \exists N \in A(\mathbf{On}(Z) \wedge \exists F : \ (\omega \times \{0\} \cup N \times \{1\}, \mathbf{R}) \xrightarrow{\cong} (Z, E)).$$

Notice that $\mathbf{W}^* \models \neg(Y \cap I^* = \mathbf{0}) \wedge (Y \neq \mathbf{0})$. Unfortunately, Y is not well-founded since

$$\mathbf{W} \models \text{"}Y = \{\omega + \mathbf{n} : \mathbf{n} \in \omega \setminus S\}\text{"}.$$

Hence Y is not in W^* and the argument shows that *ERepl* does not hold in $\mathbf{W}^*$.

From Theorem 3.2 it follows that $\mathbf{NRFST}^*$ is not a conservative extension of **RFST** (otherwise one would have $con(\mathbf{ZFC}) \Leftrightarrow con(\mathbf{RFST})$). This might be interpreted as a symptom of excessive strength of the external axioms that we have assumed. Indeed, the plausibility of those axioms has not been sufficiently motivated in relation to the notion of infinity that we have proposed. Thus the next step would be to investigate which external axioms can be reasonably included in our theory.

4. Beginning Mathematics

If, on one hand, our goal in introducing the theory $\mathbf{NRFST}^*$ is to have a theory of sets with a new notion of infinity, on the other hand such a theory should be able to support existing mathematics.

Starting from the standard and internal natural numbers one can easily define the internal integers as ordered pairs of internal natural numbers having at least one zero component. Among them, the standard integers are those corresponding to standard pairs.

We remark that the usual construction of the integers as equivalence classes would produce external sets in this context, thus we resort to selecting a peculiar representative for each equivalence class.

Operations on internal integers can be defined in order to have a structure of commutative ring in which the natural numbers are embedded.

Similarly, one can introduce the internal rationals as ordered pairs of internal integers whose second component is greater than zero and whose components are relatively prime. The standard rational numbers are those corresponding to standard pairs.

Here again, we do not consider equivalence classes for the same reason remarked above.

Operations on internal rationals can be defined in order to have a structure of field in which the integers are embedded.

In nonstandard analysis the internal rationals are called *hyperrationals*, whilst the name *rationals* is deserved for the standard ones.

We notice that, given any two nonempty external sets A and B of rationals such that $a \leq b$ for all $a \in A$ and all $b \in B$, then there exists a hyperrational $\mathbf{c}$ such that $a \leq \mathbf{c} \leq b$ for all $a \in A$ and all $b \in B$. The proof of this fact is an application of the axiom *Sat* and it rests on the fact that any nonempty external finite set of rationals is standard (by Theorem 1.3), so it has a greatest element.

Furthermore, if we call *infinitesimals* the hyperrationals that are in absolute value smaller than the inverse of any positive standard natural number, the infinitesimals form an ideal I in the ring F of *finite* hyperrationals, where by *finite* hyperrational we mean any hyperrational that is bound in absolute value by some standard natural number.

The ideal I is maximal in the ring F, hence the quotient F/I is a field that can be proved to be ordered and complete (completeness follows from what has been noticed above), and thus it is isomorphic to the usual real numbers.

However, F/I is an external set, but we would rather use internal elements, and the idea is that the elements of F/I can be replaced in every calculation by adequate hyperrationals approximations of them, without affecting the final result – up to infinitesimals – if the approximations are changed.

Along the same line, we would replace real functions (which are external objects in **NRFST***) by internal elements. We propose to use internal functions among hyperrationals. Indeed, if there is a way to approximate a real function by a standard sequence of standard functions among rationals, applying *Sat* we should get an internal function among hyperrationals that should be as useful as the given real function.

5. Conclusion

While the axioms for memberships, standard and internal sets are related to the notion of infinity that we are proposing, the external axioms were introduced in order to compare **NRFST** with **ZF**.

From the results obtained in the second part of this paper,it follows that **NRFST*** is equiconsistent with **ZF**. Thus the notions that we have introduced are no more foundationally demanding than the usual set theoretic notions. However, one might still try to conceive different axioms for external sets, whose consistency be strictly less demanding than that of **ZF**, and, at the same time, being strong enough to offer a base for the development of modern mathematics. We also feel that the attitude that we are proposing can provide a good environment where to search for the origins of the classical critical aspects of **ZF**. All these goals will be pursued in future researches.

Appendix

In this section we give a detailed account of the heuristic motivations lying behind the formal theory that we have proposed. Even if the content of this appendix is not needed in order to understand the technical developments, nonetheless it gives some insights that may better motivate our work.

A small overlapping with the *Introduction* cannot be avoided.

Once accepted that the collection of the natural numbers is not a set but a proper class (since otherwise we would be assuming that an endless process has to come to an end yielding a single object, i.e. a set), one is led to a more careful analysis of the notion of natural number.

The usual intuition that the natural numbers are exactly those entities obtained starting from zero and repeating the operation of adding one needs to be better specified. For instance, think of the way of defining the order relation between two nonzero natural numbers m and n: we have to inspect the chains $\mathcal{C}_1$ and $\mathcal{C}_2$ of predecessors of m and n, respectively. So doing, we assume of being able to recall the entire process of repeatedly adding one to obtain each of the two numbers.

This "recalling" aspect of the notion of natural number allows us to say that m is smaller than n if, when n is constructed, one has memory (he recalls) that, during the process of construction, m was already constructed.

We claim that this recalling is a fundamental feature of the notion of natural number, even though we do not know of anyone stressing this aspect. Indeed, there is a general attitude that one should recall the whole process of construction of every natural number. If it is so, there is no point in underlining the recalling feature of the notion of a natural number.

The attitude of recalling everything is a clear consequence of an illuministic standpoint and we see no reason for accepting it. For instance, with reference to the everyday experience, we see that it is very difficult to keep track of very long computations, even with the help of computers, and a lot of techniques have been developed to deal with this problem. Our position relies on the following stand-point: there is an agent – an observer – who knows of being limited in his capability of recalling, although he cannot determine where his limits are; also he believes that there is something beyond his limits.

So we will admit the existence of natural numbers, i.e. things obtained from zero by repeatedly adding one, for which the operation of adding one has been repeated so many times that it is impossible to recall simultaneously all of the applications of that operation.

The two features of the notion of natural number, namely adding one and recalling the steps done, are independent in the sense that, even though one may not recall the entire process that led to a certain number, one can still conceive of the possibility of adding one once more.

The natural numbers of which we do not recall the entire process of construction can be considered unreachable. From the point of view that we are proposing they may be regarded as infinite, and finite will be the reachable natural numbers.

So we get another notion of infinity.

So far we have considered the natural numbers because we are critical of the notion of infinity of **ZF**, based on the acceptance of the collection of natural numbers as forming a set, but we should extend our consideration to any set. As usual, sets are thought of as collections that are also viewed as single elements and can thus become subject of further mental operations or attributes. Collections themselves are regarded as the mental act of collecting some elements (possibly none).

In our previous paper [BF93], we already stated which collections we accept to be considered as sets: those that are bijective with a (unique) natural number. By means of such a bijection the elements of a set can be indexed by the natural numbers preceding the number which is the range of the bijection, and the set itself can be viewed as obtained through a constructing process of adding the next indexed element until the indices are exhausted.

Now that we are considering finite (reachable) and infinite (unreachable) natural numbers, we can also divide the sets in finite and infinite ones, according to whether the natural number to which a set is bijective is finite or infinite, or, equivalentiy, to whether the process of adding the next indexed element is finite or infinite in our sense.

This notion takes into account the role of the observer: he regards a set as infinite if he is not able to recall its entire construction.

But then, how to distinguish between finite and infinite sets?

For instance, with respect to a fixed computer, one might define the numbers that can be recalled as those less or equal to the largest number that the machine can handle; however such numbers depend on the machine. But we do not want to commit ourselves to the choice of a specific observer, so we want to keep undefined and open to any reasonable interpretation the notion that permits to perform the distinction between finite and infinite sets.

Thus, in the set theory that we propose there is, in addition to the usual membership predicate, an undefined unary predicate, *standard*, that will be denoted by st and whose intended interpretation is that it holds on the elements that are reachable, finite. We could have chosen the name *finite* instead of *standard* for this predicate, but then we should always specify "finite in our sense". Furthermore, the choice of the name *standard* is related to the homonymous predicate of non standard analysis and non standard set theory to which we are proposing to give the meaning of finite in our sense.

As already remarked, the interpretation of the predicate standard does not involve just the observed world (i.e. the environment described by the set theory introduced in [BF93]), but it is a relationship between the observer and the observed world. Somehow, it singles out the elements of the observed world that the observer can know as they actually are.

All the elements of the observed world form the *internal universe* and, among them, those that are too large to be grasped by the observer will be called *non standard*.

We will assume that the observer has a vague notion of the non standard elements of the observed world, and, from the point of view of the set theory that we propose, we want to be able to consider these vague notions as well. To be more precise, since we are going to work inside a set theory, we want to be able to deal with *extensions of vague notions*. So, in the sequel we will replace "vague notions" by "extensions of vague notions". Thus we need a further undefined unary predicate, *internal*, denoted by in, that will distinguish between the objects of the observed world and the extensions of vague notions associated to them.

Of course, standard elements are internal, and the extensions of the vague notions related to them coincides with what they actually are, but the situation is different for non standard elements.

Consider, for instance, two different infinite natural numbers, i.e. two sets obtained by repeating the process of adding one - starting from zero - too many times to be recalled. The observer will only notice that the standard natural numbers belong to both sets and that no other standard element belongs to either set, and might have the same vague notion of both of them.

Furthermore, in order to compare **NRFST** with **ZF**, for the moment being, we want the possibility of fully operating with these non internal sets, so we require the non internal elements to be "closed" under all the usual set-theoretic operations.

We will call *external* all the objects that we will deal with, whether they are obtained from the observed world or from a vague perception of the observed world or by an operation on external sets previously obtained.

References

[BF93] Stefano Baratella and Ruggero Ferro, *A theory of sets with the negation of the axiom of infinity*, Mathematical Logic Quarterly **39** (1993), 338–352.

[CK] C.C. Chang and H.J. Keisler, *Model Theory*, North-Holland, Amsterdam.

[CW83] C.-T. Chong and M.J. Wicks (eds.), *Southeast Asian Conference on Logic*, North-Holland, 1983.

[Kaw83] T. Kawai, *Nonstandard analysis by axiomatic method*, In Chong and Wicks [CW83], pp. 55–76.

[Nel77] E. Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bull. Am. Math. Soc. **83** (1977).

(S. Baratella) *Dipartimento di Matematica, Università degli Studi di Trento, Via Sommarive, I-38050 Povo TN (Italy)*

E-mail address: baratella@ itnvax.science.unitn.it

(R. Ferro) *Dipartimento di Matematica, Università degli Studi di Lecce, Via per Arnesano, I-63100 Lecce (Italy)*

E-mail address: ferro@ingle01.unile.it