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ON THE EXISTENCE OF FINITELY DETERMINATE MODELS FOR SOME THEORIES IN STATIONARY LOGIC

by

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The following is a write up of series of talks that the author gave at the Siena Logic Seminar in April 1982. The material is part of the work that the author carried out as a graduate student at Stanford University under the supervision of Prof.S.Feferman. Many thanks are due to him for his guidance. Also, we would like to thank the Siena logicians, especially A. Ursini, C. Bernardi and F. Montagna, for inviting us at Siena and for their interest in the subject. Finally, we are grateful to L. Harrington, M. Kaufman and A. Mekler for helpful talks on stationary Logic.

Let $P_{<}(A)$ be the set of subsets of A of cardinality less than that of A. As shown by Kueker and Jech, there exists a notion of stationary set on $P_{<}(A)$.

Shelah, Barwise, Kaufman, Makkai have developped a logic with an extra quantifier where one can express the fact that a formula holds true on a stationary set. A model is finitely determinate if it is not possible to find a formula φ such that both φ and $\neg\varphi$ holds true on a stationary set (this definition is due to M.Kaufman). The following investigates the existence of finitely determinate models for some theories in stationary Logic.

More precisely,

an aa_1-theory is one whose every axiom says in effect that a first order formula $\theta(S_o,\ldots,S_{n-1})$ holds true on a closed unbounded set. The main result of this paper shows that if an aa_1-theory has models of large cardinality, then it has a finitely determinate one (theorem 2.3.2).

A substantial part of the material presented here is introductory. §§1.2,1.3 are independent of the rest of this paper and are only supposed to give the reader who is not familiar with the subject a feel for stationary sets on $P_<(A)$. It is organized as follows. Part I deals with combinatorics on $P_<(A)$. It ends up with a technical fact needed for the next part. Part 2 contains a short introduction to stationary logic and finite determinacy along with the above-mentioned existence theorem. Part 3 introduces a notion of Skolem ultrapower for finitely determinate weak models and puts it to use in order to derive some Hanf number results.

I. SOME COMBINATORICS ON $P_<(A)$.

1.1. Basic definitions and facts. (Jech, Kueker)

We assume throughout this paper that A is a set whose cardinal number k is uncountable and regular.

$P_<(A)$ is the set of subsets of A whose cardinality is $< k$. A set $C \subseteq P_<(A)$ is closed unbounded iff $\forall a \in P_<(A)\ \exists a' \in P_<(A)$ such that $a \subseteq a' \in C$ and C is closed under unions indexed by a cardinal $< k$. The closed unbounded sets constitute a filter basis. $D_<(A)$, the closed unbounded filter is the filter generated by this basis. $X \subseteq P_<(A)$ is stationary iff $P_<(A) - X \notin D_<(A)$. Let $X \subseteq P_<(A)$ and $f : X \to A$; f is regressive iff $\forall a \in X\ f(a) \in a$. Let $(Z_x)_{x \in A}$ be a family of subsets of $P_<(A)$. ΔZ_x, the diagonal intersection of (Z_x), is the set of $a \in P_<(A)$ such that $\forall x \in a\ \ a \in Z_x$.

facts.

(a) $D_<(A)$ is non principal, i.e. every set of the form $\{S \in P_<(A) : a \subseteq S\}$ $(a \in P_<(A))$ is in $D_<(a)$.

(b) $D_<(A)$ is k-complete, i.e. it is closed under intersections indexed by a cardinal $< k$. In fact,

(c) $D_<(A)$ is normal, i.e. every regressive function on a stationary set is constant on a stationary set.

(d) $D_<(A)$ is not an ultra filter, i.e. there exists a stationary subset which is not closed unbounded.

(In fact, Solovay proved a result to the effect that $D_<(A)$ is not even k-saturated).

1.2. Stationary sets on $P_<(A)$ and on k (Folklore)

Let $\mathcal{D}(k)$ be the filter generated by the closed unbounded subsets of k in the usual sense, i.e. cofinal in k and such that whenever $\xi < k$ and $C \cap \xi$ is cofinal in ξ, $\xi \in C$. $X \subseteq k$ is stationary if $k - X$ is not closed unbounded. We shall investigate the relationship between $D_<(A)$ and $\mathcal{D}(k)$.

A filtration of A is a family $(a_\xi)_{\xi<k}$ of elements of $P_<(A)$ such that (a_ξ) is increasing, $a_\alpha = \bigcup_{\xi<\alpha} a_\xi$ for every limit ordinal $\alpha < k$ and $A = \bigcup_{\xi<k} a_\xi$.

Now, pick any filtration (a_ξ) of A. The map

$$\xi \in k \mapsto a_\xi \in P_<(A)$$

induces a dual map

$$X \in P\,P_<(A) \mapsto F(X) = \{\xi < k : a_\xi \in X\} \in P(k)$$

Moreover, the map

$$\text{class of } X \in P\,P_<(A) \text{ modulo } D_<(A) \mapsto \text{class of } F(X) \in P(k) \text{ modulo } \mathcal{D}(k)$$

is well defined, does not depend on the choice of (a_ξ) and defines an isomorphism between the boolean algebras $P\,P_<(A)/D_<(A)$ and $P(k)/\mathcal{D}(k)$.

Let $X \subseteq P_<(A)$. Define the ordinal content $\tilde{X}$ of X to be the class of $F(X)$ modulo $\mathcal{D}(k)$.

Theorem (Solovay).

The cardinal number of $P(k)/_{\mathcal{D}(k)}$ is 2^k.

1.3 Some typical closed unbounded and stationary sets.

1.3.1 Example. (Lowenheim, Skolem, Tarski, ..., Kueker).
Let $\mathfrak{A} = \langle A, \ldots \rangle$ be a first order structure for a countable language. Let $X = \{a \in P_<(A) : \mathfrak{A}$ restricted to a is an elementary substructure of $\mathfrak{A}\}$. Then X is closed unbounded.

1.3. 2. Example (Conway). Let A be a dense, k-like, linear ordering (an ordering is k-like if has cardinality k but all its bounded initial segments have cardinality $< k$). Let X(A) be the set of $a \in P_<(A)$ which have a lowest upper bound. It can easily be shown that for each $[S] \in P(k)/_{\mathcal{D}(k)}$ there exists a dense k-like linear ordering A such that $\widetilde{X(A)} = [S]$. Therefore, by theorem 1.2, there are 2^k such orderings (Remember that $k > \omega$).

For example, if $k = \omega_1$ and

$A = \mathbb{Q} + \mathbb{Q} + \ldots$ (ω_1 times)

$B = \mathbb{Q} + (1+\mathbb{Q}) + (1+\mathbb{Q}) + \ldots$ (ω_1 times)

then, by picking the right filtration, the reader can convince himself that $\tilde{X}(A)=0$ and $\tilde{X}(B)=1$. So A and B are not isomorphic. For more on the subject, see Eklof-Mekler [1979], § 7.

1.3.3. Example. Suppose $k=\omega_1$ and let A be an abelian group. Also, assume that A is an ω_1-group, i.e. that every countable subgroup of A is free. Notice that the set of countable subgroups is closed unbounded so that we can identify it with $P_<(A)$.
Now, let Y(A) be the set of $a \in P_<(A)$ such that $\exists b \in P_<(A)$: $a \subseteq b$ and b/a is not free. It is easy to see that A is free iff $\tilde{Y}(A)=0$. Eklof has shown that for every $[S] \in P(\omega_1)/\mathcal{D}(\omega_1)$ there exists an ω_1-group A such that $[S]=\tilde{Y}(A)$. As above, it follows that are 2^{ω_1} non isomorphic ω_1-group of cardinality ω_1.
For more information on the subject the reader is referred to the original paper (Eklof [1975]).

1.4. n-ineffability.

We shall now introduce a large cardinal property and prove a statement about it which we shall need later on.

Let $a,b \in P_<(A)$. We shall write $a < b$ for $a \in P_<(b)$ and $a \leq b$ for $a \in P_<(b) \cup \{b\}$. Also, if $X \subseteq P_<(A)$, $[X]^n$ will be the set of sequences $a = \langle a_0,\ldots,a_{n-1}\rangle$ such that $a_i \in X$ $(0 \leq i < n)$ and $a_i < a_{i+1}$ $(0 \leq i < n-1)$.

1.4.1. Definitions (Baumgartner). Let $X \subseteq P_<(A)$ and $n \geq 1$. An (X,n) - sequence is a family (S_a), where $a = \langle a_0,\ldots,a_{n-1}\rangle$ ranges over $[X]^n$, such that, for every such a, $S_a \subseteq a_0$.
A set $H \subseteq X$ is homogeneous for (S_a) if for any $a = \langle a_0,\ldots,a_{n-1}\rangle$ and $b=\langle b_0,\ldots,b_{n-1}\rangle$ in $[H]^n$

$$a_0 \leq b_0 \Rightarrow S_b \cap a_0 = S_a$$

This amounts to saying that if $S=\bigcup S_a$, where a ranges over H, then $S_a = S \cap a_0$.
A set X is n-ineffable iff every (X,n)-sequence admits a stationary homogeneous set. k is an n-ineffable cardinal iff $P_<(A)$ is an n-ineffable set.
In Baumgastner [1973] it is shown that

$$\text{measurable} \underset{\not\Leftarrow}{\Rightarrow} n{+}1 - \text{ineffable} \underset{\not\Leftarrow}{\Rightarrow} n\text{-ineffable} \underset{\not\Leftarrow}{\Rightarrow} \text{weakly compact.}$$

1.4.2. Definition. (Abramson, Harrington, Kleinberg, Swicker).

Let X be a set and $(Z_x)_{x\in A}$ be a family of subsets of X. A flip of (Z_x) is a family (Z'_x) such that $\forall x\in A$, Z'_x is either Z_x or $X-Z_x$.

1.4.3. Definition. Let $n \geqslant 1$ and (Z_x) be a family of subsets of $[P_<(A)]^n$. $Z=\Delta Z_x$, the diagonal intersection of (Z_x), is the set of $a=\langle a_0,\ldots,a_{n-1}\rangle \in [P_<(A)]^n$ such that

$$\forall x\in a_0 \quad a\in Z_x$$

The following result is proved in Abramson et al. [1977] for the case n = 1.

1.4.4. Proposition. Let k be n-ineffable. Then for any family $(Z_x)_{x\in A}$ of subsets of $X=[P_<(A)]^n$ there exist a flip (Z'_x) of (Z_x) and a stationary set $H\subseteq P_<(A)$ such that $[H]^n\subseteq \Delta Z'_x$.

Proof. For $a=\langle a_0,\ldots,a_{n-1}\rangle\in[P_<(A)]^n$, put

$$S_a=\{x\in a_0 : a\in Z_x\}$$

Since $P_<(A)$ is n-ineffable, there exists a stationary set which is homogeneous for (S_a). Let $S=\bigcup S_a$ where a ranges over H (so $S\cap a_0 = S_a$ whenever $a\in H$). Now put

$$Z'_x = Z_x \quad \text{if} \quad x\in S$$
$$= X - Z'_x \qquad \text{otherwise.}$$

It remains to show that H and (Z'_x) are as desired. Let $a=\langle a_0,\ldots,a_{n-1}\rangle\in[H]^n$ and $x\in a_0$.

Show $a\in Z'_x$.

2 cases may arise:

case 1: $Z'_x=Z_x$. Then $x\in S$ so $x\in S_{a_0}$. Therefore $a\in Z_x$.

case 2: $Z'_x=X-Z_x$. Then $x\notin S$ so $x\notin S_{a_0}$. Therefore $a\notin Z_x$.

This finishes the proof.

2. STATIONARY LOGIC AND THE EXISTENCE OF FINITELY DETERMINATE MODELS.

2.1. Stationary logic and natural models.

So far, we have assumed that the cardinal number k of A is regular and uncountable. Now, we drop the condition that k be uncountable. Instead, for $k\leqslant\omega$ we put $P_<(A) = P_<(A\cup\omega_1)$ and $D_<(A)=D_<(A\cup\omega_1)$. This is a way of saying that we do not intend to deal seriously with the situation.

Let L be a first order language. We enlarge it by adding second order variables s, s', ... t, t'... which can occur only in contexts of the form $s(\tau)$ where τ is term. Also, we add a new quantifier aas (read: for almost all sets s).

The language so obtained is called L^{aa}.

Stat s (read: for stationary many sets s) is defined to be the dual quantifier ¬aas¬ . Given a formula φ of L^{aa} whose free variables are among $\vec{s},\vec{v}$, the closure of φ (in L^{aa}) is aas$\vec{s}\,\forall\vec{v}\,\varphi$ (Warning: $\forall\vec{v}$ aas$\vec{s}\,\varphi$ will not be equivalent in our logic and would not do the job).

A natural model for L^{aa} is a first order structure $\mathfrak{A} = \langle A, \ldots \rangle$ for L where it is understood that the satisfaction relation obeys the extra rules $((\vec{a}, b \in P_{<}(A), x \in A)$:

$$\mathfrak{A} \models b(\tau(\vec{a},\vec{x})) \text{ iff } \tau^{\mathfrak{A}}(\vec{a},\vec{x}) \in b$$

$$\mathfrak{A} \models \text{aas}\,\varphi(s,\vec{a},\vec{x}) \text{ iff } \{b \in P_{<}(A) : \mathfrak{A} \models (b,\vec{a},\vec{x})\} \in D_{<}(A)$$

Notice that it follows from our conventions that the cardinal of a natural model is regular.

Clearly, the following formulae are valid (i.e. their closure is true) in every natural model

aas $(\varphi(s) \to \psi(s)) \wedge$ aas $\varphi(s) \to$ aas $\psi(s)$

$\forall v$ aas $s(v)$

$\forall v$ aas $\varphi(s,v) \to$ aas $\forall v \in s\ \varphi(s,v)$

They say that $D_{<}(A)$ is a filter, that it is not principal and it is closed under diagonal intersection. The closure of these formulae constitute the axioms for stationary logic. Stationary logic is obtained from first order logic by adding these axioms and the new rule:

from $\varphi \to \psi(s)$, infer $\varphi \to$ aas $\psi(s)$ (s not free in φ).
Define Q$v\,\varphi(v)$ to be ¬aas $\forall v(\varphi(v) \to s(v))$. Q is the Chang Quantifier in the sense that, if $\mathfrak{A}$ is an uncountable model,

$$\mathfrak{A} \models Qv\ \varphi(v) \text{ iff the cardinality of } \varphi^{\mathfrak{A}} \text{ is } k.$$

Completeness theorem (Barwise, Kaufman, Makkai). Let T be a countable consistent theory in L^{aa}. Then T has a natural model of cardinality $\leq \omega_1$.

Clearly this natural model is uncountable iff it satisfies the sentence $Qv(v=v)$.

2.2. Finite determinacy.

2.2.1 Definition (M. Kaufman). A natural model $\mathfrak{A}$ is finitely determinate if it satisfies the schema of determinacy, i.e. the closure of all the formulae of the form aas $\varphi \vee$ aas¬ φ (φ being in L^{aa}).

2.2.2. Example. Let V be an uncountable vector space over a countable field K. We think of it as a structure of the form $\langle V, +, \lambda \rangle_{\lambda \in K}$ where each λ is a unary function symbol. The techniques of Eklof-Mekler[1979]show that V is finitely determinate.

2.2.3. Example. For each limit ordinal $\alpha < \omega_1$, let $(\alpha_n)_{n<\omega}$ be a sequence of ordinals $<\alpha$ such that

$\sup \alpha_n = \alpha$. Let R be

$\{\langle \alpha, n, \alpha_n \rangle : n < \omega,\ \alpha$ is a limit ordinal$\}$ and

$\mathfrak{A} = \langle \omega_1, \omega, <, R \rangle$.

$\mathfrak{A}$ is not finitely determinate. To see this, pick $N < \omega$ such that the regressive function $\alpha \mapsto \alpha_N$ (α limit) is not almost bounded i.e. such that

$\forall \beta \quad \{\alpha$ limit : $\alpha_N \geq \beta\}$ is stationary

(or look at any proof that $\mathcal{D}(\omega_1)$ is not an ultrafilter).

Make $\beta = 0$. Since $\mathcal{D}(\omega_1)$ is normal, this provides a stationary set X_o and an ordinal ξ_o such that $\forall \alpha \in X_o \ \alpha_N = \xi_o$. Now, make $\beta = \xi_o + 1$. This yields a stationary set X_1 and an ordinal ξ_1 such that $\forall \alpha \in X_1 \ \alpha_N = \xi_1$ and $\xi_o \neq \xi_1$. From this (remember § 1.2) it follows that

$\mathfrak{A} \models$ stat s R(sup s, N, ξ_o)$\wedge$ stat s $\neg$ R(sup s, N, ξ_o). This proves the desired result.

2.2.4. <u>Example</u>. Let A be a dense k-like linear ordering. Then :

If X(A) (as in 1.3.2) is 0 or 1, A is finitely determinate.

If X(A) $\neq$ 0,1, then A $\models$stat s (sup s exists)$\wedge$ stat s$\neg$(sup s exists). Therefore A is not finitely determinate.

2.2.5. <u>Proposition</u>. The schema of determinacy is equivalent to the family of Schemas

Det$_n$: closure of aa$\vec{s}\vartheta \vee$ aa$\vec{s}\neg\vartheta$, where $\vec{s}$ is a sequence of lenght n and ϑ is a first order formula (i.e. does not contain any occurrence of aa or stat)

Sketch of proof. By induction on n , we show that

Det$_{n-1} \Rightarrow$ finite determinacy for formulae having at most n occurrences of aa (or stat).

<u>Conclusion</u>. Canonical mathematical structures, like the field of real numbers, are always finitely determinate. The construction of a non finitely determinate structure , like that of a stationary set which is not closed unbounded, requires the use of the uncountable axiom of choice. The resulting structures are just as pathological as a set of reals which is not Lebesgue measurable.

2.3 <u>On the existence of finitely determinate models</u>.

2.3.1. <u>Definition</u>. A theory is called an <u>aa$_1$ theory</u> iff all its axioms are of the form aa$\vec{s}$ θ, where θ is a first order formula. A formula of this form is called an <u>aa$_1$ formula</u>.

Problem : when does an aa$_1$ theory have a finitely determinate model?

Eklof and Mekler [1979] prove that every first order theory has a finitely determinate model. Shelah (unpublished) recently showed that every theory in L(Q) has a finitely determinate model, thus answering a question of the above mentioned article. Notice that a theory in L(Q) is equivalent to an aa_1 theory.

On the other hand, if $\mathfrak{A} = \{\omega_1, \omega, <, R\}$ is as in 2.2.3, the reader can check that we have actually shown that the first order theory of $\mathfrak{A} \cup \{aas\ (\sup s \text{ exists})\}$ has no finitely determinate model. Of course, this is an aa_1 theory.

2.3.2. Theorem. Let T be a countable aa_1 theory. If for every n, T has a natural model whose cardinality is n-ineffable, then T has a finitely determinate (natural) model.

Proof of the theorem. We show that for every n, $T \cup Det_n$ is consistent. By the Completeness Theorem and 2.2.5, this will be enough.

Let $n \geq 1$ be given and $\mathfrak{A}$ be an n-ineffable natural model. Let $(Z_x)_{x \in A}$ be a list of all the sets of the form

$$\{ b \in [P_<(A)]^n : \mathfrak{A} \models \vartheta(b, \vec{a}, \vec{x}) \}$$

where ϑ is a first order formula, $\vec{a} \in P_<(A)$ and $\vec{x} \in A$. This is possible since k, the cardinal number of A, is n-ineffable and so is strongly inaccessible whence $P_<(A)$ and A have the same cardinality. Notice that the family (Z_x) is not supposed to be uniform.

By proposition 1.1.4, there exists a flip (Z'_x) of (Z_x) and a stationary set $H \subseteq P_<(A)$ such that $[H]^n \subseteq \Delta Z'_x$. Now, define a new "model" $\mathfrak{B}$ by keeping the same first order structure as in $\mathfrak{A}$ but changing the satisfaction rule for second order formulae into

$\mathfrak{B} \models aas\ \varphi(s, \vec{a}, \vec{x})$ iff there exists a closed unbounded set C such that:

$$C \cap H \subseteq \{ b \in P_<(A) : \mathfrak{A} \models \varphi(b, \vec{a}, \vec{x}) \}.$$

Now,

fact a). $\mathfrak{B}$ satisfies the axioms for stationary logic since H is stationary. Hence, it will do the job for a consistency proof.

fact b). $\mathfrak{A} \models$ T. In fact every aa_1-statement (with parameters from $A \cup P_<(A)$) which is true in $\mathfrak{A}$ remains so in $\mathfrak{B}$.

fact c). $\mathfrak{A} \models Det_n$. To see this, it is enough to find for each Z'_x a closed unbounded set C_x such that

$[C_x \cap H]^n \subseteq Z'_x$. Put $C_x = \{a \in P_<(A) : x \in a\}$.

Then

$$b = \langle b_0, \dots, b_{n-1} \rangle \in [H \cap C_x]^n \Rightarrow b \in \Delta Z'_x$$
$$\Rightarrow \forall y \in b_0 \quad b \in Z'_x$$
$$\Rightarrow b \in Z'_x \text{ since } x \in b_0$$

This finishes the proof.

3. SOME HANF NUMBERS

3.1 Weak models.

Now, we allow the language L^{aa} to be uncountable. Also, we drop the requirement that L^{aa} be derived from a first order language L^{aa}. Instead we allow primitive function symbols to occur in contexts of the form f(s), this being considered a first order term. Now let $\mathfrak{A} = \langle \mathfrak{A}', a_i, f_j, \models \rangle_{i \in I, j \in J}$ be a structure where $\mathfrak{A}' = \langle A, \dots \rangle$ is a first order structure, each a_i is a subset of A, each f_j is a function on $\{a_i : i \in I\}$ (possibly empty) into A (we need a magic convention to the effect that even in this case $f_j \neq f_{j'}$ whenever $j \neq j'$) and $\models$ is an extension of the usual first order satisfaction relation to formulae from L^{aa}. The first order parameters of $\mathfrak{A}$

are the $x \in A$. The second order parameters are the a_i's .

$\mathfrak{A}$ is a weak model, if beside the obvions rules

$$\mathfrak{A} \models f_j(a_i) = x \quad \text{iff} \quad f_j(a_i) = x$$
$$\mathfrak{A} \models a_i(x) \quad \text{iff} \quad x \in a_i$$

it obeys the crucial ones

$$\mathfrak{A} \models \neg Qv\, a_i(v) \quad \text{for each } i$$

$\mathfrak{A} \models \varphi$ where φ is an instance of an axiom of stationary logic with parameters.

Example: if $\mathfrak{B}$ is as in the proof of 2.3.2 then $\langle \mathfrak{B}, a \rangle_{a \in P_<(A)}$ is a weak medel.

A weak model is tidy if for every $\varphi = \varphi(\vec{s}, \vec{v}, t, w)$ in L^{aa} (without parameters), there exists a function symbol $f_{aat \exists w \varphi}(\vec{s}, \vec{v}, t)$ s.t. the following is satisfied

$$aa\vec{s}\ \forall \vec{v}\ (\ aat\ \exists w \varphi(\vec{s}, \vec{v}, t, w) \rightarrow aat\ \varphi(\vec{s}, \vec{v}, t, f_{aat \exists w \varphi}(\vec{s}, \vec{v}, t)).$$

3.1.1. Definitions. A finitely determinate weak model is one which satisfies the schema of determinacy. Let $\mathfrak{A}$ be a finitely determinate weak model which is tidy. Ult $(\mathfrak{A}) = \langle \text{Ult}\ (A), \dots, \underline{A}, \dots \rangle$ is defined as follows

(a) universe: Let U be the set of function symbols $f = f_{aas \exists v \varphi}$. Put $f \sim g$ iff

$\mathfrak{A} \models$ aat $f(t) = g(t)$. Ult (A) is the set of equivalence classes $[f]$ modulo $\sim$, where $f \in \mathcal{U}$. In what follows, we shall indentify $[f]$ and f.

(b) first order structure: it is given by the rules

Ult$(\mathfrak{A}) \models R(f_o,..,f_{n-1})$ iff $\mathfrak{A} \models$aat $R(f_o(t), ..., f_{n-1}(t))$;

Ult $(\mathfrak{A}) \models \underline{A}$ (f) iff $f \sim f_x$ for some $x \in A$, f_x being $f_{\text{aat } \exists v(v = x)}$.

(c) Second order structure: it is given by the rule

Ult $(\mathfrak{A}) \models$ aas $\varphi(A,s,f_o,...,f_{n-1})$ iff $\mathfrak{A} \models$aat aas $\varphi(t,s,f_o(t),..,f_{n-1}(t))$.

3.1.2. <u>The fundamental theorem on Ult</u>. Let $\mathfrak{A}$ be a tidy, finitely determinate weak model. Then

(a) For any formula $\varphi(t,v_o,...,v_{n-1})$ Ult $(\mathfrak{A}) \models \varphi(\underline{A},f_o,...,f_{n-1})$ iff $\mathfrak{A} \models$ aat $\varphi(t,f_o(t),...,f_{n-1}(t))$.

(b) The canonical embedding $x \mapsto f_x$ is L^{aa}--elementary and so we can identify $\underline{A}^{\text{Ult}(\mathfrak{A})} = \{f_x : x \in A\}$ and A.

From this, it follows immediately that

(c) A is a second order parameter of $\mathfrak{A}$, i.e.

$\mathfrak{A} \models \neg Qv\ \underline{A}(v)$

(d) Ult $(\mathfrak{A})$ is a <u>blunt extention</u> of $\mathfrak{A}$, i.e. for every φ with parameters from $\mathfrak{A}$

Ult $(\mathfrak{A}) \models \varphi(A) \Rightarrow \mathfrak{A} \models$aat $\varphi(t)$.

The proof is easy and is left to the reader.

3.2. <u>The Hanf number calculation</u>.

3.2.1. <u>Theorem 2.3.2 strengthned</u>. In theorem 2.3.2, we can require that the determinate model be tidy.

Proof. Let $\mathfrak{A}$ be as in the proof of 2.3.2. Construct $\mathfrak{A}_o$ by adding for each formula $\varphi(s,v,t,w)$ without parameters where all the free variables are displayed, a function $F = F_{\text{aat}\exists w\varphi}$ such that

$\forall \vec{a}, b \in P_<(A)\ \forall \vec{x} \in A(\mathfrak{A} \models \exists w\varphi(\vec{a},x,b,w) \rightarrow \varphi(\vec{a},x,b,F(\vec{a},x,b)))$.

Let L_o^{aa} be the language of $\mathfrak{A}_o$. Iterate this operation ω times and let $\mathfrak{A}_\infty$ be the limit structure and L_∞^{aa} be the corresponding language. Clearly, if $\mathfrak{B}_\infty$ is to $\mathfrak{A}_\infty$ as $\mathfrak{B}$ is to $\mathfrak{A}$, then

$\mathfrak{B}_\infty \models T \cup \text{Det}_n \mathcal{U}$ axioms for tidyness.

This completes the proof.

Now, we can use our previous results to compute some Hanf numbers. 3.2.2 generalizes a theorem in Schmerl [1976].

3.2.2. Theorem. Let T be a countable aa_1-theory. If for every n, T has a natural model whose cardinal is n-ineffable, then T has a finitely determinate natural model in every uncountable regular cardinality.

Call a theory T finitely determinate iff $T \vdash$ schema of determinacy. Clearly, a finitely determinate theory is equivalent to an aa_1-theory, whence

3.2.3. Corollary. Let T be finitely determinate and countable. If for every n, T has a natural model which is n-ineffable, then T has a natural model in every uncountable regular cardinality.

Proof of 3.2.2. Let T be as in the assumption and k be a regular uncountable cardinal. Since weak models are essentially first order objects, the Lowenhein Skolem Theorem holds for them. Therefore, by 2.3.2, T has a countable, tidy, finitely determinate weak model $\bar{\mathfrak{A}}$. By iterating the Ult operation k times (this means that we take direct limits at limit stages) we obtain a sequence $(\bar{\mathfrak{A}}^{(\xi)})_{\xi<k}$ whose direct limit $\bar{\mathfrak{A}}^{(k)}$ is the desired model.

To see this, we have only to ensure that if $\alpha \leq k$ is a limit ordinal, then $\bar{\mathfrak{A}}^{(\alpha)}$ is finitely determinate and satisfies T (it is obviously tidy). These facts are proved by showing that, if $\xi_0 < \ldots < \xi_{n-1} < \alpha$ and $\vec{x} \in A^{(\xi_{i+1})}$ $(0 \leq i < n-1)$ then

$$\mathfrak{A}^{(\alpha)} \models \varphi(A^{(\xi_0)}, \ldots, A^{(\xi_{n-1})}, x)$$

iff $\mathfrak{A}^{(\alpha)} \models aas_0 \ldots aas_{n-i-2}\varphi(A^{(\xi_0)}, \ldots, A^{(\xi_i)}, s_0, \ldots, s_{n-i-2}, x)$

iff $\mathfrak{A}^{(\xi_{n-1}+1)} \models$ same formula.

(remember that the $A^{(\xi)}$, $\xi < \alpha$ are the second order parameters of $\mathfrak{A}^{(\alpha)}$).

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