Estratto da

C. Bernardi e P. Pagli (a cura di), *Atti degli incontri di logica matematica* Volume 2, Siena 5-8 gennaio 1983, 6-9 aprile 1983, 9-12 gennaio 1984, 25-28 aprile 1984.

Disponibile in rete su http://www.ailalogica.it

STRONG AXIOMS OF INFINITY

ELLIOTT MENDELSON

The first definition of the notion of infinite set seems to have been given by Peirce and Dedekind in the 1880's. Independently, they defined a set A to be infinite if there is a one-one mapping of A onto a proper subset of itself.

In 1908, Zermelo included in his first axiomatization of set theory an axiom of infinity: There is a set which contains the empty set and is closed under the operation $x \mapsto \{x\}$. In many later axiomatizations, a similar assumption is made, but with the successor operation $x \mapsto x \cup \{x\}$ replacing $x \mapsto \{x\}$. (This probably originated with von Neumann.) In von Neumann's system, the natural numbers are identified with the finite ordinal numbers. 0 is the empty set, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$, and, in general, every natural number is the set of all "smaller" natural numbers. This definition has the advantage that every natural number n is a set having exactly n members. A finite set can be defined as a set which can be put into one-one correspondence with a natural number, and an infinite set is simply a set which is not finite. A set which is infinite in the Peirce-Dedekind sense is infinite in this new sense also, but the converse implication holds only under the assumption of the axiom of choice.

Within Zermelo-Fraenkel set theory (ZF), the axiom of infinity (AI) is unprovable from the other axioms (assuming the consistency of the theory ZF - (AI)). Proof: Assume (AI) provable in ZF - (AI). The set V_ω of all sets of finite rank (that is, the set of all hereditarily finite sets) satisfies all axioms of ZF - (AI). Hence, (AI) would be provable in V_ω. More precisely, we would have $\vdash_{ZF-(AI)} (AI)^{(V_\omega)}$, that is, the relativization of (AI) to V_ω is provable within ZF - (AI). But, $\vdash_{ZF-(AI)} \neg (AI)^{(V_\omega)}$. Hence,

ZF - (AI) would be inconsistent. ∎

Within an alternative set theory, Quine's NF (New Foundations), Specker proved in 1953 that an axiom of infinity can be derived. NF has as axioms only Extensionality and the Class Existence Schema for stratified formulas, both of which are plausible. The fact that Specker also showed the axiom of choice to be disprovable in NF has been a mark against NF. But, recently, set theorists in the ZF tradition have been flirting with axioms which are incompatible with the axiom of choice, such as the axiom of determinacy. Thus, the unacceptability of NF is no longer quite so clear.

The ZF theory is based upon the cumulative hierarchy: $V_0 = 0$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit ordinals λ ; $V = \bigcup_{\alpha \in On} V_\alpha$. (Here, $\mathcal{P}(x)$ is the power set of x : $\{u \mid u \subseteq x\}$.) People in the ZF tradition think of the cumulative hierarchy as an intuitively clear justification of ZF. Of course, the hierarchy is definable within ZF, and one of the axioms of ZF, the axiom of regularity, asserts that the universe of all sets is identical with $V = \bigcup_{\alpha \in On} V_\alpha$.

Once (AI) is assumed, the power set axiom (PS) and the replacement axiom (Rep) seem to have important consequences with regard to the "size" of the universe.

(PS) cannot be proved in ZF - (PS), if the latter is consistent. Proof: If (PS) were profable in ZF - (PS), then the set $H(\omega_1)$ consisting of all hereditarily countable sets (sets for which they and all members of their transitive closure are finite or denumerable) would be a model of ZF - (PS). Within this model, (PS) is false. Thus, $\vdash_{ZF-(PS)} \neg(PS)^{(H(\omega_1))}$.

But, if (PS) is provable in ZF - (PS), $\vdash_{ZF-(PS)} (PS)^{(H(\omega_1))}$, and ZF - (PS) would be inconsistent. ∎

Similarly, the replacement axiom (Rep) cannot be proved in the theory Z obtained from ZF by

replacing (Rep) by the selection schema

(Sel): $(\forall z)(\exists y)(\forall x)(x \in y \Longleftrightarrow [x \in z \wedge \mathcal{Q}(x)])$,

where $\mathcal{Q}$ is any formula. In fact, if (Rep) could be proved in Z, then $V_{\omega+\omega}$ would be a model of Z in which (Rep) is false. (There is a formula determining within $V_{\omega+\omega}$ a mapping from ω onto $\omega+\omega$. But, $\omega+\omega \notin V_{\omega+\omega}$.) Thus, (Rep) requires a certain ordinal "length" to the universe (considerably more than $\omega+\omega$).

In these three cases (infinity, power set, and replacement), inclusion of the axiom permits construction of a set-model in which the axiom fails. In each case, the enlarged theory can prove the consistency of the original theory. Thus, arithmetic is extended in each case, since the consistency assertion can be formulated as a statement of arithmetic. This seems to be characteristic of strong axioms of infinity.

Kanamori and Magidor have given a sort of "metadefinition" of a large cardinal property: satisfaction of the property implies: (a) the existence of a cardinal which is "large", in the sense that it is a fixed point of reasonable "thinning procedures, like Mahlo's", beginning from weaker cardinals ; (b) the strengthening of set theory by, for example, the "emergence of new combinatorial properties". According to Kanamori and Magidor, "the adoption of strong axioms is thus a theological venture, involving basic questions of belief concerning what is true about the universe". We know, by the Gödel Incompleteness Theorem, that ZF is incomplete. Large cardinal axioms might be the best way to enrich ZF. Those axioms which show themselves to be very fruitful, by giving simple derivations of many previously known results and by yielding solutions of many previously unsolved problems, might eventually be accepted as "true".

What is the motivation for particular strong axioms of infinity ? One motivating force comes from consideration of the smallest infinite ordinal ω , whose existence was guaranteed by the

original axiom of infinity. ω has various relationships to the set of smaller ordinals, the finite ordinals. If we think it unlikely that these relationships characterize ω , then we are led to assume that there are uncountable ordinals which have these relationships to smaller ordinals. (The simplest examples of such ordinals will be the inaccessible ordinals.) This process of generalization calls for, in the words of Kanamori and Magidor, "a reasonable induction from familiar situations to higher orders with the concommitant confidence in the recurring richness of the cumulative hierarchy". The key word is "reasonable". Certain properties of ω lead to inconsistency when generalized, as we shall see later.

Another motivating force is the reflection principle. The ordinary form of this principle, provable in ZF, states (in part) that, for any sentence φ , there are arbitrarily large ordinals β for which $\varphi^{(V_\beta)} \Longleftrightarrow \varphi$. So, any property holding in the universe already holds in some V_β. It may then seem "reasonable" to assume that there are cardinals which satisfy analogous downward reflection principles.

EARLY MATHEMATICAL EVENTS INVOLVING LARGE CARDINALS

Cardinal numbers were initially studied in a purely mathematical, non-axiomatic setting. (We shall identify cardinals with initial ordinals, ordinals which are not equinumerous with any smaller ordinal. In the presence of AC, the axiom of choice, every set is equinumerous with a unique initial ordinal.) Hausdorff was one of the best practitioners of this kind of work. In his 1908 paper, Grundzüge einer Theorie der geordneten Mengen (Math. Annalen, Vol. 65), he considered a notion equivalent to the notion of weakly inaccessible cardinal: a regular fixed point of the $\aleph$ function. (A cardinal κ is regular $\Longleftrightarrow$ κ is not the union of fewer than κ sets of power less than κ $\Longleftrightarrow$ κ is not the supremum of a

sequence of smaller ordinals indexed by a smaller ordinal. Regularity is an obvious property of ω. It holds also for every successor cardinal $\aleph_{\gamma+1}$ (if we assume AC): If $\bigcup_{\alpha<\tau} A_\alpha = \aleph_{\gamma+1}$ with $\bar{\bar{A}}_\alpha < \aleph_{\gamma+1}$ and $\tau < \aleph_{\gamma+1}$, then

$$\overline{\overline{\bigcup_{\alpha<\tau} A_\alpha}} \leq \sum_{\alpha<\tau} \bar{\bar{A}}_\alpha \leq \sum_{\alpha<\tau} \aleph_\gamma = \bar{\bar{\tau}} \cdot \aleph_\gamma \leq \aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma,$$

a contradiction. Therefore, it is natural to ask whether any cardinal $\aleph_\lambda$, with λ a limit ordinal, is regular. Such cardinals are the weakly inaccessible ordinals. The $\aleph$ function enumerates the infinite cardinals. A fixed point of this function is an ordinal κ such that $\aleph_\kappa = \kappa$.) The standard definition of a weakly inaccessible cardinal is that it is a regular $\aleph_\alpha$ with α a limit ordinal. The regularity of $\aleph_\alpha$ then implies that $\alpha = \aleph_\alpha$. The connection with models of axiomatic set theory was realized only much later. This is also true, of course, of the fact that the existence of weakly inaccessible cardinals is not provable in ZFC (that is, in ZF + (AC)), if the latter is consistent.

The notion of weakly inaccessible cardinal admits of various generalization, for example, the hyperinaccessibles. Call the weakly inaccessible cardinals 0-weakly hyperinaccessible. A cardinal is $(\alpha+1)$-weakly hyperinaccessible if it is a regular limit of α-weakly hyperinaccessibles ; that is, κ is $(\alpha+1)$-weakly hyperinaccessible $\Longleftrightarrow$ κ is regular and there are κ α-weakly hyperinaccessibles less than κ . For limit λ , κ is λ-weakly hyperinaccessible if and only if κ is α-weakly hyperinaccessible for all $\alpha < \lambda$. One can then prove by induction that any α-weakly hyperinaccessible cardinal is also γ-weakly hyperinaccessible for all $\gamma < \alpha$.

The process for generating weakly hyperinaccessibles can be described in the following way. A subset Y of a cardinal κ is said to be unbounded in κ if, for any $\alpha < \kappa$, there exists

$\gamma \in Y$ such that $\alpha < \gamma$. Define $W[X] = \{\kappa \in X \mid X \cap \kappa$ is unbounded in $\kappa\}$. Then one generates the hyper-inaccessibles by starting with the regular cardinals and iterating the operation W ; at limit ordinals, one takes the intersection of all the previous classes.

This approach was taken much further by P. Mahlo in 1911-1913. To understand Mahlo's idea, we need the notion of a normal function. A function $f: \gamma \to \kappa$ is normal on κ if it is increasing and continuous and its range is unbounded in κ ; that is, (i) $\alpha < \beta < \gamma \Longrightarrow f(\alpha) < f(\beta)$; (ii) for limit $\lambda < \gamma$, $f(\lambda) = \lim_{\alpha<\lambda} f(\alpha) = \sup\{f(\alpha) \mid \alpha < \lambda\}$; (iii) $f''\gamma$ is unbounded in κ .

If κ is such that every normal function on κ has a fixed point, then $\kappa > \omega$ and κ is regular, and conversely. Proof: There is a normal function on ω without fixed points, namely, $f(n) = n+1$. Hence, $\kappa > \omega$. Assume κ not regular. Then there is a function $f: \gamma \to \kappa$, for some $\gamma < \kappa$, such that γ is a cardinal, f is increasing, and $\lim_{\alpha<\gamma} f(\alpha) = \kappa$. Define $g(\alpha) = \gamma \dot{+} f(\alpha)$ for $\alpha < \gamma$. Then g is a normal function on κ such that $g(\alpha) > \gamma$ for all α. Hence, g cannot have a fixed point. So, κ must be regular. Conversely, assume $\kappa > \omega$ and κ is regular. Let f be any normal function on κ. Define $h(0) = f(0)$ and $h(n+1) = f(h(n))$ for $n < \omega$. Let $\tau = \lim_{n<\omega} h(n)$. Then $\tau < \kappa$ and τ is a fixed point of f. ∎ (A similar proof shows that, if $\kappa > \omega$ and κ is regular, then the set of fixed points of a normal function on κ is unbounded in κ.)

Mahlo considered cardinals κ such that every normal function on κ has a regular fixed point. Such cardinals are now called weakly Mahlo. By the preceding paragraph, a weakly Mahlo cardinal must be $> \omega$ and regular. Moreover, every normal function on a weakly Mahlo cardinal κ not only has a regular fixed point, but has κ regular fixed points. (Proof: Let f be a

normal function on κ, and let $\alpha < \kappa$. Define $g(\gamma) = f(\alpha \dot{+} \gamma)$ for $\gamma < \kappa$. Then g is a normal function on κ, and, therefore, g has a fixed point τ. Then, $f(\alpha \dot{+} \tau) = g(\tau) = \tau \leq \alpha \dot{+} \tau \leq f(\alpha \dot{+} \tau)$. So, $\alpha \dot{+} \tau$ is a fixed point of f. Thus, f has an unbounded in κ set of fixed points. Since κ is regular, there will be κ such fixed points. ∎)

There is another way of describing weakly Mahlo cardinals. We say that a set $A \subseteq \kappa$ is closed in κ if every $\alpha < \kappa$ which is a supremum of a subset of A is in A. A is said to be club in κ if A is closed and unbounded in κ. Club subsets of κ correspond to normal functions on κ: the enumerating function of a club subset of κ is a normal function on κ, and the range of any normal function on κ is a club subset of κ. (The enumerating function of a club subset A is defined as follows: For $\alpha < \kappa$, define $f(\alpha)$ inductively as the least member of A not in $f''\alpha$.)

In the definition of a weakly Mahlo cardinal κ, it suffices to assume that every normal function on κ contains a regular ordinal in its range, or, equivalently, that every club subset of κ contains regular members. Proof: Let g be a normal function on κ. Let Φ be the normal function on κ that enumerates all limit ordinals $< \kappa$. Then $g \circ \Phi$ is a normal function on κ. Assume τ is a regular ordinal in the range of $g \circ \Phi$. Let $g(\Phi(\sigma)) = \tau$ with $\sigma < \kappa$. Now, every increasing function $f: \kappa \to \kappa$ satisfies $f(\alpha) \geq \alpha$ for all α. (Assume γ is the first ordinal in κ for which $f(\gamma) < \gamma$. Since f is increasing, $f(f(\gamma)) < f(\gamma)$, contradicting the minimality of γ.) Hence, $\tau = g(\Phi(\sigma)) \geq \Phi(\sigma)$. If $\Phi(\sigma) < \tau$, then $g \upharpoonright \Phi(\sigma)$, the restriction of g to $\Phi(\sigma)$, would be an increasing sequence of ordinals with domain $< \tau$ and supremum τ, contradicting the regularity of τ. Hence, $\Phi(\sigma) = \tau$ and $\tau = g(\tau)$. ∎

It is easy to see that every weakly Mahlo cardinal κ must be weakly inaccessible. (Proof: We already know that $\kappa > \omega$ and κ is regular. Let $\kappa = \lambda'_\alpha$. We must show that α is a limit ordinal. Assume $\alpha = \beta + 1$. Let $f(\gamma) = \lambda'_\beta \dot{+} \gamma$ for all $\gamma < \kappa$. This is a normal function on κ. But f has no regular fixed point, since, for $0 < \gamma < \kappa$, $\lambda'_\beta \dot{+} \gamma$ is not a cardinal, and, therefore, cannot be regular. ∎) Not only are weakly Mahlo cardinals κ weakly inaccessible, but they are also the limit of κ smaller weakly inaccessible cardinals. (Proof: Let A be the set of infinite limit cardinals $< \kappa$. A is club in κ, since $\kappa = \lambda'_\kappa$. Hence, A contains an unbounded set in κ of regular cardinals. These are weakly inaccessible and their limit is κ. ∎) Thus, weakly Mahlo cardinals are 1-weakly hyperinaccessible.

Even more can be asserted: Every weakly Mahlo cardinal κ is α-weakly hyperinaccessible for all $\alpha \leq \kappa$. Proof: By induction on α. Abbreviate weakly hyperinaccessible by whi. We already know the theorem for $\alpha = 0$ and $\alpha = 1$. Case 1: α is a limit ordinal. Then the theorem is immediate. Case 2: $\alpha = \beta+1$. Case 2a: $\beta = \gamma+1$. Then κ is $\gamma+1$-whi. So, κ is a limit of κ smaller γ-whi ordinals. Let f enumerate the closure of the set B of γ-whi ordinals $< \kappa$. (In other words, let f(0) be the least member of B ; let $f(\delta+1)$ be the least member of B greater than $f(\delta)$; and let $f(\lambda)$ be the supremum of all $f(\delta)$ for $\delta < \lambda$, when λ is a limit ordinal.) Since f is normal on κ and κ is weakly Mahlo, f has κ regular fixed points, each of which is $\gamma+1$-whi. (This requires a brief verification.) So, κ is a regular limit of $\gamma+1$-whi ordinals. Therefore, κ is $\beta+1$-whi. Case 2b: β is a limit ordinal. So, κ is δ-whi for all $\delta < \beta$. For $\delta < \beta$, let A_δ be the closure of $\{\sigma < \kappa \mid \sigma \text{ is } \delta\text{-whi}\}$. Since κ is $\delta+1$-whi,

$\overline{\overline{A_\delta}} = \kappa$ and all regular members of A_δ are δ-whi. Now, $\bigcap_{\delta<\beta} A_\delta$ is closed, since the intersection of closed subsets is closed. Hence, if we can show that $\bigcap_{\delta<\beta} A_\delta$ is unbounded in κ, it will follow that $\bigcap_{\delta<\beta} A_\delta$ contains κ regular fixed points, each of which is β-whi, and, therefore, κ would be $\beta+1$-whi. To see that $\bigcap_{\delta<\beta} A_\delta$ is unbounded in κ, assume $\sigma < \kappa$. For each $\delta < \beta$, let γ_δ be the least member of $A_\delta - \sigma$. Let $\gamma = \sup_{\delta<\beta} \gamma_\delta$. By regularity of κ, $\gamma < \kappa$. For $\delta < \beta$, γ is a limit of members of A_δ. (For, if $\delta < \tau < \beta$, $\gamma_\tau \in A_\delta$.) Hence, $\gamma \in A_\delta$. Thus, $\gamma \in \bigcap_{\delta<\beta} A_\delta$ and $\gamma > \sigma$. ∎

The weakly Mahlo cardinals can be viewed in still another way. A subset X of a cardinal κ is said to be stationary in κ if X intersects every club subset of κ. By the cofinality $cf(\kappa)$ of κ we mean the least ordinal β such that there is an increasing β-sequence of ordinals with limit κ. (Clearly, $\mathrm{cf}(\kappa) \leq \kappa$. κ is regular if and only if $\mathrm{cf}(\kappa) = \kappa$.) By a final segment of κ is meant a set $\{\gamma \mid \alpha < \gamma < \kappa\}$ for some $\alpha < \kappa$.

If $\mathrm{cf}(\kappa) = \omega$, then X is stationary in κ if and only if X includes a final segment of κ. Proof: $\Leftarrow$ is obvious. $\Rightarrow$: Assume X stationary in κ. Let $\lim_{n<\omega} \alpha_n = \kappa$. Assume X includes no final segment of κ. Define β_0 = least element of $\kappa - X$ which is $> \alpha_0$; β_{n+1} = least element of $\kappa - X$ which is $> \max(\beta_n, \alpha_{n+1})$. Then, $\lim_{n<\omega} \beta_n = \kappa$, $\{\beta_n \mid n<\omega\}$ is club in κ, but $\{\beta_n \mid n<\omega\} \cap X = \emptyset$, which is a contradiction. ∎

Consider the case now where $\mathrm{cf}(\kappa) > \omega$. Notice that the intersection of two club subsets of κ is again club in κ. (Proof: Let C_1 and C_2 be club in κ. It is obvious that $C_1 \cap C_2$ is closed in κ. Assume $\alpha < \kappa$. Define $\alpha_0 = \alpha$; α_{2k+1} = the least member of C_1 which is $> \alpha_{2k}$; α_{2k+2} = the least member of C_2 which is $> \alpha_{2k+1}$. Then $\beta = \lim_{n<\omega} \alpha_n < \kappa$; $\beta \in C_1 \cap C_2$; $\beta > \alpha$. Hence, $C_1 \cap C_2$ is unbounded in κ. ∎)

Therefore, every club subset of κ is stationary in κ. However, the converse is false ; every stationary subset of κ is unbounded in κ but not necessarily closed in κ. (Example: Let λ be a limit ordinal $< \kappa$. Then $\kappa - \{\lambda\}$ is stationary but not closed in κ.)

By a previous theorem, we know that κ is weakly Mahlo if and only if $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary in κ. Now, consider the so-called Mahlo operation $M[X] = \{\kappa \in X \mid X \cap \kappa \text{ is stationary in } \kappa\}$. If X is the class of regular cardinals, $M[X]$ is the class of weakly Mahlo cardinals. This operation can be transfinitely iterated: Mahlo_0 = the class of weakly Mahlo

cardinals ; $\mathrm{Mahlo}_{\alpha+1} = M[\mathrm{Mahlo}_\alpha]$;
$\mathrm{Mahlo}_\lambda = \bigcap_{\alpha<\lambda} \mathrm{Mahlo}_\alpha$ for limit λ .

The reason for the term "stationary" is the following. Let $A \subseteq \kappa$. A function $f: A \longrightarrow \kappa$ is said to be regressive if $f(\alpha) < \alpha$ for $\alpha \neq 0$, and $f(0) = 0$. The following theorem was proved in essence by Alexandroff and Urysohn in 1929 and rediscovered by Fodor in 1956: If f is a regressive function on a stationary subset of a regular cardinal κ , then there exists some β in κ such that $f^{-1}[\{\beta\}]$ is unbounded in κ (in fact, even stationary).

STRONG FORMS.

A strongly inaccessible (briefly, inaccessible) cardinal is defined to be a regular cardinal κ such that $\alpha < \kappa \Longrightarrow 2^\alpha < \kappa$. Every inaccessible cardinal is easily seen to be weakly inaccessible. Given the generalized continuum hypothesis GCH, the classes In and WIn of inaccessible and weakly inaccessible cardinals are the same. The

inaccessible cardinals are identical with the regular fixed points of the beth function:
$\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\lambda = \lim_{\alpha<\lambda} \beth_\alpha$
for limit λ .

A strongly Mahlo (briefly, Mahlo) cardinal is a cardinal κ such that every normal function on κ has an inaccessible fixed point. (equivalently, every normal function on κ has an inaccessible cardinal in its range, or every club subset of κ contains an inaccessible cardinal). As before, if κ is Mahlo, every normal function on κ has an unbounded in κ class of inaccessible fixed points. Every Mahlo cardinal is weakly Mahlo, and is, therefore, weakly inaccessible. It follows easily that every Mahlo cardinal is inaccessible, since it is the limit of inaccessible cardinals. The class of Mahlo cardinals is the class $M[\mathrm{In}] = \{\kappa \in \mathrm{In} \mid \mathrm{In} \cap \kappa$ is stationary in $\kappa\}$. The class of weakly Mahlo cardinals could have been defined as $M[\mathrm{WIn}] = \{\kappa \in \mathrm{WIn} \mid$

$W\text{In} \cap \kappa$ is stationary in $\kappa \}$.

Drake has remarked that postulating the existence of Mahlo cardinals "can be regarded as saying that the class of strongly inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class ; however we climb through κ , provided we are continuous at limits, we shall eventually have to hit a strongly inaccessible cardinal".

MEASURE THEORY.

A measure on a set x is a mapping $\mu : \mathcal{P}(x) \longrightarrow [0,1]$ such that: (a) $\mu(x) = 1$; (b) μ is countably additive. (Countable additivity means that, if $\{u_n \mid n<\omega\}$ is a sequence of pairwise-disjoint subsets of x, then $\mu(\bigcup_{n<\omega} u_n) = \sum_{n<\omega} \mu(u_n)$.) The following properties hold for measures.

(i) If $u \subseteq v$, then $\mu(u) \leq \mu(v)$.

(ii) If $u_1 \subseteq u_2 \subseteq u_3 \subseteq \ldots$, then

$$\mu(\bigcup_{n<\omega} u_n) = \lim_{n<\omega} \mu(u_n)$$

(iii) If $u_1 \supseteq u_2 \supseteq u_3 \supseteq \ldots$, then

$$\mu(\bigcap_{n<\omega} u_n) = \lim_{n<\omega} \mu(u_n) .$$

(iv) $\mu(\emptyset) = 0$.

A measure μ on x is said to be non-trivial if $\mu(\{a\}) = 0$ for all $a \in x$. By countable additivity, if μ is non-trivial, $\mu(y) = 0$ for every countable $y \subseteq x$. Therefore, there is no non-trivial measure on a countable set.

A measure μ is two-valued if the range of μ is $\{0,1\}$. If we allow trivial measures, we can easily define many two-valued measures: $\mu(y) = 1$ if $a \in y$ and $\mu(y) = 0$ if $a \notin y$, where a is some fixed element.

Example: The set ${}^{\omega}2$ of all functions from ω into $2 = \{0,1\}$ has no non-trivial measure.

Proof: Define a sequence $\{u_n\}$ in ${}^{\omega}2$ as follows:

given $u_0, u_1, \ldots, u_{n-1}$ such that $\mu(\{f \in {}^{\omega}2 \mid f(0)=u_0 \wedge \ldots \wedge f(n-1)=u_{n-1}\}) = 1$, let

$Y_0 = \{f \in {}^{\omega}2 \mid f(0)=u_0 \wedge \ldots \wedge f(n-1)=u_{n-1} \wedge f(n)=0\}$,

$Y_1 = \{f \in {}^{\omega}2 \mid f(0)=u_0 \wedge \ldots \wedge f(n-1)=u_{n-1} \wedge f(n)=1\}$.

$Y_0 \cap Y_1 = \emptyset$, and, by hypothesis, $\mu(Y_0 \cup Y_1) = 1$. Hence, either $\mu(Y_0) = 1$ or $\mu(Y_1) = 1$. Let $u_n = 0$ if $\mu(Y_0) = 1$ and $u_n = 1$ if $\mu(Y_1) = 1$.

So, there is a sequence $W_0 \supseteq W_1 \supseteq \ldots$ such that $\mu(W_i) = 1$ and $W_i = \{f \in {}^{\omega}2 \mid f(0)=u_0 \wedge \ldots \wedge f(i)=u_i\}$.

Then $\mu(\bigcap_{n<\omega} W_n) = \lim_{n<\omega} \mu(W_n) = 1$. But, $\bigcap_{n<\omega} W_n$ consists of a single point. Hence, $\mu(\bigcap_{n<\omega} W_n) = 0$, which yields a contradiction. ∎

This result has an obvious generalization: If there is no non-trivial two-valued measure on κ, there is no such measure on ${}^{\kappa}2$. (In general, ${}^{a}b$ denotes the set of all mappings from a into b.)

Proof: Let $A = {}^{\kappa}2$, and assume μ is a non-trivial two-valued measure on A. For each $\alpha < \kappa$, let $A^0_\alpha = \{f \in A \mid f(\alpha)=0\}$ and $A^1_\alpha = \{f \in A \mid f(\alpha)=1\}$. $A^0_\alpha \cap A^1_\alpha = \emptyset$ and $A^0_\alpha \cup A^1_\alpha = A$. Choose n_α so that $\mu(A^{n_\alpha}_\alpha) = 1$. Then $\mu(\bigcap_{\alpha<\kappa} A^{n_\alpha}_\alpha) = 1$.

(Otherwise, one can define a non-trivial two-valued measure μ^* on κ as follows: for $B \subseteq \kappa$, let $\mu^*(B) = \mu(\bigcup_{\alpha \in B} \overline{A^{n_\alpha}_\alpha})$.) But $\bigcap_{\alpha<\kappa} A^{n_\alpha}_\alpha$ consists of a single point of ${}^{\kappa}2$. So, $\mu(\bigcap_{\alpha<\kappa} A^{n_\alpha}_\alpha) = 0$, which is a contradiction. ∎

ORIGIN OF THE NOTION OF MEASURE.

Lebesgue measure is a non-trivial countably additive "measure" on a certain set of subsets of the real number system, the Lebesgue-measurable sets. G. Vitali proved in 1905 that not all sets of real numbers could be Lebesgue-measurable.

Vitali's Theorem. There is no non-trivial translation-invariant countably additive measure on all sets of real numbers.

Proof. Assume such a measure μ exists and consider only its restriction to subsets of $[0,1]$. (The original measure did not satisfy our definition, since the measure of the entire set of real numbers is $+\infty$. However, $\mu([0,1])=1$. In addition, the meaning of translation invariance is that $\mu(A) = \mu(\{a+r \mid a \in A\})$ for every constant r.) Define an equivalence relation: $x \sim y$ if and only if $x-y$ is rational. Each equivalence class is countable. (If we assume AC, the number of equivalence classes has the power of the continuum.) By AC, let Y be a set which contains exactly one element from each equivalence class. For each rational q with $0 \leq q < 1$, let $Y_q = Y +_1 q = \{y +_1 q \mid y \in Y\}$. (Here, $+_1$ is addition modulo 1: $y +_1 q = y + q$ if $y + q \leq 1$, and $y +_1 q = y + q - 1$ if $y + q > 1$.) It follows from translation invariance that $\mu(Y_q) = \mu(Y)$. It is easy to verify that $[0,1] = \bigcup_{0 \leq q < 1} Y_q$ and that $Y_{q_1} \cap Y_{q_2} = \emptyset$ for $q_1 \neq q_2$. Hence, $1 = \mu([0,1]) = \sum_{0 \leq q < 1} \mu(Y_q)$. Since all the

$\mu(Y_q)$ are equal, it follows that $\mu(Y_q) = 0$ for all q. Then $1 = \sum_{0 \leq q < 1} 0 = 0$, which is a contradiction. ■

AC was used in this proof in a form in which it yields a choice function on the set of equivalence classes. The strength of the form of AC required has important consequences for the axiomatics of set theory. Let DC (the Principle of Dependent Choice) assert that, if r is a non-empty relation whose range is included in its domain, then there is a denumerable sequence $\{a_n \mid n < \omega\}$ such that, for all $n < \omega$, $\langle a_n, a_{n+1} \rangle \in r$. Solovay [1970] has shown that, if the existence of a non-Lebesgue measurable set can be proved in ZF + DC, then there are no inaccessible cardinals. Shelah has proved the converse. Hence, "ZFC + There is an inaccessible cardinal" is equiconsistent with "ZF + DC + All sets of reals are Lebesgue measurable".

Long after Vitali's proof, in 1929, Banach and

Kuratowski raised the question of the existence of a non-trivial countably additive measure on the reals (and other sets), without imposing a property like translation invariance. Substantial results were obtained by Ulam in 1930. We shall outline some of his work. From now on, we shall assume that the word <u>measure</u> refers to a non-trivial countably additive measure.

(U1) If $\kappa = \bigcup_{\alpha<\tau} A_\alpha$, where the A_α's are pairwise disjoint, and if there is no measure on each A_α or on τ, then there is no measure on κ.

Proof. Assume μ is a measure on κ. Then $\mu(A_\alpha) = 0$ for all α. (Otherwise, if $\mu(A_\alpha) \neq 0$, then μ would induce a measure on A_α.) Define a measure on τ as follows: for $B \subseteq \tau$, let $\nu(B) = \mu(\bigcup_{\alpha \in B} A_\alpha)$. ∎

(U2) If there is no measure on ω_α, then there is no measure on $\omega_{\alpha+1}$.

Proof. Assume there is a measure on $\omega_{\alpha+1}$. By the proof of (U1), we know that there is no

partition of $\omega_{\alpha+1}$ into λ'_α sets of measure 0. Hence, every partition of $\omega_{\alpha+1}$ into λ'_α sets has at least one component with non-zero measure. Form an infinite matrix (A^σ_τ) with the column index σ ranging over $\omega_\alpha \leq \sigma < \omega_{\alpha+1}$, and the row index τ ranging over $0 \leq \tau < \omega_\alpha$. For each δ with $\omega_\alpha \leq \delta < \omega_{\alpha+1}$, pick an ω_α-enumeration of δ: $\delta = \{\beta^\delta_0, \beta^\delta_1, \ldots, \beta^\delta_\tau, \ldots\}_{\tau<\omega_\alpha}$. For an arbitrary τ^{th} row, put δ into the set $A^{\beta^\delta_\tau}_\tau$ in the β^δ_τ th column, and into no other set in this row. Then the sets in each row are pairwise disjoint. Consider the σ^{th} column. For each $\delta > \sigma$, $\sigma = \beta^\delta_\tau$ for some $\tau < \omega_\alpha$. Hence, $\delta \in A^\sigma_\tau$. So, $\bigcup_{\tau<\omega_\alpha} A^\sigma_\tau = \omega_{\alpha+1} - (\sigma+1)$. Thus, $\bigcup_{\tau<\omega_\alpha} A^\sigma_\tau$ is a set of power $\omega_{\alpha+1}$ which is partitioned into ω_α sets. Therefore, at least one set in the σ^{th} column has positive measure. Hence, there are $\lambda'_{\alpha+1}$ entries in the matrix with positive measure. So, there must be a row with uncountably many entries of positive measure. Since these entries are pairwise disjoint, we obtain a contradiction. (In fact, assume there exist uncountably many pairwise disjoint sets of positive measure. For $n < \omega$, let $\mathcal{A}_n$ be the set of all such sets of measure $\geq \frac{1}{n}$. Since every set of positive measure is in at least one $\mathcal{A}_n$, there must be some $\mathcal{A}_n$ containing uncountably many sets. Hence, such an $\mathcal{A}_n$ contains denumerably many pairwise disjoint sets $Z_1, Z_2, \ldots$. Then $\mu(\bigcup_{i<\omega} Z_i) = \sum_{i<\omega} \mu(Z_i) \geq \sum_{i<\omega} \frac{1}{n} = +\infty$, which is impossible.) ∎

(U3) It follows from (U1)-(U2) that all cardinals less than the first weakly inaccessible cardinal (if such a cardinal exists) do not have a measure. Hence, if $2^{\aleph_0}$ is less than the first weakly inaccessible cardinal, then there is no measure on the reals, and, therefore, there exists a non-Lebesgue measurable set.

(U4) If μ is a measure on κ, then a subset $B \subseteq \kappa$ is called a μ-atom if $\mu(B) > 0$ and every subset of B has measure either equal to 0 or to $\mu(B)$.

(a) Assume that μ is a measure on κ for which there are no μ-atoms. Then, for every $\varepsilon > 0$, there exist sets $E_1, \ldots, E_n$ for which $\mu(E_i) \leq \varepsilon$ for all i, and $\kappa = E_1 \cup \ldots \cup E_n$, with the E_i pairwise disjoint.

Proof. It suffices to prove the theorem for $\varepsilon = \frac{1}{2}$, for it can then be iterated to $\frac{1}{4}, \frac{1}{8}$, etc. Assume the result false. We shall derive the consequence that there is a $\supsetneqq$-descending ω_1-sequence $\{P_\alpha \mid \alpha < \omega_1\}$ such that $\gamma > \delta \Rightarrow \mu(P_\gamma) < \mu(P_\delta)$ and $\mu(P_\alpha) > \frac{1}{2}$ for all α, and this is impossible. (Otherwise, $\{P_\alpha - P_{\alpha+1} \mid \alpha < \omega_1\}$ would form a non-denumerable collection of pairwise disjoint sets of positive measure.) Let $P_0 = \kappa$. Assume P_γ defined for all $\gamma < \delta$. Let $B = \bigcap_{\gamma<\delta} P_\gamma$. We may assume $\mu(B) \geq \frac{1}{2}$. (Otherwise, consider

$$P_0 - B = (P_0-P_1) \cup (P_1-P_2) \cup \ldots \cup \left(\bigcap_{n<\omega} P_n - P_\omega\right) \cup (P_\omega - P_{\omega+1}) \cup \ldots .$$

This is a countable partition of $P_0 - B$ into sets of measure $< \frac{1}{2}$. Rearrange so that $\mu(P_0 - B)$ is the infinite sum of measures of these partition sets. Go far enough into the series so that the sum of the remaining measures is $< \frac{1}{2}$. This yields a finite partition of P_0 into sets of measure $< \frac{1}{2}$.)

If $\mu(B) = \frac{1}{2}$, then $\kappa = (\kappa - B) \cup B$, with $\mu(B) = \mu(\kappa - B) = \frac{1}{2}$. So, we may assume $\mu(B) > \frac{1}{2}$. Since B cannot be an atom, there exist B_1, B_2 such that $B = B_1 \cup B_2$ and $\mu(B_1) \geq \mu(B_2) > 0$. Then $\mu(B_1) > \frac{1}{2}$; otherwise, $\kappa = (\kappa - B) \cup B_1 \cup B_2$ would yield the theorem. Now let $P_\delta = B_1$. Then $\mu(P_\delta) = \mu(B_1) < \mu(B) = \mu(\bigcap_{\gamma<\delta} P_\gamma) \leq \mu(P_\gamma)$ for $\gamma < \delta$. ∎

(b) If μ is a measure on κ, but there is no two-valued measure on κ, then, for every positive integer n, κ can be partitioned into finitely many sets, each of measure $\leq \frac{1}{n}$.

Proof. By (a), it suffices to prove that there are no μ-atoms. Assume there is a subset A of positive measure such that every subset of A has measure 0 or $\mu(A)$. Let $\nu(C) = \mu(C \cap A)/\mu(A)$ for all $C \subseteq \kappa$. Then ν would be a two-valued measure on κ. ∎

(U5) Assume there is no weakly inaccessible cardinal $\leq 2^{\aleph_0}$.

(I) If there is a measure on κ, then there is a two-valued measure on κ.

(II) If there is no measure on κ, there is no measure on 2^κ.

(III) Every cardinal less than the first strongly inaccessible cardinal has no measure.

Proof. (I) Let μ be a measure on κ. Assume there is no two-valued measure on κ. By (b) of (U4), for each n there is a partition $\kappa = A_1^n \cup A_2^n \cup \ldots \cup A_{k(n)}^n$ with $\mu(A_j^n) \leq \frac{1}{n}$. Then

$$\kappa = \bigcap_{n=1} \bigcup_{j=1}^{k(n)} A_j^n = \bigcup_{j_n \leq k(n)} (A_{j_1}^1 \cap A_{j_2}^2 \cap \ldots \cap A_{j_n}^n \cap \ldots).$$

Each term on the right has measure 0, and there are a continuum of them. By hypothesis and (U3), there is no measure on $2^{\aleph_0}$. By the method of proof of (U1), we obtain a contradiction. ∎

(II) It suffices to prove that there is no two-valued measure on 2^{κ}. But this has been proved earlier.

(III) Let κ be the least cardinal less than the first strongly inaccessible cardinal on which there is a measure. By (U1), κ cannot be singular. So, there is a cardinal $\tau < \kappa$ for which $2^{\tau} \geq \kappa$. By (II), there is no measure on 2^{τ}. But this contradicts the fact that, if $x \subseteq y$ and there is a measure on x, then there is an extension of this measure to a measure on y. (In fact, if $x \subseteq y$ and μ is a measure on x, define $\nu(A) = \mu(x \cap A)$ for all $A \subseteq y$. Then ν is a measure on y, and $\nu(A) = \mu(A)$ for $A \subseteq x$.) ∎

(U6) If κ is the smallest cardinal on which there is a measure, then either $\kappa \leq 2^{\aleph_0}$ or there is a two-valued measure on κ (but not both).

Proof. By the method of proof of (I) of (U5), if there is no two-valued measure on κ, then there is a measure on $2^{\aleph_0}$. So, $\kappa \leq 2^{\aleph_0}$. ∎

A measure μ is said to be κ-additive if, for any collection $\{x_\alpha \mid \alpha < \tau\}$ of pairwise disjoint subsets with $\tau < \kappa$, $\mu(\bigcup_{\alpha<\tau} x_\alpha) = \sum_{\alpha<\tau} \mu(x_\alpha)$. (Notice that every measure is automatically ω_1-additive.)

Theorem. Let κ be the least cardinal having a measure. Then the measure μ on κ is κ-additive.

Proof. Observe first that, for any $\tau < \kappa$, if $\{x_\alpha \mid \alpha < \tau\}$ is a collection of pairwise disjoint subsets of κ of measure 0, then $\bigcup_{\alpha<\tau} x_\alpha$ has measure 0. If not, let $b = \mu(\bigcup_{\alpha<\tau} x_\alpha) > 0$, and define a measure ν on τ as follows:

for $z \subseteq \tau$, $\nu(z) = \frac{1}{b}\mu(\bigcup_{\alpha \in z} x_\alpha)$. This would contradict the minimality of κ. Now, assume $\{x_\alpha \mid \alpha < \tau\}$, $\tau < \kappa$, is a pairwise disjoint collection of subsets of κ. As we have proved before, the set $W = \{\alpha < \tau \mid \mu(x_\alpha) > 0\}$ is countable. Let $Y = \tau - W$. Then, for all $\alpha \in Y$, x_α has measure 0, and

$$\mu(\bigcup_{\alpha<\tau} x_\alpha) = \mu(\bigcup_{\alpha \in Y} x_\alpha \cup \bigcup_{\alpha \in W} x_\alpha) =$$

$$\mu(\bigcup_{\alpha \in Y} x_\alpha) + \mu(\bigcup_{\alpha \in W} x_\alpha) =$$

$$0 + \sum_{\alpha \in W} \mu(x_\alpha) = \sum_{\alpha < \tau} \mu(x_\alpha) . \blacksquare$$

(A similar proof shows that the least cardinal with a two-valued measure is κ-additive.)

Corollary. The least cardinal κ with a measure must be regular.

Proof. By the minimality of κ, every subset of κ of cardinality $< \kappa$ has measure 0. (Otherwise, one could define a measure on such a subset.) If κ were the union of τ subsets

$\{x_\alpha \mid \alpha < \tau\}$ with $\overline{\overline{x}}_\alpha < \kappa$ and $\tau < \kappa$, then $\mu(\kappa) = \mu(\bigcup_{\alpha<\tau} x_\alpha) = \sum_{\alpha<\tau} \mu(x_\alpha) = \sum_{\alpha<\tau} 0 = 0$. $\blacksquare$

Similarly, the least cardinal with a two-valued measure is regular.

A cardinal κ is said to be real-valued measurable if there is a κ-additive measure on κ. A cardinal κ is said to be measurable if there is a two-valued κ-additive measure on κ.

Notice that the property of being real-valued measurable or measurable is stronger than just having a measure or two-valued measure. It ensures, for one thing, that the measure is not just an inessential extension of a measure on a smaller cardinal.

Theorem (Ulam). If κ is measurable, then κ is strongly inaccessible.

Proof. The regularity of κ follows by a proof similar to that of the Corollary immediately above. Now, assume $\tau < \kappa$ and $2^\tau \geqslant \kappa$. Let μ be a two-valued κ-additive measure on κ. This extends to a two-valued κ-additive measure μ^* on ${}^{\tau}2$. Define a τ-sequence $f \in {}^{\tau}2$ in the following manner. Assume that $\beta < \tau$ and that $f \restriction \beta$ has been defined so that the set $U(f \restriction \beta) = \{g \in {}^{\tau}2 \mid f(\alpha) = g(\alpha) \text{ for } \alpha < \beta\}$ has measure 1. Let $U^0 = \{g \in U(f \restriction \beta) \mid g(\beta) = 0\}$ and $U^1 = \{g \in U(f \restriction \beta) \mid g(\beta) = 1\}$. Let $f(\beta) = i$ $\mu^*(U^i) = 1$. (For limit ordinals β, $U(f \restriction \beta) = \bigcap_{\alpha<\beta} U(f \restriction \alpha)$. Since μ^* is κ-additive and $\beta < \kappa$, $\mu^*(U(f \restriction \beta) = 1$.) [In general, if $\beta < \kappa$ and a β-sequence $\{A_\alpha \mid \alpha < \beta\}$ is increasing, then $\mu^*(\bigcup_{\alpha<\beta} A_\alpha) = \sup_{\alpha<\beta} \mu^*(A_\alpha)$, while, if $\{A_\alpha \mid \alpha < \beta\}$ is decreasing, $\mu^*(\bigcap_{\alpha<\beta} A_\alpha) = \inf_{\alpha<\beta} \mu^*(A_\alpha)$.] Hence, $\mu^*(\{f\}) = \inf_{\beta<\tau} \mu^*(U(f \restriction \beta)) = 1$. But, by non-triviality, $\mu^*(\{f\}) = 0$. ■

Corollary. If ZFC is consistent, the existence of a measurable cardinal is unprovable in ZFC.

For a long time, it was not known whether the smallest inaccessible cardinal is measurable. The negative answer obtained by Tarski and Hanf will be presented later.

In the theory of filters, a filter is said to be κ-complete if the intersection of fewer than κ members of the filter is also a member of the filter. It is easy to see that κ is measurable if and only if $\kappa > \omega$ and there exists a κ-complete non-principal ultrafilter on κ. (Notice that, by the Prime Ideal Theorem, there exists a non-principal ω-additive ultrafilter on ω. In this perspective, measurability is a generalization of a property of ω.)

Exercise. Every real-valued measurable cardinal is weakly inaccessible.

Scott proved in 1961 that the existence of measurable (or real-valued measurable) cardinals is incompatible with V = L. We shall sketch the ideas used in his proof.

Assume that μ is a two-valued κ-additive measure on κ. For any f and g in ${}^{\kappa}V$ (that is, f and g are κ-sequences), define

$$f \sim g \Longleftrightarrow \mu(\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}) = 1 .$$

It is easy to see that $\sim$ is an equivalence relation. Scott devised a trick of forming the "equivalence-set" of a given κ-sequence f, consisting of all κ-sequences which have the least rank among κ-sequences equivalent to f. (The equivalence classes of κ-sequences would be proper classes and could not be elements of a model.)

$$[f] = \{g \mid g \sim f \wedge (\forall h)(h \sim f \Rightarrow rk(g) \leq rk(h))\}.$$

Here, rk is the rank function: rk(x) is the least α such that $x \in V_\alpha$. The key property of equivalence classes holds: $[f] = [g] \Longleftrightarrow f \sim g$.

Let $V^\kappa/\mu = \{[f] \mid f \in {}^{\kappa}V\}$. Now define a membership relation E_μ on V^κ/μ:

$$[f]\ E_\mu\ [g] \Longleftrightarrow \mu(\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\}) = 1 .$$

Theorem. E_μ is well-founded.

Proof. Assume there is an infinitely descending E_μ-sequence $\{[f_i] \mid i < \omega\}$, with $[f_{i+1}]\ E_\mu\ [f_i]$. Let $A_i = \{\alpha < \kappa \mid f_{i+1}(\alpha) \in f_i(\alpha)\}$. $\mu(A_i) = 1$. Hence, $\mu(\bigcap_{i<\omega} A_i) = 1$. Let $\alpha \in \bigcap_{i<\omega} A_i$. Then $f_{i+1}(\alpha) \in f_i(\alpha)$, and $\{f_i(\alpha) \mid i < \omega\}$ would be an infinitely descending $\in$-sequence, contradicting the axiom of regularity. ■

Now we can prove a special case of Łoś's Theorem on ultraproducts.

Łoś's Theorem. If φ is any formula of ZF with free variables among $x_1, \ldots, x_n$,

$$\vdash (\forall f_1, \ldots, f_n \in {}^{\kappa}V)[\varphi^{\langle V^\kappa/\mu,\, E_\mu\rangle}([f_1], \ldots, [f_n])$$

$$\Longleftrightarrow \mu(\{\alpha < \kappa \mid \varphi(f_1(\alpha), \ldots, f_n(\alpha))\}) = 1].$$

Proof. By induction on the complexity of φ. The theorem is clear for atomic φ, and the induction step for $\neg$ and $\vee$ is easy. So, assume that φ is $(\exists x_0)\psi$. Assume first that

$\varphi([f_1],\ldots,[f_n])$ holds in $\langle V^\kappa/\mu, E_\mu\rangle$.

Then $(\exists g \in {}^\kappa V)\,\psi([g],[f_1],\ldots,[f_n])$ holds in $\langle V^\kappa/\mu, E_\mu\rangle$. By inductive hypothesis,

$\mu(\{\alpha<\kappa \mid \psi(g(\alpha),f_1(\alpha),\ldots,f_n(\alpha))\}) = 1$,

and, therefore,

$\mu(\{\alpha<\kappa \mid (\exists x_0)\psi(x_0,f_1(\alpha),\ldots,f_n(\alpha))\}) = 1$.

Conversely, assume the last equation. For $\alpha<\kappa$, choose $g(\alpha)$ so that $\psi(g(\alpha),f_1(\alpha),\ldots,f_n(\alpha))$ if there is such a $g(\alpha)$; otherwise, let $g(\alpha) = 0$. Then, by inductive hypothesis, $\psi([g],[f_1],\ldots,[f_n])$ holds in $\langle V^\kappa/\mu, E_\mu\rangle$. Hence, $(\exists x_0)\psi$ also holds. ■

We shall define an isomorphic embedding of V into V^κ/μ. For any $x\in V$, define $c_x \in {}^\kappa V$ by setting $c_x(\alpha) = x$ for all $\alpha<\kappa$. Let $i(x) = [c_x]$. Then $x\in y \Leftrightarrow i(x)\; E_\mu\; i(y)$ and $x = y \Leftrightarrow i(x) = i(y)$.

Embedding Theorem. For every formula φ,

$$\vdash [\varphi^{\langle V^\kappa/\mu, E_\mu\rangle}(i(x_1),\ldots,i(x_n))] \Leftrightarrow \varphi(x_1,\ldots,x_n).$$

Proof.

$\{\alpha<\kappa \mid \varphi(c_{x_1}(\alpha),\ldots,c_{x_n}(\alpha))\} =$

κ if $\varphi(x_1,\ldots,x_n)$, and 0 if $\neg\varphi(x_1,\ldots,x_n)$.

Now use Łoś's Theorem. ■

We shall assume familiarity with the following well-known result.

Mostowski's Collapsing Theorem. If R is an extensional and well-founded relation on a class A, then there is a transitive class M and an isomorphism π of $\langle A,R\rangle$ onto $\langle M,E(M)\rangle$, where $E(M)$ is the membership relation restricted to M.

Hence, there is an isomorphism π of $\langle V^\kappa/\mu, E_\mu\rangle$ onto a transitive class M. In particular, $\pi([f]) = \{\pi([g]) \mid [g]\; E_\mu\; [f]\}$. We can take the composition of i and π:

$$j(x) = \pi(i(x)).$$

j is an elementary embedding of V into M:

$$\vdash \varphi^{(M)}(j(x_1),\ldots,j(x_n)) \Leftrightarrow \varphi(x_1,\ldots,x_n).$$

Theorem. (a) $\alpha < \kappa \Rightarrow j(\alpha) = \alpha$.
(b) $j(\kappa) > \kappa$. (c) $\mathcal{P}(\kappa) \subseteq M$.
(d) ${}^{\kappa}M \subseteq M$.

Proof. (a) Since j is an elementary embedding, $j(\alpha)$ is an ordinal and j is strictly increasing on On. Hence, $j(\alpha) \geq \alpha$. Assume $j(\alpha) > \alpha$. For $\beta < j(\alpha)$, $\beta \in M$ by transitivity of M. So, $\beta = \pi([f])$ for some $f \in {}^{\kappa}V$ such that $[f]\ E_\mu\ [c_\alpha]$. Hence, $\mu(\{\delta < \kappa \mid f(\delta) \in \alpha\}) = 1$. For $\gamma < \alpha$, let $A_\gamma = \{\delta < \kappa \mid f(\delta) = \gamma\}$. So, $\mu(\bigcup_{\gamma<\alpha} A_\gamma) = 1$. By κ-additivity, $\mu(A_\gamma) = 1$ for some $\gamma < \alpha$. Hence, $[f] = [c_\gamma]$ and $\beta = \pi([c_\gamma]) = \pi(i(\gamma)) = j(\gamma)$. Thus, $j(\alpha) = \{j(\gamma) \mid \gamma < \alpha\}$. By inductive hypothesis, $j(\alpha) = \alpha$.

(b) Let I_κ be the identity mapping on κ. For $\alpha < \kappa$, $[c_\alpha]\ E_\mu\ [I_\kappa]$. Hence, $\alpha \in \pi([I_\kappa])$, since $\pi([c_\alpha]) = \alpha$. Now, $[I_\kappa]\ E_\mu\ [c_\kappa]$. Hence, $\pi([I_\kappa]) \in \pi([c_\kappa]) \in$ On. So, $\pi([I_\kappa]) \in$ On. Therefore, $\kappa \leq \pi([I_\kappa])$ and

$\pi([I_\kappa]) < j(\kappa)$. Thus, $\kappa < j(\kappa)$.

(c) Since $\kappa < j(\kappa) \in M$, $\kappa \in M$ and there exists $g_\kappa \in {}^{\kappa}V$ such that $\pi([g_\kappa]) = \kappa$. Since $\pi([g_\kappa]) = \kappa \leq \pi([I_\kappa])$, we can choose g_κ so that $g_\kappa(\alpha) \leq \alpha$ for all $\alpha < \kappa$. (Let $A = \{\alpha < \kappa \mid g_\kappa(\alpha) \leq I_\kappa(\alpha) = \alpha\}$. Then $\mu(A) = 1$. For $\alpha \notin A$, change $g_\kappa(\alpha)$ to 0.) Now, assume $x \subseteq \kappa$. Let $f(\alpha) = x \cap g_\kappa(\alpha)$ for $\alpha < \kappa$. Let us prove $\pi([f]) = x$. (i) Assume $\delta < \kappa$. $[c_\delta]\ E_\mu\ [f] \Leftrightarrow \mu(\{\beta < \kappa \mid \delta \in x \cap g_\kappa(\beta)\}) = 1$. If $\delta \notin x$, $\mu(B) = 0$. If $\delta \in x$, $\mu(B) = \mu(\{\beta < \kappa \mid \delta \in g_\kappa(\beta)\}) = 1$. (For, $\delta < \kappa = \pi([g_\kappa])$. Hence, $\pi([c_\delta]) < \pi([g_\kappa])$ and $[c_\delta]\ E_\mu\ [g_\kappa]$.) Thus, $\delta = \pi([c_\delta]) \in \pi([f]) \Leftrightarrow [c_\delta]\ E_\mu\ [f] \Leftrightarrow \delta \in x$. Therefore, $\pi([f]) \cap \kappa = x$. (ii) Now assume $[g]\ E_\mu\ [f]$. Then $\mu(\{\beta < \kappa \mid g(\beta) \in x \cap g_\kappa(\beta)\}) = 1$ and so, $\mu(\{\beta < \kappa \mid g(\beta) \in g_\kappa(\beta)\}) = 1$. Hence, $[g]\ E_\mu\ [g_\kappa]$. So, $\pi([g]) \in \kappa$, and, therefore, $\pi([f]) \subseteq \kappa$. This proves that $\pi([f]) = x$.

(d) Assume $\{\pi([s_\alpha]) \mid \alpha < \kappa\} \subseteq M$. Let $g_\kappa : \kappa \to \kappa$ be such that $\pi([g_\kappa]) = \kappa$. For each $\beta < \kappa$, let $h(\beta)$ be a function with domain $g_\kappa(\beta)$ such that $(h(\beta))(\alpha) = s_\alpha(\beta)$ for $\alpha \in g_\kappa(\beta)$. Therefore, in M,

$\pi([h])$ is a function with domain $\pi([g_\kappa]) = \kappa$

$\Leftrightarrow$ $[h]$ is a function with domain $[g_\kappa]$ (in V^κ/μ)

$\Leftrightarrow \mu(\{\beta < \kappa \mid h(\beta)$ is a function with domain $g_\kappa(\beta)\}) = 1$. So, $\pi([h])$ is a function with domain κ. We must show, for $\alpha < \kappa$, that $(\pi([h]))(\alpha) = \pi([s_\alpha])$. This is equivalent to:

$(\pi([h]))(\pi([c_\alpha])) = \pi([s_\alpha]) \Leftrightarrow$

$([h])([c_\alpha]) = [s_\alpha] \Leftrightarrow$

$\mu(\{\beta < \kappa \mid (h(\beta))(c_\alpha(\beta)) = s_\alpha(\beta)\}) = 1 \Leftrightarrow$

$\mu(\{\beta < \kappa \mid (h(\beta))(\alpha) = s_\alpha(\beta)\}) = 1$.

One must verify here that α is in the domain of $h(\beta)$ for a set of β's of measure 1. But, since $\pi([c_\alpha]) = \alpha < \kappa = \pi([g_\kappa])$, $[c_\alpha] \; E_\mu \; [g_\kappa]$. Therefore, $\mu(\{\beta < \kappa \mid \alpha < g_\kappa(\beta)\}) = 1$. ∎

Tarski-Hanf Theorem. Let κ be measurable. Let $\mathcal{A}(x)$ be any Π_1^{ZFC} formula (that is, any formula equivalent in ZFC to a formula of the form $\forall u_1, \dots, u_n \, \Phi(x, u_1, \dots, u_n)$, where Φ is built up from atomic formulas by propositional connectives and bounded quantifiers). Assume $\mathcal{A}(\kappa)$. Then there are κ ordinals $< \kappa$ for which $\mathcal{A}$ holds.

Proof. Form V^κ/μ and M as above. Since M is a transitive model of ZFC, and $\mathcal{A}$ is Π_1^{ZFC}, $M \models \mathcal{A}(\kappa)$. Let g_κ be as above. Since π is an isomorphism, $V^\kappa/\mu \models \mathcal{A}([g_\kappa])$. By Łoś's Theorem, $\mu(\{\alpha < \kappa \mid \mathcal{A}(g_\kappa(\alpha))\}) = 1$. Then $\{g_\kappa(\alpha) \mid \alpha < \kappa \wedge \mathcal{A}(g_\kappa(\alpha))\}$ is unbounded in κ. (If not, assume the set is bounded by $\tau < \kappa$. Then $[g_\kappa] \; E_\mu \; [c_\tau]$. So, $\kappa = \pi([g_\kappa]) \in \pi([c_\tau]) = \tau$, contradicting $\tau < \kappa$.) By regularity of κ, $\mathcal{A}$ holds for κ ordinals $< \kappa$.

Corollary. Let κ be measurable. Then there are κ inaccessible cardinals $< \kappa$, and there are κ Mahlo cardinals $< \kappa$. (The properties of being inaccessible and of being Mahlo are Π_1^{ZFC}.) Hence, the least inaccessible cardinal cannot be measurable.

Scott's Theorem. $V = L$ is incompatible with the existence of a real-valued measurable cardinal.

Proof. We assume familiarity with the basic properties of the class L of constructible sets. Let κ be the least real-valued measurable cardinal. By (U6), either κ is two-valued measurable or there is a real-valued measure on $2^{\aleph_0}$. But, by $V = L$, $2^{\aleph_0} = \aleph_1$. By (U2), there is no measure on $\aleph_1$. Hence, κ has a two-valued measure μ. Construct V^κ/μ. It is an elementary extension of V under the embedding $x \mapsto [c_x] = i(x)$. Hence, V^κ/μ is a well-founded model of ZFC. Mostowski's collapse π maps V^κ/μ isomorphically onto a transitive model M. Hence, since $V = L$ holds in the original

universe, it holds in V^κ/μ, and, therefore, also in M. But, $L^{(M)} = L$. Hence, $M = V^{(M)} = L^{(M)} = L = V$. We know that, for any formula φ,

$$\vdash \varphi^{(M)}(j(x_1),\ldots,j(x_n)) \Longleftrightarrow \varphi(x_1,\ldots,x_n).$$

Hence, since $M = V$,

$$(*) \quad \vdash \varphi(j(x_1),\ldots,j(x_n)) \Longleftrightarrow \varphi(x_1,\ldots,x_n).$$

We already know that $\alpha < \kappa \Longrightarrow j(\alpha) = \alpha$, and $j(\kappa) > \kappa$. We shall obtain a contradiction by proving that $j(\kappa) = \kappa$. Let $\varphi(x)$ be a formula asserting that x is the least measurable cardinal. By (*), $\vdash \varphi(j(\kappa)) \Longleftrightarrow \varphi(\kappa)$. But, $\vdash \varphi(\kappa)$. Hence, $\vdash \varphi(j(\kappa))$. Since $\vdash \varphi(x) \Longrightarrow x = \kappa$, we obtain $\vdash j(\kappa) = \kappa$. ∎

"Converse" of Scott's Theorem. If there is an elementary embedding $j\colon V \longrightarrow M$, where M is some transitive model and j is not the identity, then there exists a measurable cardinal.

Proof. Let x be a set of minimal rank such that $j(x) \neq x$. Now, $u \in x \Rightarrow j(u) \in j(x) \Rightarrow u \in j(x)$, since $j(u) = u$. Hence, $x \subseteq j(x)$. So, $rk(x) \leq rk(j(x))$. Assume $rk(x) = rk(j(x))$. Assume $z \in j(x)$. Then $rk(z) < rk(x)$. Hence, $j(z) = z$. So, $j(z) \in j(x)$. Then $z \in x$. Thus, $j(x) \subseteq x$. Therefore, $x = j(x)$, contradicting $j(x) \neq x$. Hence, $rk(x) < rk(j(x)) = j(rk(x))$. (The fact that $rk(j(x)) = j(rk(x))$ follows from the absoluteness of "rk" for transitive models: $\beta = rk(x) \Longleftrightarrow j(\beta) = rk^{(M)}(j(x)) \Longleftrightarrow j(\beta) = rk(j(x))$. Hence, $j(rk(x)) = rk(j(x))$.) Now, let $\kappa = rk(x)$. Then $\kappa < j(\kappa)$. Let F be the set $\{x \mid x \subseteq \kappa \wedge \kappa \in j(x)\}$. Then F is a κ-complete non-principal ultrafilter on κ, and, therefore, κ is measurable. (To show that F is a κ-complete non-principal ultrafilter, the hard part is the κ-completeness. So, assume f is a function with domain $\tau < \kappa$ and range $\subseteq F$. Let

$z = \bigcap_{\alpha<\tau} f(\alpha)$. We must prove $z \in F$. Clearly, $z \subseteq \kappa$. We want to prove $\kappa \in j(z)$. It suffices to prove: (%) $\bigcap_{\alpha<\tau} (j(f))(\alpha) \subseteq j(z)$. (For, since j is an embedding and $j(\tau) = \tau$, $j(f)$ is a function with domain τ. For $\alpha < \tau$, $\langle \alpha, f(\alpha) \rangle \in f$. Hence, since $j(\alpha) = \alpha$, $\langle \alpha, j(f(\alpha)) \rangle \in j(f)$, that is, $(j(f))(\alpha) = j(f(\alpha))$. By hypothesis, $\kappa \in j(f(\alpha))$. Hence, $\kappa \in (j(f))(\alpha)$. Therefore, $\kappa \in \bigcap_{\alpha<\tau} (j(f))(\alpha)$. By (%), $\kappa \in j(z)$.)

(%) follows from the fact that j is an embedding. In V, we have: $(\forall u)[(\forall\alpha)(\alpha < \tau \Rightarrow u \in f(\alpha)) \Rightarrow u \in z]$. Hence, in M, $(\forall u)[(\forall\alpha)(\alpha < \tau \Rightarrow u \in (j(f))(\alpha)) \Rightarrow u \in j(z)]$. (Here, we use $j(\tau) = \tau$.) Hence, $\bigcap_{\alpha<\tau} (j(f))(\alpha) \subseteq j(z)$. ∎

AXIOMATIC CONSIDERATIONS.

Let (In) stand for the assertion that there exists an inaccessible cardinal.

Theorem. If ZF is consistent, then ZFC + $\neg$(In) is consistent, that is, the existence of inaccessible cardinals cannot be proved in ZFC.

Proof. Assume (In) provable in ZFC. Let κ be the least inaccessible cardinal. V_κ is a model of ZFC. But, for any $\alpha \in V_\kappa$,

$$(\alpha \text{ is inaccessible})^{(V_\kappa)} \Longleftrightarrow \alpha \text{ is inaccessible.}$$

Hence, $\neg(\text{In})^{(V_\kappa)}$. But, if (In) is provable in ZFC, $(\text{In})^{(V_\kappa)}$. Hence, ZFC would be inconsistent. ∎

Strong axioms of infinity usually are preserved in standard independence proofs.

Examples. (1) If ZF + (In) is consistent, so is ZF + (In) + V = L. Proof: κ is inaccessible $\Rightarrow$ $(\kappa \text{ is inaccessible})^{(L)}$. ∎

(2) If ZF + $(\exists\kappa)\text{Mahlo}(\kappa)$ is consistent, so is ZF + $(\exists\kappa)\text{Mahlo}(\kappa)$ + V = L. Proof: Assume $A \subseteq \kappa$, $A \in L$, and $(A \text{ is club in } \kappa)^{(L)}$. Then A is club in κ. By $\text{Mahlo}(\kappa)$, A contains an inaccessible cardinal τ. Then, $(\tau \in A \wedge \tau \text{ is inaccessible})^{(L)}$. ∎

Whether (In) is consistent with ZFC is an open problem. (If ZFC is ω-consistent, then the relative consistency of ZFC + (In) is not provable in ZFC. For, assume $\text{Con}_{ZFC} \rightarrow \text{Con}_{ZFC + (In)}$ provable in ZFC. Since V_κ is a model of ZFC when κ is inaccessible, Con_{ZFC} is provable in ZFC + (In). Hence, $\text{Con}_{ZFC + (In)}$ is provable in ZFC + (In). By Gödel's Second Incompleteness Theorem, ZFC + (In) would be inconsistent. So, $\neg$(In) would be provable in ZFC. Then, $\neg\,\text{Con}_{ZFC + (In)}$ would be provable in ZFC, and, therefore, so would $\neg\,\text{Con}_{ZFC}$. But then, by part of Gödel's First Incompleteness Theorem, ZFC would be ω-inconsistent.) Solovay proved in 1970 that, if the existence of non-Lebesgue measurable sets is provable in ZF + DC, then (In) is disprovable in ZFC. The converse has been recently proved by Shelah.

REFLECTION PRINCIPLES.

Strong axioms of infinity often are related to reflection principles of one sort or another. A naive reflection principle would state that whatever holds for the universe of all sets must also hold for some set. For example, the universe is infinite; therefore, there must be an infinite set. The universe satisfies the standard set-theoretic axioms. Hence, there must be a set with the same property. Such a set must be of the form V_κ, where κ is an inaccessible cardinal. (This is not quite the whole story. If the axioms are those of the theory NBG, it is accurate; but it is not correct if the axioms are those of ZF.) Hence, there must exist inaccessible cardinals. This way of obtaining large cardinals has been made more precise by Lévy [1960] and Bernays [1961]. The argument can be extended further. The class On of all ordinals is an "inaccessible" greater than any given ordinal α; hence, there must be an inaccessible cardinal $> \alpha$. Therefore, there are arbitrarily large

inaccessible cardinals. Moreover, since there are "On-many" inaccessibles $<$ On, there must be an inaccessible cardinal κ preceded by κ inaccessibles (that is, a hyperinaccessible). Further, assume F is a normal function from On into On. Then $F(\alpha) \geq \alpha$ for all α. We can imagine that F(On) is defined. Then F(On) $\geq$ On and F(On) $\leq$ On. Hence, F(On) = On, that is, On is a fixed point of every normal function. So, there must be an ordinal κ with the same property (relative to normal functions on κ): every normal function on κ has an inaccessible fixed point. Then κ would be a Mahlo cardinal.

A reflection principle can be proved for a finite number of formulas: There are arbitrarily large V_β's for which a given finite number of formulas are absolute. Let us make this more precise.

Tarski-Vaught Lemma. Let M_1 and M_2 be classes such that $M_1 \subseteq M_2$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be a list of formulas closed under taking of subformulas. Then $\varphi_1,\ldots,\varphi_n$ are absolute for M_1, M_2 if and only if, for any formula in the list of the form $(\exists u)\,\varphi_i(u,x_1,\ldots,x_k)$,

$$(\forall x_1,\ldots,x_k\in M_1)\,[(\exists u\in M_2)\,\varphi_i^{(M_2)}(u,x_1,\ldots,x_k)$$
$$\Rightarrow (\exists u\in M_1)\,\varphi_i^{(M_2)}(u,x_1,\ldots,x_k)]\ .$$

(A formula $\varphi(x_1,\ldots,x_r)$ is said to be absolute for M_1, M_2 if :

$$(\forall x_1,\ldots,x_r\in M_1)\,[\varphi^{(M_1)}(x_1,\ldots,x_r)\Leftrightarrow \varphi^{(M_2)}(x_1,\ldots,x_r)].)$$

Proof. $\Rightarrow$ is obvious. $\Leftarrow$ follows easily by induction on the number of connectives and quantifiers in the formula. ■

Reflection Principle for ZF. Let $\varphi_1,\ldots,\varphi_n$ be any formulas whose variables occur among the list of variables $x_1,\ldots,x_k$.

$$\vdash_{ZF} (\forall\alpha)(\exists\beta>\alpha)(\forall x_1,\ldots,x_k\in V_\beta)$$
$$[\varphi_1\Leftrightarrow\varphi_1^{(V_\beta)}\wedge\ldots\wedge\varphi_k\Leftrightarrow\varphi_k^{(V_\beta)}]\ .$$

Proof. We follow the argument in Kunen[1980]. We may assume the list $\varphi_1,\ldots,\varphi_n$ closed with respect to taking of subformulas. (If not, add to the list all subformulas of formulas in it.) For each φ_i, define a function $\Phi_i:\mathrm{On}\to\mathrm{On}$ in the following manner. If φ_i is not an existentional formula, let $\Phi_i(\alpha)=0$ for all α. If φ_i is $(\exists u)\,\varphi_j(u,x_1,\ldots,x_k)$, let $\Psi_i(x_1,\ldots,x_k)$ be the least τ such that $(\exists u\in V_\tau)\,\varphi_j^{(V_\tau)}(u,x_1,\ldots,x_k)$, or let it be 0 if there is no such τ. Let $\Phi_i(\alpha)=\sup\{\Psi_i(x_1,\ldots,x_k)\mid x_1,\ldots,x_k\in V_\alpha\}$. If we can find a limit ordinal β for which $(\forall\alpha<\beta)(\Phi_i(\alpha)<\beta)$, then the Tarski-Vaught Lemma implies that $\varphi_1,\ldots,\varphi_n$ are absolute for V_β. Define a denumerable sequence $\{\beta_r\mid r<\omega\}$ as follows: $\beta_0=\alpha$ and $\beta_{r+1}=\max(\beta_r+1,\Phi_1(\beta_r),\ldots,\Phi_n(\beta_r))$. Let $\beta=\sup_{r<\omega}\beta_r$. If $\alpha<\beta$, $\alpha<\beta_r$ for some r. Then $\Phi_i(\alpha)\le\Phi_r(\beta_r)\le\beta_{r+1}<\beta$. ■

The Reflection Principle can be extended in various ways. (1) Instead of using the natural models V_β, one can use any continuous, increasing sequence $\{A_\beta\}_{\beta \in On}$ and then one obtains absoluteness for $(A_\beta, \bigcup_{\alpha \in On} A_\alpha)$ instead of (V_β, V). An example would be the constructible hierarchy $\{L_\beta\}$. (2) One can assume that V_β is chosen to contain any given finite number of sets $u_1,\ldots,u_m$. Simply take β greater than α and the maximum of the ranks of $u_1,\ldots,u_m$.

Observe that the Reflection Principle implies that no consistent extension of ZF (including ZF itself) is finitely axiomatizable.

The Reflection Principle yields some of the axioms of ZF if applied to weaker theories. For example, if we assume the Reflection Principle for the theory ZF - (AI), then (AI) is derivable. (Use the formula $x \in On \Rightarrow (\exists\delta)(\delta \in On \wedge x \in \delta)$.) Likewise, the Replacement Schema can be derived from a Reflection Principle for the theory obtained from ZF by substituting the Selection Schema for the Replacement Schema. (Given a

functional relation $\psi(x,y)$ and a set z, let $\alpha = rk(z)$. Apply the Reflection Principle to the formulas $\psi(x,y)$ and $(\exists y)\psi(x,y)$. One can prove $(\exists\beta)[z \in V_\beta \wedge (\forall x)(x \in z \wedge (\exists y)\psi(x,y) \Rightarrow (\exists y \in V_\beta)\psi(x,y))]$, and then one uses the Selection Schema. Since the existence of the V_β's depends upon the Replacement Schema, one has to include within the Reflection Principle a description of V_β.)

The Reflection Principle for ZF suggested to Lévy the possibility of strengthening the principle so as to obtain strong axioms of infinity. Let $Scm^{ZF}(u)$ assert that u is a transitive model of ZF with the Replacement Schema replaced by a "second-order" axiom:

$$(\forall z)(\forall f)[(\forall t,v,w)(\langle t,v\rangle \in f \wedge \langle t,w\rangle \in f \Rightarrow v=w)$$
$$\Rightarrow (\exists y)(\forall x)(x \in y \Leftrightarrow (\exists t)(t \in z \wedge \langle t,x\rangle \in f))].$$

Lévy's Schema (N). For any formula $\mathcal{Q}$ with free variables among $x_1,\ldots,x_k$,

$$(\exists u)(Scm^{ZF}(u) \wedge (\forall x_1,\ldots,x_k)(x_1,\ldots,x_k \in u \Rightarrow$$
$$[\mathcal{Q} \Leftrightarrow \mathcal{Q}^{(u)}])).$$

(This can be shown to be equivalent to a principle in which there may be several $\mathcal{Q}_i$'s and in which u is required to contain a given set.)

Lévy's Schema (N) is equivalent to each of the following.

(N''') $(\exists\alpha)(\mathrm{Scm}^{ZF}(V_\alpha) \wedge (\forall x_1,\ldots,x_k)$
$(x_1,\ldots,x_k \in V_\alpha \Rightarrow [\mathcal{Q} \Longleftrightarrow \mathcal{Q}^{(V_\alpha)}]))$.

(Note that, in the presence of (AC), $\mathrm{Scm}^{ZF}(V_\alpha)$ is equivalent to α being inaccessible.)

(M) (Mahlo's Schema) Every normal function on On has inaccessible cardinals in its range.

From (M) one can prove the existence of Mahlo cardinals of every order.

Bernays [1961] simplified Lévy's approach. He adopted a partial reflection schema:

(*) $\mathcal{A} \Rightarrow (\exists y)(\mathrm{Trans}(y) \wedge \mathcal{A}^{(y)})$,

where $\mathcal{A}$ is allowed to have class quantifiers. From (*), together with Extensionality, Selection, Regularity, and Choice, he derived all of Morse-Kelly set theory plus Mahlo's Schema (M).

INFINITARY LANGUAGES.

Let α and β be infinite cardinals. Let $\mathcal{L}_{\alpha,\beta}$ be the ordinary language of ZF set theory,

augmented by additional variables and relation, function, and constant symbols, and permitting formation of infinite conjunctions and disjunctions and prefixing of infinitely many quantifiers of the same type. More precisely, for any $\tau < \alpha$, we allow conjunctions $\bigwedge_{\gamma<\tau} \mathcal{Q}_\gamma$ and disjunctions $\bigvee_{\gamma<\tau} \mathcal{Q}_\gamma$ of a sequence of τ formulas ; for any $\delta < \beta$, we can prefix to any formula a δ-sequence of all universal quantifiers or all existential quantifiers, obtaining formulas of the form $(\forall y_\gamma)_{\gamma<\delta}\mathcal{Q}$ and $(\exists y_\gamma)_{\gamma<\delta}\mathcal{Q}$. In particular, the language $\mathcal{L}_{\omega,\omega}$ is the ordinary first-order language, with only finite conjunctions and disjunctions and finite sequences of quantifiers.

It is well-known that the language $\mathcal{L}_{\omega,\omega}$ satisfies the Compactness Theorem: If every finite subset of a set $\mathcal{S}$ of sentences has a model, then $\mathcal{S}$ has a model. (This is a direct consequence of the Completeness Theorem: Every consistent theory has a model.) It is natural to wonder to what extent the Compactness Theorem generalizes to infinitary languages.

Let κ be a non-denumerable cardinal. Then κ is said to be weakly compact if, for any set $\mathcal{S}$ of sentences of $\mathcal{L}_{\kappa,\kappa}$ such that $\overline{\overline{\mathcal{S}}} \leq \kappa$, if every subset of $\mathcal{S}$ of cardinality $< \kappa$ has a model, then $\mathcal{S}$ has a model. κ is said to be strongly compact if, for any set $\mathcal{S}$ of sentences of $\mathcal{L}_{\kappa,\kappa}$, if every subset of $\mathcal{S}$ of cardinality $< \kappa$ has a model, then $\mathcal{S}$ has a model. Clearly, strong implies weak compactness.

No cardinal $\aleph_{\gamma+1}$ is weakly compact.

Proof. Let $\sigma = \aleph_\gamma$ and $\tau = \aleph_{\gamma+1}$. Add to $\mathcal{L}_{\tau,\tau}$ individual constants $(a_\alpha)_{\alpha<\tau}$ and $(b_\beta)_{\beta<\sigma}$.

Consider the following set $\mathcal{S}$ of τ formulas.

$$\mathcal{S} = \{ \bigvee_{\beta<\sigma} a_\alpha = b_\beta \mid \alpha < \tau \} \cup$$
$$\{ a_{\alpha_1} \neq a_{\alpha_2} \mid \alpha_1 \neq \alpha_2 ; \alpha_1, \alpha_2 < \tau \} .$$

Every subset of $\mathcal{S}$ of cardinality $< \tau$ has a model, but $\mathcal{S}$ has no model. ■)

No singular cardinal κ is weakly compact.

(Proof. Let WO stand for the formula

$$(\forall x,y)(x<y \vee y<x \vee x=y) \wedge \neg (\exists x_n)_{n<\omega} (\bigwedge_{n<\omega} (x_{n+1} < x_n)).$$

WO holds in a model when and only when $<$ is a well-ordering of the model. Let $A_\alpha(x)$ stand for

$$(\exists x_\gamma)_{\gamma<\alpha} [\bigwedge_{\delta<\alpha} \bigwedge_{\rho<\delta} (x_\rho < x_\delta) \wedge (\forall z)(z<x \Rightarrow \bigvee_{\delta<\alpha} (z=x_\delta)].$$

If $<$ well-orders the model, $A_\alpha(x)$ holds when and only when x is the α th element in the well-ordering. Since κ is singular, we can write $\kappa = \bigcup_{\gamma<\tau} \nu_\gamma$, $\nu_\gamma < \kappa$, $\tau < \kappa$. Let B be a new monadic predicate letter. Consider the following set $\mathcal{S}$ of κ formulas.

$$\{WO\} \cup \{ \bigvee_{\gamma<\tau} (\exists x_\delta)_{\delta<\nu_\gamma} [(\forall z)(B(z) \Rightarrow \bigvee_{\delta<\nu_\gamma} (z=x_\delta))] \}$$
$$\cup \{ (\exists z)(B(z) \wedge A_\alpha(z)) \mid \alpha < \kappa \} .$$

$\mathcal{S}$ has no model, but every subset of $\mathcal{S}$ of cardinality $< \kappa$ has a model. ■)

Thus, every weakly compact cardinal is weakly inaccessible.

Here are some of the most important properties of weakly and strongly compact cardinals.

1) Every weakly compact cardinal is weakly Mahlo, weakly hyperMahlo, etc.

2) (Hanf) Every weakly compact cardinal which is inaccessible also is Mahlo. The first strongly inaccessible cardinal is not weakly compact.

3) Every measurable cardinal κ is weakly compact (and, in fact, there are κ weakly compact cardinals $< \kappa$).

4) For regular cardinals κ , κ is strongly compact if and only if any κ-complete filter on a set can be extended to a κ-complete ultrafilter on that set.

Corollary of 4). If κ is strongly compact, κ is measurable. (Proof. Let F be the filter of all subsets of κ with complements of power $< \kappa$. Extend F to a κ-complete ultra-filter F^* on κ . F^* is clearly non-principal and determines a two-valued measure on κ . ■)

On the other hand, if the existence of measurable

cardinals is consistent with ZFC, then it is consistent with ZFC to assume the existence of a measurable cardinal which is not strongly compact. So, strong compactness is stronger than measurability. Nevertheless, contrary to what usually happens with large cardinals, Magidor has shown that it is possible for the least measurable cardinal to be strongly compact.

5) (Solovay) If κ is strongly compact, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every singular strong limit cardinal $\aleph_\alpha > \kappa$. (A cardinal τ is said to be a strong limit cardinal if $\sigma < \tau \Rightarrow 2^\sigma < \tau$.) Hence, the existence of a strongly compact cardinal implies the existence of a proper class of cardinals for which the generalized continuum hypothesis holds. This seems to be the only known case in which a strong axiom of infinity has some effect on the GCH (cf. Drake[1974], pp. 306-310).

6) If κ is strongly inaccessible, then weak compactness of κ implies that every totally ordered set of cardinality κ has a subset of power κ such that either it has order type κ or order type the inverse of κ . Conversely,

this property implies that κ is strongly inaccessible and weakly compact.

7) Weak compactness does not contradict $V = L$. In fact, if κ is weakly compact, then κ is weakly compact in L (cf. Jech[1978], p. 389).

INFINITE COMBINATORICS.

Much of the most interesting work on large cardinals revolves around the notions of partition relations and trees.

For any set X of ordinals and any ordinal γ, let $[X]^\gamma = \{Y \subseteq X \mid Y \text{ has order type } \gamma\}$. The basic partition relation $\alpha \rightarrow (\beta)^\gamma_\delta$ asserts that, if $f: [\alpha]^\gamma \rightarrow \delta$, then there exists W in $[\alpha]^\beta$ which is homogeneous for f, that is, f takes the same value on every member of $[W]^\gamma$.

The classical result is due to Ramsey (1929). Remember that $[X]^n$ stands for all n-element subsets of X.

Ramsey's Theorem. If X is infinite and $[X]^n$ is partitioned into k subsets $D_1, \ldots, D_k$, then X contains an infinite subset Y such that $[Y]^n \subseteq D_i$ for some i.

Proof. It suffices to prove this for $k = 2$. The proof is by induction on n. It is obvious for $n = 1$, by the infinite pigeon-hole principle. Now assume the theorem is true for n and for any infinite X. Assume $[X]^{n+1} = D_L \cup D_R$ with

$D_L \cap D_R = \emptyset$. For each a in X, define
$X_L(a) = \{W \in [X]^n \mid W \cup \{a\} \in D_L\}$ and
$X_R(a) = \{W \in [X]^n \mid W \cup \{a\} \in D_R\}$. Then
$[X - \{a\}]^n = X_L(a) \cup X_R(a)$ and $X_L(a) \cap X_R(a) = \emptyset$.
By inductive hypothesis, there exists an infinite set $Z \subseteq X - \{a\}$ for which either $[Z]^n \subseteq X_L(a)$ or $[Z]^n \subseteq X_R(a)$.

Process L. Define a sequence of infinite sets $L_0 \supseteq L_1 \supseteq \ldots$ and a sequence of elements $a_0, a_1, \ldots$, as follows: First, let $L_0 = X$. Given L_i, choose (if possible) a_i in L_i for which there exists an infinite $Z \subseteq L_i$ such that $[Z]^n \subseteq X_L(a_i)$. Let $L_{i+1} = Z$. If this process continues for all $i < \omega$, let $Y = \{a_i \mid i < \omega\}$. Then $[Y]^{n+1} \subseteq D_L$. If process L comes to a halt with some set L_m, begin the following new process.

Process R. Choose a sequence of infinite sets $R_0 \supseteq R_1 \supseteq \ldots$ and a sequence of elements $b_0, b_1, \ldots$ in the following manner. Let $R_0 = L_m$. Given R_j, choose $b_j \in R_j$ such that there exists an infinite set $Z \subseteq R_j$ for which $[Z]^n \subseteq X_R(b_j)$, and let $R_{j+1} = Z$. (There always exists such a set Z. In fact, $R_j \subseteq L_m$ and R_j is infinite. Choose any $b_j \in R_j$. $[R_j - \{b_j\}]^n = X_L(b_j) \cup X_R(b_j)$. By inductive hypothesis, there exists infinite $Z \subseteq R_j$ for which either $[Z]^n \subseteq X_L(b_j)$ or $[Z]^n \subseteq X_R(b_j)$. Since process L broke off at L_m and $R_j \subseteq L_m$, we must have $[Z]^n \subseteq X_R(b_j)$.) Now, let $Y = \{b_j \mid j < \omega\}$. Then $[Y]^{n+1} \subseteq D_R$. ∎

A special case of Ramsey's Theorem is the partition relation $\omega \rightarrow (\omega)^n_k$ for all positive integers n and k.

Exercises. 1. The countable AC implies Ramsey's Theorem. 2. Ramsey's Theorem implies the following weak axiom of choice: For any infinite set of finite non-empty sets, there is a choice function that selects one element from each

of infinitely many of the finite sets.

Now let us look for a moment at trees. A tree is a partially ordered set $\langle T, <_T \rangle$ such that, for each t in T, the initial segment $\{s \in T \mid s <_T t\}$ is well-ordered. For each t in T, the tree rank $r_T(t)$ is the order type of $\{s \in T \mid s <_T t\}$. For any ordinal α, the α^{th} level of T is the set of all elements of T of rank α, and we set $T_\alpha = \{t \in T \mid r_T(t) < \alpha\}$. By the height of T we mean the least α such that $T = T_\alpha$. A branch of T is a maximal chain in T. (Its <u>length</u> is its order type.) An antichain of T is a set of pairwise incomparable elements. (Every level is an antichain.) T is finitary if every element of T has only finitely many immediate successors and T has only finitely many minimal elements.

A classical result about trees is due to König (1927).

König's Lemma. Any infinite finitary tree has an infinite branch.

Proof. It follows easily by induction that all finite levels have only finitely many elements. Choose t_0 in T_0 so that t_0 has infinitely many successors. Choose t_1 to be an immediate successor of t_0 with infinitely many successors, etc. Thus, $t_0 <_T t_1 <_T \dots$. Extend this ω-chain to a branch, by Zorn's Lemma. ∎

A special case of König's Lemma is that any infinite tree all of whose finite levels are finite has an infinite branch. This special case was the original form of the lemma.

König's Lemma is equivalent to the axiom of choice for a countable set of non-empty finite sets. König's Lemma also enables us to give another proof of Ramsey's Theorem, which we shall give here because the technique will be useful elsewhere.

Ramsey's Theorem Again. Assume $f: [X]^n \to k$, where X is infinite and $0 < k$, $n < \omega$. Then there exists a homogeneous infinite subset of X for f, that is, a subset $Y \subseteq X$ for which $f''[Y]^n$ consists of a single element.

Proof. (Drake) Well-order X: $\{x_\beta \mid \beta < \alpha\}$. We shall define a relation $\prec$ on X for which $\langle X, \prec \rangle$ is a tree. Let $x_\gamma \prec x_\beta$ mean: $\gamma < \beta$ and, for any $\gamma_0 < \dots < \gamma_{n-2}$ such that $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-2}} \prec x_\gamma$,

$$f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\gamma) = f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\beta).$$

$\prec$ is obviously a partial order. To prove that every initial segment $\{x_\tau \mid x_\tau \prec x_\beta\}$ is well-ordered, it suffices to show that, if $x_\delta \prec x_\beta$ and $x_\gamma \prec x_\beta$ and $\delta < \gamma$, then $x_\delta \prec x_\gamma$. [Use induction on δ. Assume $x_\rho \prec x_\beta$ and $\rho < \delta$ imply that x_ρ is comparable with any other predecessor of x_β. Assume $\gamma_0 < \dots < \gamma_{n-2}$ and $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-2}} \prec x_\delta$. By inductive hypothesis, $x_{\gamma_{n-2}} \prec x_\gamma$. So, $f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\delta) = f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\beta)$ and $f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\gamma) = f(x_{\gamma_0}, \dots, x_{\gamma_{n-2}}, x_\beta)$. Therefore, $x_\delta \prec x_\gamma$.]

Thus, $\langle X, \prec \rangle$ is a tree. Notice that $x_0 \prec x_\delta$ for all $\delta > 0$; $x_1 \prec x_\delta$ for all $\delta > 1$; $\dots$; $x_{n-2} \prec x_\delta$ for all $\delta > n-2$. Hence,

$x_0 \prec x_1 \prec x_2 \prec \dots \prec x_{n-2}$ are the first n-1 levels.

Assume now that x_γ is at level σ, $\sigma \geq n-2$. Assume x_{β_1}, x_{β_2} are distinct immediate successors of x_γ, with $\beta_1 < \beta_2$. Assume also that $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-2}} \prec x_\gamma$. Then

$f(x_{\gamma_0},\dots,x_{\gamma_{n-2}},x_{\beta_1}) = f(x_{\gamma_0},\dots,x_{\gamma_{n-2}},x_\gamma) =$

$f(x_{\gamma_0},\dots,x_{\gamma_{n-2}},x_{\beta_2})$. But, $x_{\beta_1} \not\prec x_{\beta_2}$. Hence, there exist $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-3}} \prec x_\gamma$ such that

$f(x_{\gamma_0},\dots,x_{\gamma_{n-3}},x_\gamma,x_{\beta_1}) \neq f(x_{\gamma_0},\dots,x_{\gamma_{n-3}},x_\gamma,x_{\beta_2})$.

The number of sequences $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-3}} \prec x_\gamma$ is at most $\overline{\overline{[\sigma]^{n-2}}}$. Hence, the number of possible values of $f(x_{\gamma_0},\dots,x_{\gamma_{n-3}},x_\gamma,x_\beta)$ is $k^{\overline{\overline{([\sigma]^{n-2})}}}$. This is the maximum number of immediate successors of x_γ. When σ is finite, this is finite, and, therefore, all finite levels are finite.

Apply König's Lemma to the tree $\langle X_\omega, \prec \rangle$ (just all finite levels). Hence, there is an

infinite branch B in $\langle X_\omega, \prec \rangle$. For any $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-2}}$, let $g(x_{\gamma_0},\dots,x_{\gamma_{n-2}}) =$ $f(x_{\gamma_0},\dots,x_{\gamma_{n-2}},x_\gamma)$ for any $x_\gamma \in B$ with $x_{\gamma_{n-2}} \prec x_\gamma$. Thus, $g: [B]^{n-1} \to k$. By induction on n, there is an infinite $Y \subseteq B$ which is homogeneous for g. Then Y is also homogeneous for f. For, if we assume $x_{\gamma_0} \prec \dots \prec x_{\gamma_{n-1}}$ in Y, then $f(x_{\gamma_0},\dots,x_{\gamma_{n-1}}) = g(x_{\gamma_0},\dots,x_{\gamma_{n-2}})$, which is constant on Y. ■

Other applications of trees to partition relations:

1) $\aleph_1 \to (\omega + 1)^n_k$ for $0 < m$, $k < \omega$.

(Outline of proof: Assume $f: [X]^n \to k$ with $X = \aleph_1$. Form a tree $\langle X, \prec \rangle$ as in the proof above of Ramsey's Theorem. Since all finite levels are finite, $X_\omega \subsetneq X$. Let $Y = \{u \mid u \prec y\}$, where y is some element of X of tree rank $\geq \omega$. For $u_0 \prec \dots \prec u_{n-2} \prec u_{n-1}$ in Y, $f(u_0,\dots,u_{n-2},u_{n-1}) = f(u_0,\dots,u_{n-2},y)$, since $u_{n-1} \prec y$. Let $g(u_0,\dots,u_{n-2}) = f(u_0,\dots,u_{n-2},y)$

if $u_0 \prec \dots \prec u_{n-2}$ in Y. So, $g: [Y]^{n-1} \to k$.

By Ramsey's Theorem, there exists $Y' \subseteq Y$ which is homogeneous for g. Then Y' has order type $\geq \omega$; $Y' \cup \{y\}$ has order type $\geq \omega + 1$ and is homogeneous for f. For, if $u_0 \prec \dots \prec u_{n-1}$ and $v_0 \prec \dots \prec v_{n-1}$ in Y', then $f(u_0,\dots,u_{n-1}) = f(u_0,\dots,u_{n-2},y) = g(u_0,\dots,u_{n-2}) = g(v_0,\dots,v_{n-2}) = f(v_0,\dots,v_{n-2},y) = f(v_0,\dots,v_{n-2},v_{n-1})$.)

2) Erdös–Rado Theorem: For τ an infinite cardinal and $n < \omega$, $(\beth_n(\tau))^+ \to (\tau^+)^{n+1}_\tau$.

Here, κ^+ denotes the next cardinal after κ, and the relative beth sequence is defined by: $\beth_0(\tau) = \tau$, $\beth_{\gamma+1}(\tau) = 2^{\beth_\gamma(\tau)}$, and, for limit λ, $\beth_\lambda(\tau) = \bigcup_{\gamma<\lambda} \beth_\gamma(\tau)$. (Cf. Drake[1974, p. 207)

Notation: $\alpha \to (\beta)^n_2$ is usually abbreviated by $\alpha \to (\beta)^n$, that is, we omit the subscript 2.

One of the most interesting partition properties is $\kappa \to (\kappa)^2$. To study it, we need a preliminary result.

Theorem. The lexicographically ordered set ${}^{\kappa}2$ has no increasing or decreasing κ^+-sequences.

Proof. Assume that $Y = \{x_\alpha \mid \alpha < \kappa^+\}$ is an increasing κ^+-sequence in ${}^{\kappa}2$. Let $\gamma \leq \kappa$ be the least γ such that the set of restrictions $\{x_\alpha \restriction \gamma \mid \alpha < \kappa^+\}$ has power κ^+. Let $Y^* \subseteq Y$ be chosen so that $\overline{\overline{Y^*}} = \kappa^+$ and $u \restriction \gamma \neq v \restriction \gamma$ for $u \neq v$ in Y^*. We may assume that $Y^* = Y$. For $\alpha < \kappa^+$, let δ_α be the least ordinal where x_α differs from $x_{\alpha+1}$. (Then $x_\alpha(\delta_\alpha) = 0$ and $x_{\alpha+1}(\delta_\alpha) = 1$.) Obviously, $\delta_\alpha < \gamma$. So, by a pigeonhole argument, there exists $\sigma < \gamma$ such that $\sigma = \delta_\alpha$ for κ^+ members x_α of Y. But, if $x_\alpha \restriction \sigma = x_\beta \restriction \sigma$ for α and β such that $\sigma = \delta_\alpha = \delta_\beta$, then $x_\beta < x_{\alpha+1}$ and $x_\alpha < x_{\beta+1}$, which implies that $x_\alpha = x_\beta$. So, $\{x_\alpha \restriction \sigma \mid \alpha < \kappa^+\}$ has power κ^+, contradicting the minimality of γ. A similar proof holds for decreasing sequences. ∎

Theorem. If $\kappa > \omega$ and $\kappa \to (\kappa)^2$, then κ is inaccessible.

Proof. (a) κ is regular. (Assume $\kappa = \bigcup \{D_\gamma \mid \gamma < \tau\}$, where the D_γ are pairwise disjoint, $\overline{\overline{D_\gamma}} < \kappa$ for all $\gamma < \tau$, and $\tau < \kappa$. Let $f: [\kappa]^2 \to \{0,1\}$ be defined by setting $f(\{\alpha,\beta\}) = 0$ when α and β belong to the same D_γ. There is clearly no homogeneous set of power κ.)

(b) Assume $\tau < \kappa$ and $\kappa \leq 2^\tau$. Since $\kappa \to (\kappa)^2$, $\kappa \to (\tau^+)^2$, and, therefore, $2^\tau \to (\tau^+)^2$. Now, let $\{g_\alpha \mid \alpha < 2^\tau\}$ be an enumeration of ${}^\tau 2$, and let $<$ denote the lexicographic ordering on ${}^\tau 2$. Define $f: [2^\tau]^2 \to 2$ by setting $f(\{\alpha,\beta\}) = 1$ when $(\alpha \in \beta \Longleftrightarrow g_\alpha < g_\beta)$. By $2^\tau \to (\tau^+)^2$, let $Y \subseteq 2^\tau$ be a homogeneous set for f of order type τ^+. Then $\{g_\alpha \mid \alpha \in Y\}$ would be an increasing or decreasing τ^+-sequence in the lexicographically ordered set ${}^\tau 2$, contradicting the theorem above. ∎

Let us consider some related concepts involving trees. By a κ-tree we mean a tree of height κ such that every level has cardinality $< \kappa$. κ is said to have the tree property if every

κ-tree has a branch of length κ. König's Lemma tells us that ω has the tree property. Aronszajn proved that ω_1 does not have the tree property.

Theorem. If $\kappa \to (\kappa)^2$, then κ has the tree property.

Proof. Assume that $\langle T, <_T \rangle$ is a tree of height κ such that every level has power $< \kappa$. We may assume that $T = \kappa$. Extend $<_T$ to a total order $\prec$ on κ as follows: if α and β are not $<_T$-comparable, let δ be the first level at which the predecessors $\alpha_\delta, \beta_\delta$ differ, and set $\alpha \prec \beta$ when $\alpha_\delta < \beta_\delta$. Now define $f: [\kappa]^2 \to 2$ by letting $f(\{\alpha,\beta\}) = 1$ when $\prec$ agrees with $<$ on $\{\alpha,\beta\}$. Since $\kappa \to (\kappa)^2$, there is a set $Y \subseteq \kappa$ of power κ which is homogeneous for f. Let B be the set of all $\sigma < \kappa$ such that $\overline{\overline{\{\alpha \in Y \mid \sigma <_T \alpha\}}} = \kappa$. Since all levels have power $< \kappa$, B intersects every level. To show that B is a branch of length κ, it remains to show that any two elements σ and τ of B are $<_T$-comparable. Assume that σ and τ are incomparable and $\sigma \prec \tau$. σ and τ have

κ successors in B. Hence, there are $\alpha < \beta < \gamma$ in Y such that $\sigma <_T \alpha$, $\tau <_T \beta$, $\sigma <_T \gamma$. Then $\alpha \not< \beta$ and $\gamma \not< \beta$. Therefore, $f(\{\alpha,\beta\}) = 1$ and $f(\{\gamma,\beta\}) = 0$, contradicting the homogeneity of B. ∎

A partial converse is also true.

Theorem. If κ is inaccessible and has the tree property, then $\kappa \rightarrow (\kappa)^2$.

Proof. Assume $f\colon [\kappa]^2 \rightarrow 2$. Construct the following tree $\langle T, \subseteq \rangle$, where T consists of $\{0,1\}$-sequences with domains $< \kappa$. Construct T by induction: $T = \{t_0, t_1, \ldots, t_\beta, \ldots\}$. First, $t_0 = \emptyset$. Assume that $\alpha < \kappa$ and t_β has been constructed for all $\beta < \alpha$. Assume that the sequence t_α has been defined for all $\delta < \gamma$. If $t_\alpha \restriction \gamma$ is not equal to some t_β with $\beta < \alpha$, let $t_\alpha = t_\alpha \restriction \gamma$. If $t_\alpha \restriction \gamma = t_\beta$ for some $\beta < \alpha$, let $t_\alpha(\gamma) = f(\{\beta,\alpha\})$. $\overline{\overline{T}} = \kappa$. By inaccessibility of κ, every level of T has power $< \kappa$; so, the height of T is κ. By the tree property, let B be a branch of T of length κ. For any $t_\alpha \in B$, let δ_α = the domain of t_α. Define

$Y_0 = \{\alpha \mid t_\alpha \in B \text{ and } t_\alpha \cup \{\langle \delta_\alpha, 0\rangle\} \in B\}$, and $Y_1 = \{\alpha \mid t_\alpha \in B \text{ and } t_\alpha \cup \{\langle \delta_\alpha, 1\rangle\} \in B\}$. Then Y_0 and Y_1 are homogeneous for f. Since $\overline{\overline{Y_0 \cup Y_1}} = \kappa$, at least one of Y_0 or Y_1 has power κ. ∎

$\kappa \rightarrow (\kappa)^2$ also turns out to be equivalent to the following interesting property: every totally ordered set of power κ has a subset of power κ which is well-ordered or inversely well-ordered by the given ordering. (The implication in one direction is very easy. Assume $\prec$ totally orders κ. Define $f\colon [\kappa]^2 \rightarrow 2$ as follows: $f(\{x,y\}) = 0$ if $(x \in y \Leftrightarrow x \prec y)$. By $\kappa \rightarrow (\kappa)^2$, let C be a homogeneous subset of κ of power κ. On C, either $\prec$ agrees with $\in$ or $\prec$ agrees with the converse of $\in$.)

Another remarkable connection is the fact that the tree property is equivalent to weak compactness. Let us show that weak compactness implies the tree property. Assume $\langle T, <_T \rangle$ is a κ-tree. For each $t \in T$, let P_t be a monadic predicate letter. Let $\mathcal{S}$ be the following set of sentences.

$$\Big\{ \bigvee_{t \in \mathrm{level}\,\alpha} (\exists x) P_t(x) \;\Big|\; \alpha < \kappa \Big\} \cup$$

$$\Big\{ \neg((\exists x) P_{t_1}(x) \wedge (\exists x) P_{t_2}(x)) \;\Big|\; t_1 \text{ and } t_2 \text{ are not } <_T\text{-comparable} \Big\}.$$

Every subset of $\mathcal{S}$ of power $< \kappa$ has a model. Since κ is weakly compact, $\mathcal{S}$ has a model $\mathcal{M}$, which yields a branch $\{ t \in T \mid \mathcal{M} \models (\exists x) P_t(x) \}$ of length κ. ■

In an earlier discussion of the motivation for strong axioms of infinity, one of the basic intuitions mentioned was that whatever holds for ω should also hold for some larger cardinals. Naturally, this cannot be understood literally ; some restriction would have to be placed on the kinds of properties of ω we are referring to. In fact, there is an important property of ω that one might believe should extend to higher cardinals and which, as we shall now show, does not do so.

Ramsey's Theorem tells us that $\omega \longrightarrow (\omega)^{\alpha}$ for

every $\alpha < \omega$. But, when $\kappa > \omega$, $\kappa \longrightarrow (\kappa)^{\omega}$ is false. In fact, let us prove that $\kappa \not\longrightarrow (\omega)^{\omega}$ for $\kappa \geq \omega$. Proof. Let the relation R well-order the set $[\kappa]^{\omega}$. Define $f\colon [\kappa]^{\omega} \longrightarrow 2$ by letting, for every denumerable subset D of κ , $f(D) = 0$ when D precedes all denumerable proper subsets of D with respect to the ordering R. Assume there is a denumerable subset Y of κ which is homogeneous for f. Let D_0 be the least member of $[Y]^{\omega}$ with respect to the well-ordering R. Then $f(D_0) = 0$. By the homogeneity of Y, $f(D) = 0$ for all $D \in [Y]^{\omega}$. Since Y is denumerable, we can enumerate Y: $Y = \{ y_n \mid n < \omega \}$. Let $E = \{ y_{2n} \mid n < \omega \}$, and $Y_n = E \cup \{ y_1, y_3, \ldots, y_{2n+1} \}$. Then $Y_0 \subsetneq Y_1 \subsetneq \ldots$. Since $f(Y_n) = 0$ for all n, we have: $\ldots \; R \; Y_2 \; R \; Y_1 \; R \; Y_0$, contradicting the fact that R is a well-ordering. ■

There has been an explosion in recent years of types of large cardinals. We may soon exhaust the supply of adjectives available to name these concepts. Here is a short list of a few of the more important notions.

(A) Ramsey and Rowbottom Cardinals. Let $[\kappa]^{<\omega} = \bigcup_{n<\omega} [\kappa]^n$, the set of all finite subsets of κ. Let $\kappa \rightarrow (\alpha)^{<\omega}_{\tau}$ mean that, for any partition $f: [\kappa]^{<\omega} \rightarrow \tau$, there exists $Y \subseteq \kappa$ such that Y has order type α and, for $n < \omega$, f is constant on $[Y]^n$.

κ is called a Ramsey cardinal if $\kappa \rightarrow (\kappa)^{<\omega}_2$

Specker gave a clever example to show that ω is not a Ramsey cardinal. For any finite subset x of ω, let $f(x) = 0$ if $\overline{\overline{x}} \in x$ and let $f(x) = 1$ otherwise. Assume Y is a denumerable subset of ω which is "homogeneous" for f. Let $n < \omega$. If $f(x) = 0$ for all $x \subseteq Y$ such that $\overline{\overline{x}} = n$, then $n \in x$ for all such x. Since $\overline{\overline{Y}} = \omega$, this is impossible. Hence, $f(x) = 1$ for all n-element subsets of Y. Hence, $n \notin x$ for all such subsets. Therefore, $n \notin Y$. This implies that $Y = \emptyset$, which is a contradiction.

It can be shown that all measurable cardinals are Ramsey cardinals and that all Ramsey cardinals are weakly compact.

A cardinal κ is said to be a Rowbottom cardinal if $\kappa > \aleph_1$ and, for any $f: [\kappa]^{<\omega} \rightarrow \tau$ with $\tau < \kappa$, there must exist $Y \subseteq \kappa$ such that $\overline{\overline{Y}} = \kappa$ and the range of f restricted to $[Y]^{<\omega}$ is countable.

It turns out that every Ramsey cardinal is a Rowbottom cardinal. Moreover, the existence of Rowbottom cardinals implies that $\mathcal{P}(\omega) \cap L$ is denumerable. Hence, V = L is incompatible with large cardinal assumptions that are weaker than the existence of a measurable cardinal.

Let $\mathcal{L}$ be a countable first-order language with a designated monadic predicate P. In a model $\mathfrak{U}$, the set of elements satisfying P is denoted $P^{\mathfrak{U}}$. A model $\mathfrak{U} = \langle M, P^{\mathfrak{U}}, \ldots \rangle$ of $\mathcal{L}$ is said to have type $\langle \kappa, \tau \rangle$ if $\overline{\overline{M}} = \kappa$ and $\overline{\overline{P^{\mathfrak{U}}}} = \tau$. It can be proved that a cardinal κ

is a Rowbottom cardinal if and only if $\kappa > \aleph_1$ and, for $\aleph_1 < \tau < \kappa$, every model of type $<\kappa, \tau>$ has an elementary submodel of type $<\kappa, \aleph_0>$. Another result is that every Rowbottom cardinal is either inaccessible or has cofinality ω.

(B) Universal Algebra. An algebra is a structure consisting of a set A and finitary operations on A. An algebra is said to be a Jonsson algebra if it has no proper subalgebras of the same cardinality. A cardinal κ is a Jonsson cardinal if there is no Jonsson algebra of power κ. It is easy to see that ω is not a Jonsson cardinal. (Take $<\omega, f>$ with $f(n) = n-1$ for $n > 0$.) Every Rowbottom cardinal is also a Jonsson cardinal. Jonsson cardinals have a more general model-theoretic characterization: κ is a Jonsson cardinal if and only if every model with countable signature and power κ has a proper elementary substructure of power κ. It has been

shown that, if κ^+ is a Jonsson cardinal, then κ is also a Jonsson cardinal. Hence, no $\aleph_n$ with $n < \omega$ is a Jonsson cardinal. Whether $\aleph_\omega$ is a Jonsson cardinal still seems to be open.

(C) Indescribability. A cardinal κ is said to be Π^n_m-indescribable if, for any $A \subseteq V_\kappa$ and any Π^n_m-sentence φ such that $<V_\kappa, \in, A> \models \varphi$, there exists $\tau < \kappa$ for which $<V_\tau, \in, A \cap V_\tau> \models \varphi$. (Similarly for Σ^n_m.) Some of the lower levels of indescribability correspond to well-known large cardinal properties. For example, κ is inaccessible if and only if κ is Π^0_m-indescribable for all m, and κ is weakly compact if and only if κ is Π^1_1-indescribable. Every measurable cardinal is Π^2_1-indescribable ; but the least measurable cardinal is Σ^2_1-describable.

Other important classes are the supercompact and huge cardinals. (See Kanamori-Magidor [1978] and Jech [1978].)

We have sampled only the most elementary parts of the study of large cardinals. There is at the present time no general theory of large cardinals. Some first steps in the direction of such a theory have been suggested by Reinhardt [1974].

The relation of large cardinals and strong axioms of infinity to the foundations of mathematics is still problematical. At first, there seem to be strong reasons for rejecting large cardinals entirely (for example, by denying the existence of inaccessible cardinals). One such reason is simplicity ; all the large cardinal properties would be false for all cardinals. A second apparent reason is that the universe of sets should be unique ; the existence of an inaccessible cardinal would yield a set which is a model of ZF set theory (and even of the class theory of Bernays-Gödel). But these reasons are not at all conclusive. In his book, From Mathematics to Philosophy, Hao Wang describes (on pp. 189-190) some of Gödel's views on strong axioms of infinity

and, in particular, on reflection principles. Gödel believed, according to Wang, that "the universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the $\in$-relation in it, expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number". Gödel's Incompleteness Theorem is one of the prime pieces of evidence for this standpoint. According to this view, any particular collection of properties of the universe which we can formulate does not completely describe the universe and must also be satisfied by some part of the universe. This is the main theological basis for the faith in large cardinals.

BIBLIOGRAPHY

The basic reference for beginners is Drake[1974]. A more recent reference with excellent coverage is Jech[1978]. Lévy[1980] contains a good introduction to infinite combinatorics. An encyclopedic, but often entertaining, survey is Kanamori-Magidor [1978]. More technically oriented is the survey by Boos [1975].

Abramson, F.G., L.A. Harrington, E.M. Kleinberg, W.S. Zwicker. Flipping properties: a unifying thread in the theory of large cardinals. Ann. Math. Logic, 12, 1977, 37-58.

Bernays, P. Zur Frage der Unendlichkeitsschemata in der axiomatischen Mengenlehre, Essays on the Foundations of Mathematics, Jerusalem, 1961, 3-49.

Boos, W. Lectures on large cardinal axioms. Logic Conference, Kiel, 1974, Lecture Notes in Math., 499, 1975, 25-88.

Chang, C.C. and H.J. Keisler. Model Theory. North Holland, 1973.

Comfort, W.W. and S. Negrepontis. The Theory of Ultrafilters. 1974, Springer.

Devlin, K.J. Aspects of Constructibility. Lecture Notes in Math., 354, 1973.

Dickmann, M.A. Large Infinitary Languages. North Holland, 1976.

Drake, F.R. Set Theory. North Holland, 1974.

Erdös, P. and R. Rado. A partition calculus in set theory. Bull. A.M.S., 62, 1965, 427-489.

Erdös, P., A. Hajnal, R. Rado. Partition relations for cardinal numbers, Acta Math. Acad. Sci. Hung., 16, 1965, 93-196.

Friedman, H. Higher set theory and mathematical practice. Ann. Math. Logic, 2, 1971, 326-357.

Gaifman, H. Elementary embeddings of set theory. Axiomatic Set Theory , Proc. Symp. Pure Math. 13 (2), A.M.S., 1974, 33-101.

Hanf, W.P. Incompactness in languages with infinitely long expressions, Fund. Math. 53, 1964, 309-324.

Jech, T.J. Trees, J. Symb. Logic, 36, 1971, 1-14.

Jech, T.J. Set Theory. Academic, 1978.

Kanamori, A., M. Magidor. The Evolution of Large Cardinal Axioms in Set Theory, Higher Set Theory. Lecture Notes in Math., 669, 1978, 99-275.

Keisler, H.J. Model Theory for Infinitary Logic. North Holland, 1971

Keisler, H.J., A. Tarski. From accessible to inaccessible cardinals. Fund. Math. 53, 1964, 225-308.

Kleinberg, E.M. Infinitary combinatorics, Cambridge Summer School in Mathematical Logic , Lecture Notes in Math., 337, 1973, 361-418.

Kunen, K. Combinatorics. Handbook of Mathematical Logic. North Holland, 1977, 430-455.

Kunen, K. Set Theory. North Holland, 1980.

Lévy, A. Axiom schemata of strong infinity, Pacific J. of Math., 10, 1960, 223-238.

Lévy, A. The sizes of the indescribable cardinals, Axiomatic Set Theory, Proc. Symp. Pure Math., A.M.S. , 13 (1), 1971, 205-218.

Lévy, A. Basic Set Theory . North Holland, 1980.

Mahlo, P. Über lineare transfinite Mengen, Ber. Verhandl. Sächs. Akad. Wiss., Leipzig, Math. Naturw. Kl., 63, 1911, 187-225.

Reinhardt, W.N. Remarks on reflection principles, large cardinals, and elementary embeddings. Axiomatic Set Theory, Proc. Symp. Pure Math., 13 (2), A.M.S. , 1974, 189-205.

Rowbottom, F. Some strong axioms of infinity incompatible with the axiom of constructibility, Ann. Math. Logic, 3, 1971, 1-44.

Scott, D.S. Measurable cardinals and constructible sets, Bull. Acad. Pol. Sci., 7, 1961, 145-149.

Sierpinski, W., A. Tarski. Sur une propriété caractéristique des nombres inaccessible, Fund. Math., 15, 1930, 292-300.

Silver, J.H. Some applications of model theory in set theory, Ann. Math. Logic, 3, 1970, 45-110.

Solovay, R.M. A model of set theory in which every set of reals is Lebesgue measurable, Ann. Math., 92, 1970, 1-56.

Solovay, R.M. Real-valued measurable cardinals. Axiomatic Set Theory , Proc. Symp. Pure Math. 13 (1), A.M.S., 1971, 397-428.

Solovay, R.M. Strongly compact cardinals and the GCH, Proc. Tarski Symp., Proc. Symp. Pure Math. 25, A.M.S., 1974, 365-372.

Solovay, R.M., W.N. Reinhardt, A. Kanamori. Strong axioms of infinity and elementary embeddings, Ann. Math. Logic, 13, 1978, 73-116.

Tarski, A. Some problems and results relevant to the foundations of set theory, Logic, Methodology, and Philosophy of Science, Stanford., 1962, 125-135.

Ulam, S. Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math., 16, 1930, 140-150.

Ulam, S. Combinatorial analysis in infinite sets, SIAM Review, 1964, 343-355.

Wang, H. From Mathematics to Philosophy, Humanities, 1974.

Wang, H. Large Sets, Proc Fifth Int. Cong. of Logic, Methodology, and Phil. of Sci., 1975, Reidel, 1977.