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The Veblen Hierarchy and the Stability Theorem for L

Andrea Cantini

Let $PRS(\omega)$ ($PRO(\omega)$) be the class of primitive recursive set(ordinal) functions (see [JK]), expanded with a constant function whose value is ω. The Stability theorem for L states:
for every $F \in PRS(\omega)$, there is a $G_F \in PRO(\omega)$, which stabilizes F in L, i.e. for every ordinal β, $a \in L_\beta$, $F(a) \in L_{G_F(\beta)}$.

We prove some uniform versions of the theorem, showing that G_F has a familiar form. Let φ be the classical Veblen Hierarchy of ordinal functions starting with exponential in base ω (see [Sc]). As typical corollaries, we have:

A. we can find a primitive recursive function g on ω such that $F(a) \in L_{\varphi g(\hat{F})(\beta+1)}$, for every $F \in PRS(\omega)$ with index $\hat{F}$, every $\beta > 0$, $a \in L_\beta$.

Let $PRS(\varphi\omega_n : n \in \omega)$ be the extension of $PRS(\omega)$ with $\varphi\omega_n$ as new initial function, for every $n \in \omega$, where $\omega_0 = \omega$, $\omega_{n+1} = \omega^{\omega_n}$. Then we have:

B. we can find a primitive recursive function g which assigns to the index $\hat{F}$ of every F in $PRS(\varphi\omega_n : n \in \omega)$ the code of an ordinal $g(\hat{F}) < \varepsilon_0$ so that $\varphi g(\hat{F})$ already stabilizes F in L .

Let KPN_1 be the theory of admissible sets above the natural numbers, which includes 1) the existence of the set of natural numbers and standard Peano axioms; 2) extensionality, pairing, union, Δ_0-separation and Δ_0-collection, Σ_1-dependent choice; 3) $\in$-induction restricted to Σ_1-formulas. KPN_1^+ is KPN_1 plus full number-theoretic induction; remark that KPN_1 already proves Π_1-$\in$-induction.

Lemma 1. If $F \in PRS(\omega)$ ($PRS(\varphi\omega_n : n\in\omega)$), we can find uniformly and primitive recursively in F, a proof in KPN_1 (KPN_1^+) of $\forall x \exists y \Delta_F(x,y)$, where Δ_F is the Σ-formula defining the graph of F.

This is mainly because Δ_0-collection and $\in$-induction for Π_1-and Σ_1-formulas are sufficient to prove a form of the recursion theorem.

Now we apply cut -elimination tecniques; we reformulate KPN_1 as a sequent calculus à la Tait (see [Schw]), where sets Γ of formulas are derived. Due to the form of the rules corresponding to Δ_0-collection, Σ_1-DC and Σ_1-induction, we cannot eliminate all the cuts; however:

Lemma 2. (Weak Normalization). If Γ is derivable in KPN_1, then Γ is derivable using cuts only on Π_1- and Σ_1-formulas. The procedure is primitive recursive.

In the case of KPN_1^+, in order to get an analogue of 2, we embed the theory in a semiformal system with ω-rule; derivations (of finite cut-rank) will be

weakly normalized as in 2, but their length will increase up to any $\alpha<\varepsilon_0$. The main step is to read off bounding functions directly from weakly normalized proofs of Π_2-statements, thus avoiding both functional interpretation (see [Ca]) and full formalized cut elimination. We produce an "asymmetric" interpretation which is suggested by the proof of the stability theorem itself (but see [Gi]).

Theorem 3. Let f be a weakly normalized proof of length $k<\omega$ in KPN_1, ending with the set Γ, where each formula in Γ is Σ' or Π. Then

$\models \Gamma_{\vec{r}}[\beta, \varphi k(\beta+|\vec{r}|+1)]$, for every $\beta>0$, every $\vec{r} \equiv r_0,\dots,r_j \in L_\beta$ (parameters are in the list $\vec{r}$).

Here $\models \Gamma_{\vec{r}}[\beta, \varphi k(\beta+|\vec{r}|+1)]$ means "the disjunction on Γ is true whenever the domain of unrestricted $\forall$ ($\exists$) is L_β ($L_{\varphi k(\beta+|\vec{r}|+1)}$) and parameters are arbitrary sets in L_β, $\in$, N are interpreted in the standard way", $|\vec{r}| = \sum_{i=0}^{j} |r_i|$ and $|r_i|$ = order of r_i in L.

If we deal with KPN_1^+, just replace k by $\alpha<\varepsilon_0$.

As a special case it follows:

3.1. if $KPN_1 \vdash \forall x \exists y\, A(x,y)$ ($A\in\Delta_0$), then it is true for every $\beta>0$,

(+) $\forall x \in L_\beta \exists y \in L_{\varphi k(\beta+1)}\, A(x,y)$, where k is primitive recursively found in the given proof. Indeed if $\beta<\varphi\omega 0$, (+) can be proven in KPN_1. An analogue holds for KPN_1^+. Propositions 3-1 yield A-B.

By similar methods, but applying Ω_1-rule of [BFPS], we can give bounding theorems for set-theoretic Σ_1-definable functions, which are total provably in KPN_p, $(p > 1)$, KPN_p being the fragment of the full admissible set theory above N, where $\in$-induction is restricted to Σ_p and Π_p conditions. For $p > 1$, we use, instead of Veblen hierarchy, Bachmann hierarchy up to any $\alpha < \Omega^{\cdot^{\cdot^{\Omega}}}\}$p-times.

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