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Kripke-definable ordinals

(P. Minari - Firenze)

0\. It is known (see e.g. [4]) that in propositional K.(ripke)semantic it holds:

PROP. 0.0 For every $n\geq 1$ there is a (negation-free) formula $W_n (= p_o \vee \bigvee_{i=o}^{n-1}(p_i \rightarrow p_{i+1}) = T_{n-1}$ of [1]) with the property: it is valid in every K.model on the poset $\mathfrak{m} = \overset{0}{\bullet}\rightarrow\overset{1}{\bullet}\cdots\rightarrow\overset{m-1}{\bullet}$ with $m\leq n$, and has a K.countermodel on $\mathfrak{n+1} = \overset{0}{\bullet}\rightarrow\overset{1}{\bullet}\cdots\rightarrow\overset{n}{\bullet}$.

But also (as a consequence of the K.characterization with f.m.p. for Dummett's logic LC):

PROP. 0.1 For no ordinal $\xi\geq\omega$ there is a formula W_ξ with the above property (w.r. to ξ).

Our aim is to investigate on this kind of connection between ordinals and formulas in first order K.semantic.

1\. We use a first order language L with denumerably many n-place predicate symbols P_o^n, $P_1^n,\ldots$ for $n\geq o$. A K.frame is a triple $F = \langle P,\leq,V\rangle$ where $(P,\leq)$ is a poset and V a domain function on it. K.realizations ϱ for L on F and the forcing relation $\models$ are defined as usual. $\underline{L}^*F$ is the set of all L-sentences α s.t. $\varrho\models\alpha$ for every ϱ on F; $\underline{L}^*C$ (C a class of K.frames) is $\bigcap_{F\in C}\underline{L}^*F$. By $\mu,\nu,\xi\ldots$ (m,n,s...) we denote ordinals (finite); by $\preccurlyeq$ their usual order relation. If $\mu\neq o$, $(\mu,\preccurlyeq)$ is the poset of all

$\nu \prec \mu$ ordered by $\preccurlyeq$. Further, for $\xi \succ 0$:

- $[\xi;] = \{\langle P,\leq,V\rangle \mid \exists_{0\prec\mu\preccurlyeq\xi}\mu.(P,\leq)\cong(\mu,\preccurlyeq)\}$.
- $[\xi;Con] = \{\langle P,\leq,V\rangle \in [\xi;] \mid V \text{ is constant}\}$.
- $[\prec\xi;] = \bigcup_{0\prec\mu\prec\xi}[\mu;]$, $[\prec\xi;Con] = \bigcup_{0\prec\mu\prec\xi}[\mu;Con]$.

Recall the special case of Cantor Normal Form Thm.:

PROP. 1.0 $\forall\xi(0\prec\xi\prec\omega^\omega \to \exists!!n,m_1\ldots m_n\geq 0, m_0\geq 1.\ \xi = \omega^n.m_0 + \omega^{n-1}.m_1 + \ldots + \omega^0.m_n)$. In short: $c(\xi) = n;m_0,m_1\ldots m_n$.

2. Looking at Prop.0.0, we say:

DEF. 2.0 An ordinal $\xi \succ 0$ is K.definable iff there exists a L-sentence W_ξ s.t. $W_\xi \in \underline{L}^*[\xi;]\setminus\underline{L}^*[\xi+1;]$.

Our first claim is a positive one (compare it with Prop. 0.0):

THEOREM 2.1 If $0\prec\xi\prec\omega^\omega$ then ξ is K.definable.

The suitable W_ξ's are defined as follows:

(a) For $n\geq 1$ and $k\geq 0$, A^n_k is the L-sentence:

$$\forall x_1\left[\begin{array}{l}((\forall x_2\ldots x_n P^n_k x_1\ldots x_n \to \forall x_1\ldots x_n P^n_k x_1\ldots x_n) \to \forall x_1\ldots x_n P^n_k x_1\ldots x_n)\\ \wedge\ \forall x_2((\forall x_3\ldots x_n P^n_k x_1\ldots x_n \to \forall x_2\ldots x_n P^n_k x_1\ldots x_n) \to \forall x_2\ldots x_n P^n_k x_1\ldots x_n)\\ \wedge\ \forall x_2 x_3((\forall x_4\ldots x_n P^n_k x_1\ldots x_n \to \forall x_3\ldots x_n P^n_k x_1\ldots x_n) \to \forall x_3\ldots x_n P^n_k x_1\ldots x_n)\\ \wedge\\ \vdots\\ \wedge\ \forall x_2 x_3\ldots x_n((P^n_k x_1\ldots x_n \to \forall x_n P^n_k x_1\ldots x_n) \to \forall x_n P^n_k x_1\ldots x_n)\end{array}\right] \longrightarrow$$

$$\to \forall x_1\ldots x_n P^n_k x_1\ldots x_n .$$

Note that A^1_k $(k\geq 0)$ is instance of schema H^0_2 considered in [1]: $\forall x((\alpha(x)\to\forall y\alpha)\to\forall y\alpha)\to\forall x\alpha$.

(b) For all $n,m_1\ldots m_n\geq 0$ and $m_0\geq 1$, $W[n;m_0,m_1\ldots m_n]$ is the L-sentence (definition by induction on n and subinduction on m_0):

$$W[0;m_0] \ :\ P^0_0 \vee \bigvee_{i=0}^{m_0-1}(P^0_i \to P^0_{i+1})$$

$$W[1;m_0,m_1]: \begin{cases} m_0=1: \begin{cases} A^1_0 \vee (\forall x P^1_0 x \to P^0_0)\ , & \text{if } m_1=0\\ A^1_0 \vee (\forall x P^1_0 x \to W[0;m_1])\ , & \text{if } m_1>0\end{cases}\\ m_0=k+1, k\geq 1:\ A^1_k \vee (\forall x P^1_k x \to W[1;k,m_1]) \end{cases}$$

$$\underset{(n\geq 1)}{W[n+1;m_0,m_1\ldots m_{n+1}]}: \begin{cases} m_0=1: \begin{cases} A^{n+1}_0 \vee (\forall x_1\ldots x_{n+1} P^{n+1}_0 x_1\ldots x_{n+1} \to P^0_0), \text{ if } m_1=\ldots=m_{n+1}=0\ ,\\ A^{n+1}_0 \vee (\forall x_1\ldots x_{n+1} P^{n+1}_0 x_1\ldots x_{n+1} \to W[n+1-s;m_s\ldots m_{n+1}],\\ \qquad \text{if } m_1=m_2=\ldots=m_{s-1}=0 \text{ and } m_s>0 .\end{cases}\\ m_0=k+1, k\geq 1:\ A^{n+1}_k \vee (\forall x_1\ldots x_{n+1} P^{n+1}_k x_1\ldots x_{n+1} \to W[n+1;k,m_1,\ldots m_{n+1}] .\end{cases}$$

(c) Finally, for $0\prec\xi\prec\omega^\omega$, we define (by Prop. 1.0): $W_\xi = W[c(\xi)]$ (note that negation is not used!).

Then it is shown (by induction on $c(\xi)$):

LEMMA 2.1.1 $W_\xi \in \underline{L}^*[\xi;]\setminus\underline{L}^*[\xi+1;Con]$ (which is stronger than $W_\xi \in \underline{L}^*[\xi;]\setminus\underline{L}^*[\xi+1;]$ requested by Def. 2.0).

Our second claim is a negative one:

THEOREM 2.2 No $\xi \succcurlyeq \omega_1$ (= first uncountable) is K.definable.

This easily follows from

PROP. 2.2.1 For every frame $F = \langle P,\leq,V\rangle$ in which $(P,\leq)$ is well ordered there exists a subframe $F' = \langle P',\leq',V'\rangle$ in which $(P',\leq')$ is well ordered, $|P|\leq\aleph_0$, $\bigcup_{i\in P}|V_i|\leq\aleph_0$, and s.t. $\underline{L}^*F' \subseteq \underline{L}^*F$.

which in turn is a corollary to a strong Löwenheim Skolem-type Thm. for K.semantic we proved in [2] (improving Ono's result in [3]).

Unfortunately we lack answers for $\{\xi \mid \omega^\omega\preccurlyeq\xi\prec\omega_1\}$ in terms of K.definability, although we see some reasons for supporting the following

CONJECTURE 2.3 No $\xi \succcurlyeq \omega^\omega$ is K.definable.

But the proof (if one!) seems to be not so easy.

REMARK 2.4 Say: $\xi (\succcurlyeq 2)$ is K^0.definable iff there exists a L-sentence $W^0_\xi \in \underline{L}^*[\prec\xi;]\setminus\underline{L}^*[\prec\xi+1;]$. It can be shown (modifying the W's of Thm.2.1) that every $\xi \prec \omega^\omega$ is K^0.definable and that no $\xi \succcurlyeq \omega_1$ is such.

REMARK 2.5 From Thm.2.1 we obtain (for notations see [1]):

(a) There exists a strictly decreasing well ordered chain of negation-free logics (finitely axiomatizable?) between TQ and BQ, with order type $\omega^\omega+1$.

(b) There exists a strictly decreasing well ordered chain of negation-free logics containing schema D (fin.ax.?) between TQ and BH_1D, with order type $\omega^\omega+1$.

We can get other results studying intermediate logics obtained from a given basis by adding W_ξ of Thm.2.1 as new axiom schema for some ξ. It turns out, e.g., that $PW_\xi \subset PH^0_2$ for all $\omega \preccurlyeq \xi \prec \omega^\omega$.

References

[1] E.Casari - P.Minari, Negation-free intermediate predicate logics, to appear in B.U.M.I., 1983.

[2] P.Minari, A strong Löwenheim Skolem-type theorem for Kripke semantic, to appear.

[3] H.Ono, A study of intermediate predicate logics, Publ. RIMS, Kyoto Univ.,8 (1972/73), 619-649.

[4] K.Segerberg, Propositional logics related to Heyting's and Johansson's, Theoria, 34 (1968),26-61.