Estratto da

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1. What is to be meant by a foundation of mathematics?

A foundation of mathematics entails a philosophy of mathematics, but a foundation has to contain something more than a philosophy needs to do. From a foundation one expects, as indeed the term itself suggests, that a basis is exhibited and that it is shown systematically that mathematics can be erected upon this basis. Accordingly, it is particular in connection with projects of the kind exemplified by Frege's and Russell's logicism and Hilbert's program that one has been speaking about the foundations of mathematics, while one may hesitate to say that Wittgenstein contributed to this field, although he made reflections that certainly belong to the philosophy of mathematics. Sometimes one uses the term the foundations of mathematics without wanting to imply that a philosophical element is involved, simply equating the foundation of a branch of mathematics with the axiomatization of it. However, such a usage often has its roots in philosophical considerations: for instance, in a formalistic philosophy of mathematics (according to which mathematics is just a formal game) or in a scepticism toward the traditional foundational schools resulting in the pessimistic belief that axiomatizations are all that can be accomplished in the way of foundations. Of course, the three foundational schools that one spoke about in the first half of the century did not merely embrace different axiomatizations of mathematics. Logicism, Hilbert's formalism, and intuitionism differed from each other with respect to the kind of axiomatizations that were considered adequate, or, more generally, the kind of basis that was sought for mathematics,and these differences depended on different philosophical views about the nature of mathematics. In summary, one may perhaps say that according to traditional usage, a foundation of mathematics shows how mathematics can be built up systematically from a basis of a specific kind and is particularly concerned with the validity and further character of this basis, aiming to reveal something about the nature of mathematics.

To have a limited topic to discuss in this round table talk , I shall confine myself to the foundations of mathematics in this traditional sense. My aim is not to insist that the term has to be used in this way,i.e. for a kind of investigation that combines mathematical and philosophical elements. But I deplore a development where foundational studies in this sense are given up and are replaced by either purely mathematical investigations of formal theories or purely philosophical discussions about mathematics.

2. <u>The traditional schools of foundation today</u>.

By defending the traditional conception of a foundation of mathematics, I do not want to subscribe to a continuation of disputes in traditional terms between the historical schools of foundation.

This once so intensive discussion has come to an end since, for good reasons, today no one adopts the view of logicism and hardly any one adheres the Hilbert's program as a scheme of foundation.

It may be worthwhile to recapitulate briefly why logicism cannot result in a satisfactory foundation. The interest of the thesis that mathematics can be reduced to logic hinges of course on the question whether any interesting line can be drawn between the two subjects. But let us assume that a sentence in predicate logic of finite order is logically true just in case it is valid in all (individual) domains (interpreting higher order quantification in the standard way when the individuals are given), and that logical consequence is defined similarly. We then know that the concepts of mathematical analysis can be defined in predicate logic of some finite order in such a way that the mathematical truths become, not logical truths, but logical consequences of Peano's 3rd and 4th axiom (or some other axiom of infinity). This is a kind of reduction of mathematics to logic and is certainly in itself of some interest; the reduction is weaker than the one conceived by Frege but is essentially the one proposed by Russell. Why can this reduction not be considered a step in the foundation of mathematical analysis?

The answer is, it seems to me, that there is no guarantee that we can recognize the logical truth of a sentence in predicate logic of order two or higher without utilizing concepts that belong not to logic but to mathematics; by Gödel's incompleteness theorem we know that adding new mathematical concepts and principles for them it may be possible to prove the logical truth of sentences that were not provable before. This illustrates that a satisfactory foundation has to be concerned with the epistemological basis of mathematics: the proposed reduction of mathematics to logic is of little interest as long as we have to rely upon mathematical insights to recognize logical truths.

Hilbert's program may be pursued in a modified form with the purpose of extracting the constructive content of mathematical theories, but it is difficult to see how it could appeal as a foundational program. In its more sophisticated formulations, one does not take an

instrumentalistic view of the so-called ideal sentences but admit to them an intended meaning, what one calls a transfinite meaning. But there is then a strange tension in the program between, on the one side, the demand of a constructive re-interpretation of the transfinite sentences, and, on the other side, the cherishing of the transfinite concepts, as reflected e.g. in the rejection of a constructive development of mathematics.

Although the majority of mathematicians and logicians do not adopt an intuitionistic view, there is today, in contrast to the situation that now holds for logicism and Hilbert's formalism, both an intensive philosophical discussion around intuitionism and an active development of more far-reaching intuitionistic systems. The philosophical discussion of intuitionism has been much renewed by the influence of Dummett, who has turned the conflict between classical and intuitionistic mathematics from an ontological issue to a semantical one. Intuitionism is better discussed in terms of what may be called a semantical foundation of mathematics.

3. A new attempt at a semantical foundation.

The idea of a semantical foundation of mathematical theories is of course not new. After Tarski's and Carnap's works in the twenties and thirties on concepts such as denotation, satisfaction, and truth, there was, I think, a widespread hope of a new kind of foundational investigations where the meanings of the primitive terms of a formal system were studied and the axioms were justified in terms of these meanings.

Tarski's ambition was to define inductively what it is for a sentence to be true, but it was soon thought that the inductive clauses of such a truth definition, which stated truth contitions for sentences of different forms, also determined the meanings of the sentences in question. Carnap spoke in this context about semantical rules that interpreted the formulas of a formal system, and the same terminology is followed in Church's classical textbook, Mathematical Logic. Since he realized that the truth condition of a sentence cannot simultaneously both define the notion of truth and give the meaning of the sentence, Church suggests that truth is to be taken as a primitive notion in the semantics of a formal system.

The failure of this idea is obvious. Once one has seen how to state the truth condition of a sentence of the object language by using the same sentence on the meta-level, this kind of semantics can be formulated more or less mechanically and gives no insight into the meaning of the sentence. (Nor is there any point in calling a sentence true if truth is a primitive concept about which nothing is known except certain truth conditions stated in terms of sentences whose meanings are to be determined).

Dummett's discussions of the form that a theory of meaning for a language should take suggest a new analysis of some key concepts in semantics and point to the possibility of another, general way to arrive at a semantical foundation of mathematics. Some of the main ideas may roughly be summarized as follows. The central question about meaning is what it is to know the meaning of an expression. To know the meaning of a sentence is primarily to know, not its truth condition, but what establishes it as true, which, at least in mathematics, is the same as to know under what condition the sentence may be correctly asserted, or what is counted as a proof of the sentence. Having determined the meaning of a sentence in this way, one may try to justify the axioms and rules of inferences of a mathematical theory in terms of the meanings of the sentences involved; more precisely, to determine the meaning of a sentence is to determine a certain direct way of proving it, but other indirect ways of proving a sentence are then also justified, viz. demonstrations showing how a direct proof could in principle be obtained.

I think it is fair to say that Dummett's general program for a meaning theory is partly inspired by the ideas behind Gentzen's intuitionistic system of natural deduction.

Gentzen's idea of an introduction rule for sentences of a given form, which according to him "represents so to say the 'definition' of the sign in question", and his idea of justifying other inference rules, the elimination rules, in terms of the introduction rules may be seen as a rough proto-type of the kind of semantics discussed by Dummett.

The way I have tried to develop Gentzen's semantical ideas within a proof-theoretical frame-work (where proof-theory is understood as the general study of proofs and not as a tool in Hilbert's program) may roughly be summarized as follows.

The introduction rules in a system of natural deduction give <u>sufficient</u> conditions for asserting the conclusions in question. For instance, the introduction rule for $\vee$

$$\frac{A_i}{A_1 \vee A_2} \qquad i = 1 \text{ or } 2$$

corresponds to the fact that a sufficient condition for asserting $A_1 \vee A_2$ is to have satisfied either the condition for asserting A_1 or the condition for

asserting A_2. But this is not a necessary condition for asserting $A_1 \vee A_2$. On the contrary, even on an intuitionistic view, one may e.g. assert for a certain natural number n that

(1) n is prime or n is not prime

having neither a proof that n is prime nor a proof that n is not prime; e.g. one may have proved by induction that (1) holds for all natural numbers n and inferred (1) for a specific n by universal instantiation. The situation just described is a quite general one. As soon as we have stated some sufficient condition for asserting a sentence, other sufficient conditions emerge, viz. to have shown that the first sufficient condition <u>can</u> be satisfied.

What the introduction rules determine is, more precisely, the <u>canonical</u> ways of proving the conclusions in question. The idea is that to have the right to assert a sentence, one has either to possess a canonical proof of the sentence or to know how to find one, i.e. know a method for finding a canonical proof. It is in this way, by determining a notion of canonical proof, that the introduction rules determine when a sentence can be correctly asserted, i.e. what it is to have shown the sentence to be true,and hence, what the meaning of the sentence is.

Having determined the meaning of a sentence in this way, it is possible to say what one has to know in order to know that an inference is correct. To know that an inference is correct is the same as to have the right of asserting the conclusion under the hypothesis of the premisses, which furthermore allows us to assert the conclusion categorically given that we can correctly assert the premisses categorically. To know that an inference is correct we must therefore know how to find a canonical proof of the conclusion from canonical proofs of the premisses, i.e. we have to know a method that applied to canonical proofs of the premisses yields a canonical proof of the conclusion. Knowing this, we have what was required above for asserting the conclusion categorically, provided that we have already satisfied the conditions for asserting the premisses categorically.

It may be appropriate here to connect these ideas with the two courses held at this Siena meeting. We may note that the methods one has to know to see in the way just described that elimination inferences in a natural deduction system are correct are just the contractions or reductions that Flavio Previale has spoken about in his course in connection with the normalization of a proof. The direct verifications that Per Martin-Löf has spoken about in his course ("to know the

meaning of a proposition one has to know what counts as a direct verification of it") are clearly closely connected with what I have called canonical proofs, and his principle that to know that a proposition is true you must know how to give direct evidence for it (you do not need already to possess such evidence) is clearly very close to the condition I gave for correctly asserting a sentence.

In a frame-work of this kind, a general program for a semantical foundation of mathematical theories emerges which sets two main tasks : firstly, to analyse the meaning of mathematical notions in terms of the condition for correctly asserting sentences of various forms, or more precisely, in terms of what counts as a canonical proof or direct evidence, and secondly, to state other deductive principles and justify them in the way described above, i.e. by showing how a canonical proof of a conclusion obtained by applying the principle can be obtained from canonical proofs of the premisses. Gentzen started in effect the work on such a program, although he only hinted at its philosophical principles very briefly and vaguely, and instead used his results mostly as a tool for furthering Hilbert's program. The analysis of logical concepts presented by Martin-Löf in his course has many aspects, one of which is that it can be seen as a contribution to the foundations of mathematics in the general direction that I have discussed here.