Estratto da

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COMPLEXITY OF REDUCTION TREES IN PREDICATE CALCULUS

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1. The strong normalization theorem for predicate calculus asserts that any reduction sequence starting from a derivation $\mathscr{D}$ stops at a normal derivation $\mathscr{D}'$. The most intuitive proof of this result uses Prawitz's concept of valid derivation [4]. By König's lemma and a choice principle the strong normalization theorem is equivalent to the stronger assertion: there exists a function f such that the length of any reduction sequence starting from a derivation $\mathscr{D}$ is $\leqslant f(\mathscr{D})$. However the standard proof of strong normalization does not provide information about such a function f. On the other hand Minc's proof [3] for the $(\rightarrow,\forall)$-fragment or Schwichtenberg's proof [5] for the $\rightarrow$ -fragment yield a primitive recursive and an $\mathscr{E}^4$ bounding function f, respectively. In this note we give a more explicit f in $\mathscr{E}^4$ for the full $(\bot,\rightarrow,\forall)$-based classical predicate calculus using Gandy's method [1] as formulated in Girard [2]. Proofs are omitted for lack of space.

2. We consider formulae of predicate calculus built up from arbitrary parameters $a_0, a_1, \ldots,$ variables $x_0, x_1, \ldots,$ constants $m_0, m_1, \ldots,$ n-ary function symbols $f^n_0, f^n_1, \ldots,$ n-ary relation symbols $R^n_0, R^n_1, \ldots$ and logical symbols $\bot, \rightarrow, \forall$. The length of a formula A, lh(A), is the number of atomic formulae and logical symbols in A.

The (natural deduction) rules for predicate calculus include $\rightarrow I, \rightarrow E, \forall I, \forall E$ and the classical falsity rule:

$$\bot_c \; \frac{\begin{matrix}[(\neg A)] \\ \\ \bot\end{matrix}}{A}$$

for arbitrary A. The length of a derivation $\mathscr{D}$, $lh(\mathscr{D})$, is the number of inferences in $\mathscr{D}$.

Proper redundant formulae in a derivation $\mathscr{D}$ are conclusions of I-inferences which are major premises of

E-inferences. Improper redundant formulae are conclusions of $\bot_c$-inferences which are non-atomic or of the form $\bot$. Normal derivations are derivations containing no redundant formulae. The reduction rules include (A) proper reductions: the usual $\to$-reductions and $\forall$-reductions; (B) improper reductions: $\bot\to$-reductions and $\bot\forall$-reductions for $\bot_c$-inferences with non-atomic conclusions (see [4]) and $\bot\bot$-reductions (suggested by P. Martin-Löf):

$$\begin{array}{c}[(\neg\bot)]\\ \mathcal{D}\\ \bot_c\ \dfrac{\bot}{\bot}\end{array} \qquad \text{red} \qquad \begin{array}{c} \to I\ \dfrac{\overset{(1)}{\bot}}{(\neg\bot)}\,(1)\\ \mathcal{D}\\ \bot\end{array}$$

We write $\mathcal{D} \succ_1 \mathcal{D}'$: $\mathcal{D}$ immediately reduces to $\mathcal{D}'$, and $\mathcal{D} \succeq \mathcal{D}'$: $\mathcal{D}$ reduces to $\mathcal{D}'$. The length of a reduction sequence is its number of elements.

3. We extend the rules of predicate calculus by introducing the following new axiom schemata:

(Ax1) $A \to \forall x_1 \dots \forall x_n (R^n_i(x_1,\dots,x_n) \to R^n_i(x_1,\dots,x_n))$

(Ax2) $A \to (\bot \to \bot)$

(Ax3) $\forall x_1 \dots \forall x_n (R^n_i(x_1,\dots,x_n) \to R^n_i(x_1,\dots,x_n))$

(Ax4) $\bot \to \bot$

for arbitrary A and R^n_i. In derivations instances of axiom schema A are denoted by $\overline{A}$. E-derivations are derivations in this extended sense. Clearly any E-derivation can be transformed into a derivation in the usual sense.

For arbitrary A and atomic P we introduce E-derivations:

$$\begin{array}{c}(2)\\(A)\\ \mathcal{B}_{A;P}\\ P\to P\end{array} \quad , \qquad \begin{array}{c}\mathcal{B}_{P}\\ P\to P\end{array}$$

as follows, where assumption class (A) contains a single element.

$$\mathcal{B}_{A;R^n_i(t_1,\dots,t_n)} =$$

$$\begin{array}{c} \forall E\ \dfrac{\to E\ \dfrac{\overset{(2)}{A} \qquad \dfrac{}{A \to \forall x_1\dots\forall x_n(R^n_i(x_1,\dots,x_n)\to R^n_i(x_1,\dots,x_n))}\,Ax1}{\forall x_1\dots\forall x_n(R^n_i(x_1,\dots,x_n)\to R^n_i(x_1,\dots,x_n))}}{\forall x_2\dots\forall x_n(R^n_i(t_1,\dots,x_n)\to R^n_i(t_1,\dots,x_n))}\\ \vdots \\ R^n_i(t_1,\dots,t_n)\to R^n_i(t_1,\dots,t_n)\end{array}$$

$$\mathcal{B}_{R^n_i(t_1,\dots,t_n)} =$$

$$\begin{array}{c} \forall E\ \dfrac{\dfrac{}{\forall x_1\dots\forall x_n(R^n_i(x_1,\dots,x_n)\to R^n_i(x_1,\dots,x_n))}\,Ax3}{\forall x_2\dots\forall x_n(R^n_i(t_1,\dots,x_n)\to R^n_i(t_1,\dots,x_n))}\\ \vdots \\ R^n_i(t_1,\dots,t_n)\to R^n_i(t_1,\dots,t_n)\end{array}$$

$$\mathcal{B}_{A;\bot} = \to E\ \dfrac{\overset{(2)}{A} \qquad \dfrac{}{A\to(\bot\to\bot)}\,Ax2}{\bot\to\bot}$$

$$\mathcal{B}_{\bot} = \dfrac{}{\bot\to\bot}\,Ax4 \quad .$$

For arbitrary A and B the expanders:

$$\begin{array}{ccc} \begin{array}{c} \overset{(1)}{(B)}\ ,\ \overset{(2)}{(A)} \\ \mathscr{A}_{A;B} \\ B \end{array} & , & \begin{array}{c} \overset{(1)}{(B)} \\ \mathscr{A}_{B} \\ B \end{array} \end{array}$$

are defined inductively as follows, where assumption classes (A) and (B) contain a single element and three elements, respectively.

(i) If P is atomic, then:

$$\mathscr{A}_{A;P} = \rightarrow E\ \dfrac{\rightarrow E\ \dfrac{\rightarrow E\ \dfrac{\overset{(1)}{P} \quad \overset{\mathscr{B}_{A;P}}{P \rightarrow P}}{P} \quad \overset{\mathscr{B}_{A;P}}{P \rightarrow P}}{P} \quad \overset{\mathscr{B}_{A;P}}{P \rightarrow P}}{P}$$

$$\mathscr{A}_{P} = \rightarrow E\ \dfrac{\rightarrow E\ \dfrac{\rightarrow E\ \dfrac{\overset{(1)}{P} \quad \overset{\mathscr{B}_{P}}{P \rightarrow P}}{P} \quad \overset{\mathscr{B}_{P}}{P \rightarrow P}}{P} \quad \overset{\mathscr{B}_{P}}{P \rightarrow P}}{P}$$

(ii)

$$\mathscr{A}_{A;C \rightarrow D} = \begin{array}{c} \rightarrow E\ \dfrac{\overset{(3)}{C} \quad \overset{(1)}{C \rightarrow D}}{(D)}\ ,\ \overset{(2)}{(A)} \\ \mathscr{A}_{A;D} \\ \rightarrow I\ \dfrac{D}{C \rightarrow D}\ (3) \end{array}$$

$$\mathscr{A}_{C \rightarrow D} = \begin{array}{c} \rightarrow E\ \dfrac{\overset{(3)}{C} \quad \overset{(1)}{C \rightarrow D}}{(D)} \\ \mathscr{A}_{D} \\ \rightarrow I\ \dfrac{D}{C \rightarrow D}\ (3) \end{array}$$

(iii)

$$\mathscr{A}_{A;\forall x C(x)} = \begin{array}{c} \forall E\ \dfrac{\overset{(1)}{\forall x C(x)}}{(C(a))}\ ,\ \overset{(2)}{(A)} \\ \mathscr{A}_{A;C(a)} \\ \forall I\ \dfrac{C(a)}{\forall x C(x)} \end{array}$$

$$\mathscr{A}_{\forall x C(x)} = \begin{array}{c} \forall E\ \dfrac{\overset{(1)}{\forall x C(x)}}{(C(a))} \\ \mathscr{A}_{C(a)} \\ \forall I\ \dfrac{C(a)}{\forall x C(x)} \end{array}$$

The <u>E-reduction</u> rules include:

(A) <u>proper E-reductions</u>

(i) $\rightarrow$ -E-red

$$
\begin{array}{c}
\\
\\
\rightarrow E
\end{array}
\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A\end{array} \quad \dfrac{\begin{array}{c}[(A)] \\ \mathcal{D}_2 \\ B\end{array}}{A \rightarrow B}\rightarrow I}{B}
\qquad \text{E-red} \qquad
\begin{array}{c}
\mathcal{D}_1 \\ (A) \\ \mathcal{D}_2 \qquad \mathcal{D}_1 \\ (B)\ ,\ (A) \\ \mathscr{A}_{A;B} \\ B
\end{array}
$$

(ii) $\forall$ -E-red

$$
\begin{array}{r} \\ \forall I \\ \forall E \end{array}
\begin{array}{c} \mathcal{D}_1(a) \\ A(a) \\ \hline \forall x A(x) \\ \hline A(t) \end{array}
\qquad \text{E-red} \qquad
\begin{array}{c} \mathcal{D}_1(t) \\ (A(t)) \\ \mathscr{A}_{A(t)} \\ A(t) \end{array}
$$

(B) improper E-reductions

Exactly like improper reductions in the usual sense.

We write $\mathcal{D} \succ_1^E \mathcal{D}'$: $\mathcal{D}$ immediately E-reduces to $\mathcal{D}'$, and $\mathcal{D} \succcurlyeq^E \mathcal{D}'$: $\mathcal{D}$ E-reduces to $\mathcal{D}'$.

4. The rank of a formula A, $\varrho(A)$, is defined by: $\varrho(A) = \max\{n : R_i^n$ occurs in $A\}$, $= 0$ if no R_i^n occurs in A. The rank of a derivation or E-derivation $\mathcal{D}$ is defined by: $\varrho(\mathcal{D}) = \max\{\varrho(A) : A$ is a (proper or improper) redundant formula in $\mathcal{D}\}$.

Lemma. For all formulae A and B,

$$3 \leqslant \mathrm{lh}(\mathscr{A}_B) < \mathrm{lh}(\mathscr{A}_{A;B}) \leqslant 2\,\mathrm{lh}(B) + 3\,\varrho(B) + 4$$

Lemma. If $\mathcal{D} \succ_1^E \mathcal{D}'$, then $\mathrm{lh}(\mathcal{D}) < \mathrm{lh}(\mathcal{D}')$.

Theorem. Suppose that $\mathcal{D}$ is an E-derivation such that if A is the conclusion of any $\bot_c$-inference, then $\mathrm{lh}(A) \leqslant m$. Then there is an E-derivation $\mathcal{D}'$ containing no im-

proper redundant formula such that $\mathcal{D} \succcurlyeq^E \mathcal{D}'$ and

$$\mathrm{lh}(\mathcal{D}') \leqslant 2^p\,\mathrm{lh}(\mathcal{D}) + \frac{2^p - 1}{2^2 - 1}, \text{ where } p = (m-1)\cdot 2(\mathrm{lh}(\mathcal{D})-1).$$

The order of an E-derivation $\mathcal{D}$, $o(\mathcal{D})$, is defined by: $o(\mathcal{D}) = \max\{\mathrm{lh}(A) : A$ is a (proper or improper) redundant formula in $\mathcal{D}\}$, $= 1$ if $\mathcal{D}$ is normal.

Theorem. For any E-derivation $\mathcal{D}$ containing no improper redundant formula, there is a normal E-derivation $\mathcal{D}'$ such that $\mathcal{D} \succcurlyeq^E \mathcal{D}'$ and

$$\mathrm{lh}(\mathcal{D}') \leqslant 2^{\mathrm{lh}(\mathcal{D})\cdot v(\mathcal{D}) + (o(\mathcal{D})-2)\cdot(v(\mathcal{D})-1)}_{o(\mathcal{D})-1}$$

where $v(\mathcal{D}) = \dfrac{2\,o(\mathcal{D}) + 3\,\varrho(\mathcal{D}) + 3}{2}$.

Corollary. For any E-derivation $\mathcal{D}$ there is a normal E-derivation $\mathcal{D}'$ such that $\mathcal{D} \succcurlyeq^E \mathcal{D}'$ and $\mathrm{lh}(\mathcal{D}') < 2^{3\max\{\mathrm{lh}(\mathcal{D}), o(\mathcal{D}), \varrho(\mathcal{D})\}^2}_{o(\mathcal{D})+2}$.

Lemma. If $\mathcal{D} \succcurlyeq^E \mathcal{D}_1$ and $\mathcal{D} \succcurlyeq^E \mathcal{D}_2$, then there is an E-derivation $\mathcal{D}'$ such that $\mathcal{D}_1 \succcurlyeq^E \mathcal{D}'$ and $\mathcal{D}_2 \succcurlyeq^E \mathcal{D}'$.

Theorem. For any E-derivation $\mathcal{D}$, all E-reduction sequences starting from $\mathcal{D}$ are of length

$$< 2^{3\max\{\mathrm{lh}(\mathcal{D}), o(\mathcal{D}), \varrho(\mathcal{D})\}^2}_{o(\mathcal{D})+2}.$$

5. An expansion of a derivation $\mathcal{D}$ is an E-derivation $\mathcal{D}'$ which differs from $\mathcal{D}$ at most for containing a part

$$\begin{array}{ccc} (1) & \mathscr{F} \\ (B) & , & (A) \\ & \mathscr{A}_{A;B} & \\ & B & \end{array} \qquad \text{or} \qquad \begin{array}{c} (1) \\ (B) \\ \mathscr{A}_{B} \\ B \end{array}$$

in place of some occurrence of formula B.

Lemma. If $\mathscr{D} \succ_1 \mathscr{D}'$ and $\mathscr{E}$ is an expansion of $\mathscr{D}$, then there is an expansion $\mathscr{E}'$ of $\mathscr{D}'$ such that $\mathscr{E} \succsim^{E} \mathscr{E}'$.

Theorem. For any derivation $\mathscr{D}$, all reduction sequences starting from $\mathscr{D}$ are of length

$$< 2^{3\max\{\mathrm{lh}(\mathscr{D}),\, o(\mathscr{D}),\, \varrho(\mathscr{D})\}^2}_{o(\mathscr{D})+2} .$$

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