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# DOMINICAL CATEGORIES AND RECURSION THEORY

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I shall introduce here a new notion of abstract recursion theory, designed to admit new examples as well as the old ones and, to that end, to describe in the simplest possible way what it is that provides the distinctive character of recursion theory. The context is that of categorical algebra: recursive functions are replaced by morphisms in a category, so that there are no underlying sets. An ultimate goal is to bring together this theory with algebraic logic (e.g. the polyadic algebras of P.R. Halmos) in such a way as to realize the program of Halmos of providing a purely algebraic treatment of incompleteness theorems. This rapprochement, however, is still in its initial phases.

I begin by describing the kind of category in which partial functions, with their attendant domains, are the primary notion. These are the "dominical categories" discussed in §1. Dominical categories with an additional property, viz. the existence of a Turing morphism (§2) are "recursive" categories. The name is justified by showing that some of the elementary theorems of recursion theory hold in such categories (§3), and, of course, that they include the classical examples. Finally, I describe briefly some nonstandard examples and indicate how they differ in important respects from the classical ones.

I am indebted to R. di Paola for his encouragement to a newcomer to the subject and, more specifically, for Theorem 2 below.

## §1. Dominical categories

The adjective "dominical" derives from the English noun "domain." A dominical category is like a category of partial maps; a morphism $\phi: X \to Y$ will be supplied with a domain "within" its source X. The definition which follows is motivated by the simplest example, viz. the category $\mathcal{P}$ of sets and partial maps.

Recall first that a pointed category $\mathcal{C}$ is one such that for each X,Y there is a $0: X \to Y$ such that for any $\phi: W \to X$ or $\psi: Y \to Z$, $0\phi = 0$, $\psi 0 = 0$. In $\mathcal{P}$, 0 is the empty partial map. A morphism f is total if $f\phi = 0$ implies $\phi = 0$. The total morphisms form a subcategory $\mathcal{C}_T$.

A dominical category is a pointed category $\mathcal{C}$ supplied with a functor $\times: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the near-product) with the following properties.

(o) if $\phi \times \psi = 0$ then $\phi = 0$ or $\psi = 0$

(i) $\times$ restricts to $\times_T: \mathcal{C}_T \times \mathcal{C}_T \to \mathcal{C}_T$ where it is a categorical product, supplied with projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$.

(ii) the associativity and commutativity isomorphisms of $\times_T$ are natural on $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$, $\mathcal{C} \times \mathcal{C}$.

(iii) if $\phi: X \to X'$ then $p_1(\phi \times Y) = \phi p_1$ and $(\phi \times \phi)\Delta_X = \Delta_{X'}\phi$ where $\Delta_X$, $\Delta_{X'}$ are the diagonals in $\mathcal{C}_T$.

The essential point is that (iii) asserts a good deal less than the naturality of the projections on $\mathcal{C} \times \mathcal{C}$.

If $\mathcal{C}$ is a dominical category and $\phi: X \to Y$ the domain dom $\phi$ of $\phi$ is the endomorphism of X provided by the composition

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{X \times \phi} X \times Y \xrightarrow{p_1} X.$$

It is not difficult to see that the operation dom has the following properties:

(i) $\mathrm{dom}(\mathrm{dom}\,\phi) = \mathrm{dom}\,\phi$;

(ii) $(\mathrm{dom}\,\phi)(\mathrm{dom}\,\psi) = (\mathrm{dom}\,\psi)(\mathrm{dom}\,\phi) = \mathrm{dom}((\mathrm{dom}\,\phi)(\mathrm{dom}\,\psi))$;

(iii) $(\mathrm{dom}\,\phi)(\mathrm{dom}\,\phi) = \mathrm{dom}\,\phi$.



Thus the dom $\phi$, for $\phi$ with source X, form (with respect to composition) a $\wedge$-semilattice Dom X with infimum 0 and supremum $1_X$. For later use I record the additional property

(iv) $\mathrm{dom}(\psi\phi) = \mathrm{dom}((\mathrm{dom}\,\psi)\phi)$.

If $\phi: X \to Y$ then $\varepsilon \mapsto \mathrm{dom}(\varepsilon\phi)$ defines a semilattice morphism $\phi^*: \mathrm{Dom}\,Y \to \mathrm{Dom}\,X$ preserving 0; if f is total $f^*$ preserves 1 as well.

§2. The classical recursive semigroupoid; Turing morphisms

We presuppose the classical theory of recursive functions and define a recursive structure on a set X as an equivalence class of bijective maps $x: X \approx \mathbf{N}$, with $x \sim x'$ whenever $x'x^{-1}$ is a recursive function. The classical recursive category $\mathbf{T}$ has as objects such sets provided with recursive structures. If X,Y are object of $\mathbf{T}$ then $\mathbf{T}(X,Y)$ consists of all partial maps $\phi: X \to Y$ such that $y\phi x^{-1}: \mathbf{N} \to \mathbf{N}$ is a partial recursive function, where x,y represent the recursive structures on X and Y.

In fact $\mathbf{T}$ is a semigroupoid, i.e. a category in which any two objects are isomorphic. Our recursion theories will all be supposed to have this property.

The near-product in $\mathbf{T}$ is defined in the following way. Any one of the standard pairing functions $b: \mathbf{N} \times \mathbf{N} \approx \mathbf{N}$ defines a recursive structure on $\mathbf{N} \times \mathbf{N}$; they are all equivalent. If X and Y have recursive structures represented by x and y then $b(x \times y)$ defines a recursive structure on the cartesian product $X \times Y$. This makes $\mathbf{T}$ a dominical category. For an $X \in \mathbf{T}$, Dom X stands for the "recursively enumerable" subsets of X in the sense that an $\varepsilon \in \mathrm{Dom}\,X$ is a subjunction of the identity defined on a subset of X isomorphic under a structure map to an r.e. subset of $\mathbf{N}$.

Those properties of $\mathbf{T}$ that seem to be attached to the notion of recursiveness are centered about one which we shall now characterize.

In any dominical semigroupoid $\mathbb{C}$ a Turing morphism is a morphism $\tau\colon X \times X \to X$ with the property that, given any $\phi\colon X \times X \to X$, there is a total $g\colon X \to X$ such that $\tau(g \times X) = \phi$.

Theorem 0: **T** possesses a Turing morphism.

Let $\tau\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be the partial function defined by the universal Turing machine, so that the domain of $\tau$ consists of those pairs (m,n) for which the Turing machine with index m halts when provided with input n, and the value $\tau(m,n)$ is the output. Then $\tau$ is a Turing morphism, the defining property being, in essence, a translation of the "$S^m_n$ theorem."

§3. Recursive semigroupoids: some classical theorems

I propose to call a recursive semigroupoid any dominical semigroupoid which has a Turing morphism. To justify this I shall show that in any such category some of the prototypical facts of recursion theory are valid. Among these perhaps the most notable is the existence of nonrecursive r.e. sets. We may translate "r.e. set" by "domain" but not "recursive r.e. set" by "complemented domain," since our domains form only a semilattice. A satisfactory substitute is the following. A domain $\varepsilon \in \mathrm{Dom}\ X$ is decidable if it has a quasi-complement, i.e. a domain $\varepsilon' \in \mathrm{Dom}\ X$ with $\varepsilon\varepsilon' = 0$ such that $\varepsilon\phi = \varepsilon'\phi = 0$ implies $\phi = 0$.

Theorem 1: If $\tau\colon X \times X \to X$ is a Turing morphism then $\kappa = \mathrm{dom}(\tau\Delta_X)$ is undecidable.

This is in fact a corollary of a more interesting statement. We shall show that $\kappa$ is in fact "creative." To interpret this we need first the notion of an index. Since the objects of $\mathbb{C}$ lack elements we define an index, relative to $\tau$, of a $\phi\colon X \to X$ as a total $g\colon X \to X$ such that $\phi p_2 = \tau(g \times X)$. A domain $\varepsilon$ is creative if there is a total $f\colon X \to X$ such that for any domain $\varepsilon'$ disjoint from $\varepsilon$ and any index g of $\varepsilon'$, $\varepsilon fg = \varepsilon' fg = 0$. If $\varepsilon$ is creative then it is of course undecidable.



Theorem 2: $\kappa$ is creative.

We take $f = 1_X$. If g is an index of some $\kappa'$ disjoint from $\kappa$ then

$$\tau\Delta_X g = \tau(g \times X)(X \times g)\Delta_X = \kappa' g$$

$$\mathrm{dom}(\tau\Delta_X g) = \mathrm{dom}(\kappa' g) = g^*\kappa'.$$

But also

$$\mathrm{dom}(\tau\Delta_X g) = \mathrm{dom}((\mathrm{dom}\ \tau\Delta_X)g) = g^*\kappa.$$

Thus $g^*\kappa = g^*\kappa' = 0$.

We may also prove the recursion theorem.

Theorem 3: If $f\colon X \to X$ is total then there is a total $g\colon X \to X$ such that $\tau(fg \times X) = \tau(g \times X)$.

Let h satisfy $\tau(h \times X) = \tau(\tau \times X)(\Delta_X \times X)$ and let w be an index of fh. Then

$$\begin{aligned}\tau(hw \times X) &= \tau(h \times X)(w \times X)\\ &= \tau(\tau \times X)((w \times w)\Delta_X \times X)\\ &= \tau(\tau(w \times X)(X \times w)\Delta_X \times X)\\ &= \tau(fhw \times X)\end{aligned}$$

so that $g = hw$ is the morphism required.

§4. Nonstandard examples

To indicate how this abstract recursion theory differs from the classical one I adduce first the notion of constant. A constant of a dominical category is a total morphism $c\colon X \to Y$ such that for any total $f,g\colon W \to X$, $cf = cg$. The constant total functions in $\mathcal{P}$ or in **T** are of course constants, and they generate, i.e. if $\phi c = \phi' c$ for all constants c then $\phi = \phi'$. Another criterion is that of choice: a dominical category $\mathbb{C}$ satisfies the axiom of choice if for any morphism $\phi$ there is a section $\sigma$ of $\phi$, i.e. a morphism such that $\phi\sigma = \mathrm{dom}\ \sigma$, $\phi\sigma\phi = \phi$. It is easy to see that $\mathbf{T}$ satisfies the axiom of choice.

Now let G be a finite group and let $\mathbf{T}^G$ be the category of G-objects in $\mathbf{T}$, i.e. of sets X with recursive structure on which G operates via morphisms in $\mathbf{T}$, together with equivariant maps in $\mathbf{T}$. If the near products are supplied with the diagonal action of G then it is evident that $\mathbf{T}^G$ is a dominical category.

Let $\mathbf{T}_1^G$ be the full subcategory of $\mathbf{T}^G$ containing those X in which the operation of G is free, and $\mathbf{T}_{0,1}^G$ the full subcategory in which all orbits are either free or trivial, and in which there are infinitely many of both. We shall omit the proof of the following theorem, which is tedious rather than difficult.

Theorem 4: $\mathbf{T}_0^G$ and $\mathbf{T}_{0,1}^G$ are recursive semigroupoids. $\mathbf{T}_0^G$ satisfies the axiom of choice but has no constants. $\mathbf{T}_{0,1}^G$ fails to satisfy the axiom of choice. It has constants but they do not generate.

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