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# NORMALIZATIONS OF PROOFS IN SET THEORY

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## 1. A natural deduction formulation of set theory.

1.1 The simplest way to see how set theory can be formulated as a system of natural deduction in Gentzen -style is perhaps to recall the following two facts :

1.1.1 Zermelo's axioms for set formation (except the axiom of choice, about which I shall have nothing to say here) can be thought of as instances of the principle of abstraction, asserting $\exists x\ \forall y(y \in x \leftrightarrow A(y))$ for formulas $A(y)$ of certain specified forms; and

1.1.2 allowing terms of the form $\{x: A(x)\}$ for certain $A(x)$, the principle of abstraction is expressed by the following introduction and elimination rules (Prawitz 1965):

$$\in I)\ \frac{A(u)}{u \in \{x:A(x)\}} \qquad\qquad \in E)\ \frac{u \in \{x:A(x)\}}{A(u)}$$

The elimination rule is the inverse of the introduction rule, giving rise to the following contraction:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A(u)\end{matrix}}{u \in \{x:A(x)\}}}{A(u)} \quad \text{contracts to} \quad \begin{matrix}\mathcal{D}\\ A(u)\end{matrix}$$

by which an occurrence of $u \in \{x:A(x)\}$ standing both as conclusion of $\in$I and as premise of $\in$E is removed (cf. the removal of what I called maximal formulas in 1st order natural deductions or of what Gentzen called cut formulas in his calculus of sequents).

1.2 Let the <u>natural deduction system for naive set theory</u> be the system, whose <u>language</u> is obtained from a 1st order language with = and $\in$ as the only predicates by adding a term-building operation $\{x:\ldots\}$ such that

1.2.1 $\{x: A(x)\}$ is a term, if $A(a)$ is a formula, and

1.2.2 $u \in t$ is a formula, if u and t are terms

(other clauses for terms and formulas as in 1st order),

and whose <u>inference rules</u> are, in addition to those of 1st order predicate logic, $\in$I and $\in$E as stated in 1.1.2 above and the following rules for =:

$$=\text{I})\ \frac{\begin{matrix}(a \in t) & (a \in u)\\ a \in u & a \in t\end{matrix}}{t = u}\ a \qquad =\text{E})\ \frac{t_1 = t_2 \quad A(t_i)}{A(t_j)}\ \begin{matrix} i=1 \text{ or } 2\\ i \neq j\end{matrix}$$

In =I, formulas shown within parentheses indicate as usual that assumptions of that form may be discharged by the inference, and the parameter <u>a</u> shown to right of the inference line indicates that <u>a</u> is to satisfy the usual restrictions of proper parameters.



In =E, $A(t_i)$ is obtained by replacing some (not necessarily all) occurrences of $t_i$ in $A(t_i)$ by $t_j$.

1.3. By a <u>fragment of naive set theory</u> is understood any system that is like naive set theory except that clause 1.2.1 is weakened so that $\{x:A(x)\}$ is a term only for $A(x)$ of certain kinds; it is only required that a weakened clause 1.2.1 is enough uniform so that $t(u)$ is always a term, if $t(a)$ and u are.

1.4 The remark in 1.1 can now be formulated more precisely: Zermelo-Fraenkel's set theory (without the axiom of choice) can be formulated as a fragment of naive set theory, where clause 1.2.1 is weakened to allow $\{x:A(x)\}$ as a term only when $A(x)$ has a form corresponding to one of the axioms of separation, pair-formation, union, power set, infinity and replacement.

1.5. Alternatively, one may introduce special symbols for the different ways of generating sets as in informal set-theoretical practice, say $\emptyset$ (for the empty set), $\{x \in t : A(x)\}$, $\{t_1, t_2\}$, $\bigcup t$, $\mathcal{P} t$, $\omega$, and $\mathcal{R}_{y,x\in t} A(x,y)$ (for the range of values of the function $y=f(x) \equiv A(x,y)$ from the biggest subset s of t such that $A(x,y)$ is many-one for $x \in s$; cf the axiom of replacement). For each of these forms, one may then state introduction and elimination rules, i.e. rules for introducing and eliminating formulas $s \in u$ where u has one of the forms shown above (for $s \in \emptyset$ there is only an elimination rule) in such a way that the elimination rule is the inverse of the corresponding introduction rule, giving rise to contractions as usual. For instance, for the power set Pt we have the rules

$$\text{PI)}\ \frac{\begin{array}{c}(a \in s)\\ a \in t\end{array}}{s \in Pt}\,a \qquad \text{PE)}\ \frac{s \in Pt \quad u \in s}{u \in t}$$

and the contraction

$$\frac{\dfrac{\begin{array}{c}[a \in s]\\ \mathcal{D}_1(a)\\ a \in t\end{array}}{s \in Pt} \quad \begin{array}{c}\mathcal{D}_2\\ u \in s\end{array}}{u \in t} \quad \text{contracts to} \quad \begin{array}{c}\mathcal{D}_2\\ [u \in s]\\ \mathcal{D}_1(u)\\ u \in t\end{array}$$

Such a natural deduction formulation of ZF, the possibility of which has probably occurred to many people [1], is obviously equivalent to the fragment of naive set theory indicated in 1.4. It will be abbreviated $ZF^{ND}$.

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1) To my knowledge, a formulation of Zermelo set theory along the lines of 1.5 was studied by Per Martin-Löf in 1970. My presentation is following Hallnäs 1983, who is also exploiting the equivalence to a fragment of naive set theory.



2. Counterexamples to full normalization theorems.

Having stated a contraction for $\in$ (i.e. for $\in$E immediately following $\in$I) in 1.1.2, one may ask whether every derivation $\mathcal{D}$ in naive set theory reduces to a normal (i.e. irreducible) derivation by the iterated operation of replacing any subderivation of $\mathcal{D}$ by its contraction. But the answer is obviously no, since naive set theory is inconsistent and there is no normal proof of $\perp$. Indeed, the derivation corresponding to Russell's paradox reduces in two stages to itself by carrying out the only performable contractions (see Prawitz 1965). For $ZF^{ND}$ (or for the fragment of naive set theory indicated in 1.4), the answer to the similar question is likewise no: we may now formulate a closely similar proof $\mathcal{D}$ of the fact that $\{x \in a : x \notin x\} \notin a$, and again the only contractions that can be performed reduce $\mathcal{D}$ in two steps to itself [2].

Recently, in a doctoral thesis at the University of Stockholm, Lars Hallnäs has noted that the counterexamples to a normalization theorem in $ZF^{ND}$ concern contractions of open derivations, depending on assumptions.

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2) I first learned about this form Marcel Crabbe ' in 1974.

He asked what happens if one restricts the notion of reduction so that contractions are only allowed for closed derivations, and was able to show that when the notion is restricted in this way, all derivations reduce to normal form. My communication here is in all essentials a presentation of his work (Hallnäs 1983).

## 3. *$\mathcal{C}$-reductions and $\mathcal{C}$-normal derivations.*

For lack of time I shall have to disregard contractions of =E following immediately after =I (much of the more difficult part of Hallnäs work is concerned with such reductions), and shall restrict myself to the intuitionistic contractions for 1st order logical constants as defined in Prawitz 1965 or 1971 and to either the $\epsilon$-contractions defined in 1.1.2 for naive set theory or the corresponding contractions exemplified in 1.5 for $ZF^{ND}$.

A derivation $\mathcal{D}_2$ is now said to be an *immediate $\mathcal{C}$-reduction* of $\mathcal{D}_1$, if $\mathcal{D}_2$ is obtained from $\mathcal{D}_1$ by replacing an initial part $\mathcal{D}$ by its contraction (as just specified), and $\mathcal{D}$ does not depend on any assum-



ption in the case of $\epsilon$-contractions (and corresponding contractions in $ZF^{ND}$)[3].

We then define the notions of *$\mathcal{C}$-reduction*, *$\mathcal{C}$-reduction sequence*, *$\mathcal{C}$-normal*, *$\mathcal{C}$-normalizable*, and *strongly $\mathcal{C}$-normalizable* as usual, except that these notions are now based on immediate $\mathcal{C}$-reductions instead of immediate reductions; e.g. $\mathcal{D}$ is *$\mathcal{C}$-normal* if there is no immediate $\mathcal{C}$-reduction of $\mathcal{D}$, and $\mathcal{D}$ is *strongly $\mathcal{C}$-normalizable* if every sequence of immediate $\mathcal{C}$-reductions terminates in a $\mathcal{C}$-normal derivation.

We can now state the following result:

*Every derivation in $ZF^{ND}$ (or in the corresponding fragment of naive set theory) is strongly $\mathcal{C}$-normalizable.*

This result is a little different from that of Hallnäs who considers several notions of $\mathcal{C}$-normalizability and $\mathcal{C}$-normal, none of which are quite comparable to the notions used here; among other things, he follows the method of Martin-Löf 1971 of considering one particular reduction sequence where one reduces

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3) Martin-Löf 1975 argued for a general and stronger restriction to the effect that $\mathcal{D}_1$ in the definition above is closed.

from below.

$\mathcal{C}$-normal derivations not depending on any assumption have the essential property common with normal derivations not depending on assumtions that they are always in $\mathcal{J}$ – form, i.e. the conclusions of such derivations are obtained by applications of introduction rules. Thus, when the set theory is based on intuitionistic logic, we immediately obtain as corollaries the usual E-theorems: $\vdash A \vee B$ only if $\vdash A$ or $\vdash B$, and $\vdash \exists x\, A(x)$ only if $\vdash A(t)$ for some term t.

I shall give a very brief sketch of Hallnäs' proof below, restricting myself to the case when = is left out; in some respects, I shall follow the proof of strong normalizability for 2nd order logic in Prawitz 1981.

## 4. Candidates of validity and well-founded fragments of naive set theory.

Hallnäs' proof is an extension of the proofs by Girard 1971 and Martin-Löf 1971 of a normalization theorem for simple type theory, and as in all proofs of normalizability for higher order logic, an essential step is to define what Girard calls "candidat de réductibilité", here called candidate of validity. In the proofs mentioned for simple type theory, this notion is defined by induction over the type. This induction is replaced in Hallnäs' proof by an induction over the provable $\in$-relation between terms, and one thus exploits



the fact that this relation is well-founded in ZF, i.e. that there is no infinite sequence $t_1, t_2, \ldots$ of terms in $ZF^{ND}$ such that for each i, $\vdash t_{i+1} \in t_i$.

The well-foundedness of the provable $\in$-relation turns out to be the only property needed in order that the proof is to go through for an arbitrary fragment of naive set theory. Such a fragment N of naive set theory where there is no infinite sequence $t_1, t_2, \ldots$ of terms such that for each i, $\vdash_N t_{i+1} \in t_i$ is called well-founded, and Hallnäs thus obtains the following more general result: In any well-founded fragment of naive set theory, all derivations are $\mathcal{C}$-normalizable (and in fact, one may add, strongly $\mathcal{C}$-normalizable).

Since all the introduction and elimination rules for the different forms that a term may assume in $ZF^{ND}$ are subsumed under just the two rules of $\in$ I and $\in$ E in naive set theory, the proof for fragments of the latter system is of course shorter to state. In the rest of this abstract, N is to be any well-founded fragment of naive set theory, and all the constructions and proofs are for such an N.

We now define:

4.1. $\alpha$ is a candidate of validity for a formula A, if $\alpha$ is a set of strongly $\mathcal{C}$-normalizable derivations of A in N and is closed under $\mathcal{C}$-reductions.

For terms we make an inductive definition as follows:

4.2. If t is a term without $\epsilon_N$-predecessor, i.e. if there is no term u such that $\vdash_N u \epsilon t$, then $\alpha$ is a <u>candidate of validity for the term</u> t if $\alpha = t$.

4.3. If t is a term with some $\epsilon_N$-predecessor, then $\alpha$ is a <u>candidate of validity for the term</u> t, if $\alpha$ is a function whose domain contains every $\beta_u$ that is a candidate of validity for a term u such that $\vdash_N u \epsilon t$, and whose value $\alpha(\beta_u)$ for such a $\beta_u$ is a candidate of validity for the formula $u \epsilon t$.

If N is based upon classical logic, a candidate of validity has to satisfy also certain other closure conditions as in Prawitz 1981. When one includes also = and its contractions, still other closure conditions have to be added.

5. <u>$\mathcal{V}_{A,\varphi}$ and $\mathcal{V}_{t,\varphi}$</u>.

We consider arbitrary assignments $\varphi$ of candidates of validity for terms to individual parameters (free variables), and define the notions <u>strong validity relative to formulas</u> A <u>or terms</u> t <u>and to assignments</u> $\varphi$, abbreviated $\mathcal{V}_{A,\varphi}$ and $\mathcal{V}_{t,\varphi}$, respectively. By $A^{\varphi}$ and $t^{\varphi}$, I denote the result of replacing certain parameters in A and t by terms: namely, every parameter <u>a</u> in A and t, respectively, to which $\varphi$ assigns a candidate of validity $\alpha_u$ for the term u is to be replaced by this term u. $\mathcal{V}_{A,\varphi}$ will be a set of derivations of $A^{\varphi}$. $\mathcal{V}_{t,\varphi}$ will either be a term or a function depending on whether t is as in clause 4.2 or 4.3, i.e. does not have or has $\epsilon_N$-predecessors. Simultaneously with the definition of $\mathcal{V}_{A,\varphi}$ and $\mathcal{V}_{t,\varphi}$, which proceeds by induction over the construction of formulas and terms, one has to prove that $\mathcal{V}_{A,\varphi}$ and $\mathcal{V}_{t,\varphi}$ are candidates of validity for $A^{\varphi}$ and $t^{\varphi}$.



The essentially new clauses in the definition (compared to the case of 2nd order logic) are:

5.1. If t is a parameter <u>a</u> assigned a value by $\varphi$, then $\mathcal{V}_{t,\varphi} = \varphi(a)$.

5.2. If 5.1 does not apply and t is a term such that $t^{\varphi}$ has no $\epsilon_N$-predecessor, then $\mathcal{V}_{t,\varphi} = t^{\varphi}$.

5.3. If 5.1 does not apply and t is a term such that $t^{\varphi}$ has some $\epsilon_N$-predecessor, in which case t must have the form $\{x:A(x)\}$, then $\mathcal{V}_{t,\varphi}$ is a function defined for every $\alpha_u$ that is a candidate of validity for a term u such that $\vdash_N u \epsilon t$, and for every such $\alpha_u$, $\mathcal{V}_{t,\varphi}(\alpha_u)$ is the set of derivations of the form

$$\frac{\begin{matrix}\mathcal{D}\\ A^{\varphi}(u)\end{matrix}}{u \epsilon \{x:A(x)\}^{\varphi}}$$

such that $\mathcal{D} \epsilon \mathcal{V}_{A(a),\varphi\binom{a}{\alpha_u}}$ for a parameter <u>a</u> not occurring in t or u; $\varphi\binom{a}{\alpha_u}$ is here to be like $\varphi$ except possibly for assigning $\alpha_u$ to a.

5.4. If A is a formula $u \in t$ and $t^{\varphi}$ has no $\in_N$-predecessor, then $\mathcal{V}_{A,\varphi}$ is the set of strongly $\mathcal{C}$-normalizable derivations in N of $A^{\varphi}$.

5.5. If A is a formula $u \in t$ and $t^{\varphi}$ has some $\in_N$-predecessor (i.e. $\mathcal{V}_{t,\varphi}$ is a function), then $\mathcal{V}_{A,\varphi}$ is the set $\Delta$ of derivations defined inductively by

5.5.1. If $\mathcal{D} \in \mathcal{V}_{t,\varphi}(\mathcal{V}_{u,\varphi})$, then $\mathcal{D} \in \Delta$.

5.5.2. If $\mathcal{D}$ is a $\mathcal{C}$-normal derivation of $A^{\varphi}$ and is either not in I-form or depends on assumptions, then $\mathcal{D} \in \Delta$.

5.5.3. If $\mathcal{D}$ is a derivation of $A^{\varphi}$ and either is not on I-form or depends on assumptions and it holds for all immediate $\mathcal{C}$-reductions $\mathcal{D}'$ of $\mathcal{D}$ that $\mathcal{D}' \in \Delta$, then also $\mathcal{D} \in \Delta$.

## 6. Validity of the $\in$-rules.

Having verified that $\mathcal{V}_{A,\varphi}$ is a candidate of validity and hence that $\mathcal{D} \in \mathcal{V}_{A,\varphi}$ implies that $\mathcal{D}$ is strongly $\mathcal{C}$-normalizable, it remains only to show as in the case of 2nd order logic that all inference rules are strongly valid, i.e. preserve validity relative to $\varphi$. To this end, we need the following substitution lemma:

$$\mathcal{V}_{A(a),\varphi(\overset{a}{\mathcal{V}_{u,\varphi}})} = \mathcal{V}_{A(u),\varphi} \quad \text{and} \quad \mathcal{V}_{t(a),\varphi(\overset{a}{\mathcal{V}_{u,\varphi}})} = \mathcal{V}_{t(u),\varphi}$$

proved by induction over the definition of $\mathcal{V}_{A(a),\varphi}$ and



$\mathcal{V}_{t(a),\varphi}$.

That the logical rules are strongly valid is then proved as in 2nd order logic; in the case of $\forall$E, one applies the substitution lemma.

To see that $\in$I is strongly valid, suppose that $t = \{x : A(x)\}$ and that

$$(1) \quad \mathcal{D} = \frac{\begin{matrix}\mathcal{D}_1 \\ A(u)^{\varphi}\end{matrix}}{(u \in t)^{\varphi}} \quad \text{where } \mathcal{D}_1 \in \mathcal{V}_{A(u),\varphi}.$$

If t has some $\in_N$-predecessor, then by applying the substitution lemma, 5.3 and 5.5.1 in this order to (1), we obtain immediately that $\mathcal{D} \in \mathcal{V}_{u \in t,\varphi}$. If t has no $\in_N$-predecessor, then since $\mathcal{D}$ is strongly $\mathcal{C}$-normalizable by the fact that $\mathcal{D}_1$ is (in virtue of belonging to a candidate of validity), $\mathcal{D} \in \mathcal{V}_{u \in t,\varphi}$ because of 5.4.

To see that $\in$E is strongly valid, suppose that $t = \{x : A(x)\}$ and that

$$(2) \quad \mathcal{D} = \frac{\begin{matrix}\mathcal{D}_1 \\ (u \in t)^{\varphi}\end{matrix}}{A(u)^{\varphi}} \quad \text{where } \mathcal{D}_1 \in \mathcal{V}_{u \in t,\varphi}.$$

Following the pattern of proof in Prawitz 1981, it is sufficient to verify that if there is a $\mathcal{C}$-contraction $\mathcal{D}'$ of $\mathcal{D}$, then $\mathcal{D}' \in \mathcal{V}_{A(u),\varphi}$.

Therefore, assume that $\mathcal{D}'$ is a $\mathcal{C}$-contraction of $\mathcal{D}$. Then $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ A(u)^{\varphi}\end{array}}{(u \in t)^{\varphi}}}{A(u)^{\varphi}}$$

and furthermore $\mathcal{D}$ must not depend on any assumption, which implies that $t^{\varphi}$ has some $\in_N$-predecessor, namely $u^{\varphi}$. It follows that $\mathcal{D}_1$ can belong to $\mathcal{V}_{u \in t, \varphi}$ only in virtue of 5.5.1, and hence, applying 5.5.1, 5.3 and the substitution lemma in this order to (2), we obtain immediately that $\mathcal{D}_2 \in \mathcal{V}_{A(u), \varphi}$, which is what was to be shown since $\mathcal{D}' = \mathcal{D}_2$.

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