Estratto da

C. Bernardi e P. Pagli (a cura di), *Atti degli incontri di logica matematica* Volume 2, Siena 5-8 gennaio 1983, 6-9 aprile 1983, 9-12 gennaio 1984, 25-28 aprile 1984.

Disponibile in rete su http://www.ailalogica.it



# A TYPE-FREE SYSTEM FOR CONSTRUCTIVE METAMATHEMATICS

Gerard R. RENARDEL DE LAVALETTE

1. My 'system of reference' is HA (Heyting's arithmetic), an intuitionistic version of Peano's arithmetic with constants for all primitive recursive functions on $\mathbb{N}$.
In HA we can define the 'Kleene-bracket' notation for general recursive functions:

$$\{e\}(x) \simeq y \iff \exists z(T(e,x,z) \wedge Uz = y).$$

This yields a *partial* application $\{e\}(x)$ between numbers. To formulate this more abstractly we use a new theory APP, inspired on work by Feferman.

2. APP is about objects (not only numbers).
Constants: those for combinatory logic: $k,s,p,p_1,p_2$;
those for $\mathbb{N}$: $0,S,Pd$(predecessor), $\Delta$(def. by cases).
Terms: constants, variables; $s,t$ terms $\Rightarrow$ $st$ term.
Prime formulae: $s = t$, $t \in \mathbb{N}$.
Axioms and rules: those for intuitionistic logic, combinatory logic and for $\mathbb{N}$ (including induction).
We can define $\lambda$-abstraction, so also recursion in APP, hence HA is a subtheory of APP.
To show that APP is conservative over HA, we try to interpret APP in HA by $xy = z \mapsto \{x\}(y) \simeq z$. At first sight this works well: we can find gödelnumbers for all the constants of APP. However, we do *not* have $\forall xy\exists z\ \{x\}(y) = z$ !

3. To make the interpretation indicated above go through, we need logic of *partial existence* with a predicate E: Et means 't

exists'.

Axioms: Ex for all free variables, Ec for all constants;

$P(t) \to Et$ (P prime, t occurs in P);

$A(t) \wedge Et \to \exists x A(x)$; $\forall x A(x) \wedge Et \to At$.

The last two axioms express that the quantifiers range over existing objects.

Using this logic we can weaken APP to $APP^E$, which is conservative over HA.

4. Although APP is essentially stronger than $APP^E$ (for $APP^E \nvdash \forall xy \exists z\ xy = z$, by the interpretation of $APP^E$ in HA), both theories prove the same arithmetical theorems. This can be shown by extending term reduction and the normal form theorem for combinatory logic to APP, and formalizing this in $APP^E$.
As a consequence we have: APP conservative over HA.

5. As might be inferred from the above, my main interest lies in interpretations and conservation results. A short general discussion follows.
Theories T are identified with their sets of theorems $\{A \mid T \vdash A\}$.
We call $T_2 \supset T_1$ *conservative over* $T_1$ [w.r.t. the set $S$ of formulae] if

$$\text{for all } A\ [\in S]\text{: } T_2 \vdash A \Rightarrow T_1 \vdash A.$$

Notation: $T_2 \succ T_1$ [$T_2 \succ_S T_1$]. So e.g. we have APP $\succ_{arithm}$ $APP^E$ HA.
Now a general proof method is: find an interpretation
$^*: T_2 \to T_1$, i.e. a translation satisfying

$$T_2 \vdash A \Rightarrow T_1 \vdash A^*$$

and take $S := \{A \mid T_1 \vdash A^* \to A\}$; then $T_2 \succ_S T_1$.



6. Let us consider realizability, an interpretation of APP into itself defined by

$$
\begin{aligned}
t\ \underline{r}\ P &= P \qquad \text{for P prime}\\
t\ \underline{r}\ A \wedge B &= p_1 t\ \underline{r}\ A \wedge p_2 t\ \underline{r}\ B\\
t\ \underline{r}\ A \vee B &= p_1 t \in N \wedge (p_1 t = 0 \to p_2 t\ \underline{r}\ A) \wedge (p_1 t \neq 0 \to p_2 t\ \underline{r}\ B)\\
t\ \underline{r}\ A \to B &= \forall x (x\ \underline{r}\ A \to tx\ \underline{r}\ B)\\
t\ \underline{r}\ \forall x A x &= \forall x\ tx\ \underline{r}\ Ax\\
t\ \underline{r}\ \exists x A x &= p_2 t\ \underline{r}\ A(p_1 t)
\end{aligned}
$$

Now the following holds:

$$APP + AC^- \vdash A \iff APP \vdash t\ \underline{r}\ A \quad \text{for some term } t,$$

where $AC^-$ is the axiom scheme

$$\forall x (A(x) \to \exists y B(x,y)) \to \exists f \forall x (A(x) \to B(x, fx)),$$

A negative, i.e. $\vee, \exists$-free.

So we have $APP + AC^- \succ_S APP$ with $S = \{A \mid APP \vdash \exists x\ x\ \underline{r}\ A \to A\}$; it is not difficult to show that $S$ contains all negative formulae.

7. We can get more, however: extend APP to $APP^\varepsilon$ by adding constants $\varepsilon_A$ (A arithmetical) with axioms

$$\exists n A(m,n) \to A(m, \varepsilon_A m).$$

Now $APP^\varepsilon \vdash A \leftrightarrow \exists x\ x\ \underline{r}\ A$ for all arithmetical A, so $APP + AC^- \succ_{arithm} APP^\varepsilon$. Using forcing in the form of an interpretation of $APP^\varepsilon$ into APP we can show $APP^\varepsilon \succ_{arithm} APP$, so we have

$$APP + AC^- \succ_{arithm} APP, \text{ hence } APP + AC^- \succ HA.$$

8. In $APP + AC^-$ we have found a theory which is, because of its flexibility, its expressive power and its being conservative over HA, very apt for constructive metamathematics.

It is e.g. possible to treat the objects of APP as choice sequences and to define an elimination translation (in the sense of Kreisel-Troelstra) without the trouble of having to code functionals by functions.

9. $APP + AC^-$ is also maximal in some sense, in view of the following facts:

i) $APP + AC \vdash \bot$ ; here AC is $AC^-$ without the restriction on A.

ii) $APP + AC^- + (\forall\neg\neg \rightarrow \neg\neg\forall) \vdash \bot$ ; so a fortiori $APP + AC^-$ conflicts with classical logic.

iii) $APP + AC^- + IP(N) \vdash \bot$ ; IP(N) is *independence of premises* for N:

$$(A \rightarrow \exists nBn) \rightarrow \exists n(A \rightarrow Bn) \qquad \text{for negative } A;$$

observe that $(A \rightarrow \exists xBx) \rightarrow \exists x(A \rightarrow Bx)$ is derivable in $APP + AC^-$ for negative A.