Estratto da

C. Bernardi e P. Pagli (a cura di), *Atti degli incontri di logica matematica* Volume 2, Siena 5-8 gennaio 1983, 6-9 aprile 1983, 9-12 gennaio 1984, 25-28 aprile 1984.

Disponibile in rete su http://www.ailalogica.it

PROVABILITY AND INTERPRETABILITY IN THEORIES CONTAINING ARITHMETIC

Per Lindström

In the following we give a survey of some recent work on provability and interpretability in theories containing P (Peano arithmetic). §1 is devoted mainly to the theory of partially conservative sentences. A sentence φ is <u>X-conservative</u> over a theory T, where X is a set of sentences, if for every $\psi \in X$, if $T + \varphi \vdash \psi$, then $T \vdash \psi$. This concept was introduced by Guaspari [2] and the basic existence theorems were established by him and Solovay (cf. [2]). The first result of this type is, however, due to Kreisel (cf. [15]) who observed that if Con_P is a "natural" formalization of "P is consistent", then $\neg Con_P$ is Π^0_1-conservative over P. Related results have also been obtained by Jensen and Ehrenfeucht [7] and Kreisel and Lévy [8]. In §2 results from §1 are applied to prove results on interpretability using the fact that if S and T are r.e. reflexive extensions of P, then S is interpretable in T iff every Π^0_1 sentence provable in S is provable in T. (For example, combining this with the above mentioned result of Kreisel we get the result of Feferman [1] that $P + \neg Con_P$ is interpretable in P. Similarly the second Gödel unprovability theorem implies that $P + Con_P$ is not interpretable in P as was first shown by Feferman [1].) Finally §3 contains a presentation of the main results on the lattice of degrees of interpretability introduced by Lindström

[9, 11] and Švejdar [17]. Most proofs are omitted or only briefly outlined.

0. Notation and terminology. In the following S, T, A, B, C are elementary theories (sets of sentences). We shall only be interested in r.e. theories. But then, by Craig's theorem, we may and shall assume that S etc. are primitive recursive. Th(T) is the set of theorems of T. We write $T \vdash X$ or $X \dashv T$, where X is a set of sentences, to mean that $X \subseteq Th(T)$. Thus $S \dashv T$ iff S is a subtheory of T. S is an X-subtheory of T, $S \dashv_X T$, if $Th(S) \cap X \subseteq Th(T)$. φ^i is φ if $i = 0$ and $\neg\varphi$ if $i = 1$. In the following Γ is either Σ^0_{n+1} or Π^0_{n+1} and $\breve{\Gamma}$ is the dual of Γ. Γ-true(x) is a Γ partial truth definition for Γ sentences, i.e. for every Γ sentence φ,

$$P \vdash \varphi \leftrightarrow \Gamma\text{-true}(\bar{\varphi}).$$

For notation and terminology not explained here see [1].

1. Partially conservative sentences. Let T be a fixed primitive recursive but otherwise arbitrary consistent extension of P and let $\tau(x)$ be a PR binumeration of T. Let $\{\Gamma\}(x,y)$ be the formula

$$\forall uv \leq y(u \text{ is } \Gamma \wedge Prf_{\tau(z) \vee z=x}(u,v) \rightarrow \Gamma\text{-true}(u)).$$

The following lemma is then easily verified.

LEMMA 1. $\{\Gamma\}(x,y)$ is a Γ formula s. t.

(i) $P \vdash \{\Gamma\}(x,y) \wedge y' \leq y \rightarrow \{\Gamma\}(x,y')$,

(ii) $T + \varphi \vdash \{\Gamma\}(\bar{\varphi},\bar{m})$ for all φ and m,

(iii) if ψ is Γ and $T + \varphi \vdash \psi$, then there is a q s. t.

$$P + \{\Gamma\}(\bar{\varphi},\bar{q}) \vdash \psi.$$

In fact these are the only properties of $\{\Gamma\}(x,y)$ that will be used in the sequel.

Using Lemma 1 we can now prove two lemmas which serve to unify the proofs of a number of results stated in

what follows (cf. [10]).

LEMMA 2. Suppose $\chi(x,y)$ is Γ. Then there is a Γ formula $\xi(x)$ s. t.

(i) $T + \xi(\bar{k}) \vdash \chi(\bar{k},\bar{m})$,

(ii) $T + \xi(\bar{k}) \dashv_{\breve{\Gamma}} T \cup \{\chi(\bar{k},\bar{q}): q \in \omega\}$.

PROOF. Case 1. $\Gamma = \Pi^0_{n+1}$. Let $\xi(x)$ be s. t.

$$P \vdash \xi(\bar{k}) \leftrightarrow \forall y(\{\Sigma^0_{n+1}\}(\overline{\xi(\bar{k})},y) \rightarrow \chi(\bar{k},y)).$$

Then (i) follows at once from Lemma 1 (ii). To prove (ii), suppose ψ is Σ^0_{n+1} and $T + \xi(\bar{k}) \vdash \psi$. By Lemma 1 (iii), there is a q s. t.

$$P + \{\Sigma^0_{n+1}\}(\overline{\xi(\bar{k})},\bar{q}) \vdash \psi.$$

Hence, by Lemma 1 (i),

$$P + \forall y \leq \bar{q}\ \chi(\bar{k},y) + \neg\psi \vdash \xi(\bar{k}).$$

But then, since $T + \xi(\bar{k}) \vdash \psi$, it follows that $T + \forall y \leq \bar{q}\ \chi(\bar{k},y) \vdash \psi$. This proves (ii).

Case 2. $\Gamma = \Sigma^0_{n+1}$. Let $\xi(x)$ be s. t.

$$P \vdash \xi(\bar{k}) \leftrightarrow \exists y(\neg\{\Pi^0_{n+1}\}(\overline{\xi(\bar{k})},y) \wedge \forall z \leq y\ \chi(\bar{k},z)).$$

LEMMA3. Suppose $\neg\chi_0(x,y)$ and $\chi_1(x,y)$ are Γ. Then there is a Γ formula $\xi(x)$ s. t. for $i = 0,1$,

(i) $T + \xi^i(\bar{k}) \vdash \forall y \leq \bar{m}\ \chi_i(\bar{k},y) \rightarrow \chi_{1-i}(\bar{k},\bar{m})$,

(ii) if ψ is $\breve{\Gamma}$ and $T + \xi^i(\bar{k}) \vdash \psi^i$, then $T \cup \{\chi_{1-i}(\bar{k},\bar{q}): q \in \omega\} \vdash \psi^i$.

Our first result on the existence of partially conservative sentences is the following theorem due to Guaspari [2] (cf. also [4,10,14]). Let Cons(Γ,T) be the set of sentences Γ-conservative over T.

THEOREM 1. Let X be any r.e. set. There is then a Γ formula $\xi(x)$ s. t.

(i) if $k \in X$, then $T \vdash \neg\xi(\bar{k})$,

(ii) if $k \notin X$, then $\xi(\bar{k}) \in Cons(\check{\Gamma},T)$.

PROOF. Let $\rho(x,y)$ be a PR binumeration of a relation $R(k,m)$ s. t. $X = \{k: \exists m R(k,m)\}$ and let $\xi(x)$ be as in Lemma 2 with $\chi(x,y) = \neg\rho(x,y)$.

A set X of sentences will be said to be monoconsistent with S if $S + \varphi$ is consistent for every $\varphi \in X$.

COROLLARY 1. If X is r.e. and monoconsistent with T, then

$$\Gamma \cap Cons(\check{\Gamma},T) - X \neq \emptyset.$$

PROOF. Let $\xi(x)$ be as in Theorem 1 and let φ be s. t. $P \vdash \varphi \leftrightarrow \xi(\bar{\varphi})$. If $\varphi \in X$, then $T \vdash \neg\xi(\bar{\varphi})$, whence $T \vdash \neg\varphi$. But this is impossible and so $\varphi \notin X$. Hence, by Theorem 1 (ii), φ is as desired.

Let

$$NX = \{\varphi: \neg\varphi \in X\},$$

$$DCons(\Gamma,T) = Cons(\Gamma,T) \cap NCons(\check{\Gamma},T).$$

We then have the following result essentially due to Solovay [16] (cf. also [2,10,14]).

THEOREM 2. Suppose X is r.e. and monoconsistent with T. Then

$$\Gamma \cap DCons(\check{\Gamma},T) - (X \cup NX) \neq \emptyset.$$

The proof is an application of Lemma 3 and is similar to the proof of Theorem 1 followed by that of Corollary 1.

Let $B(\Sigma^0_k)$ be the set of Boolean combinations of Σ^0_k sentences. By a similar construction we can then prove the following result due to Hájek [4].

THEOREM 3. $\Delta^0_{k+2} \cap Cons(B(\Sigma^0_{k+1}),T) \cap NCons(\Pi^0_{k+1},T) - NCons(\Sigma^0_{k+1},T) \neq \emptyset$.

Combining Lemma 3 and Theorem 2 (with $X = \emptyset$) and using some additional tricks we can also prove the

following result which will be applied later to answer a question on interpretability raised by Orey [12].

THEOREM 4 ([10]). There are sentences φ_i s. t. φ_i, $\neg(\varphi_0 \wedge \varphi_1) \in Cons(\Gamma,T) - NCons(\Pi^0_1,T)$.

The following interesting result has recently been proved by C. Bennet.

THEOREM 5. There are Σ^0_{n+1} sentences θ_i s. t. $T \nvdash \theta_0 \vee \theta_1$ and for $i = 0,1$,

$$\theta_i \in Cons(\Pi^0_{n+1}, T + \neg\theta_{1-i}).$$

PROOF. Let $\xi_i(x)$ be s. t.

$$P \vdash \xi_i(\bar{k}) \leftrightarrow \exists z(\neg\{\Pi^0_{n+1}\}(\overline{\eta_i(\bar{k})},z) \wedge \forall u \leq z \neg Prf_\tau(\bar{k},u))$$

where $\eta_i(x)$ is

$$\exists z(\rho_i(x,z) \wedge \forall u \leq_i z\, \neg\rho_{1-i}(x,u))$$

($\leq_0$ is $\leq$ and $\leq_1$ is $<$) where in turn $\rho_i(x,y)$ is PR and s.t.

$$P \vdash \xi_i(x) \leftrightarrow \exists z \rho_i(x,z).$$

Using Lemma 1 it can be shown that

(1) if $T \vdash \varphi$, then $T \vdash \neg\eta_i(\bar{\varphi})$,

(2) if $T \nvdash \varphi$, then $\eta_i(\bar{\varphi}) \in Cons(\Pi^0_{n+1}, T + \neg\eta_{1-i}(\bar{\varphi}))$.

Now let ψ be s. t.

$$P \vdash \psi \leftrightarrow \eta_0(\bar{\psi}) \vee \eta_1(\bar{\psi})$$

and let $\theta_i = \eta_i(\bar{\psi})$. Then, by (1) and (2), the sentences θ_i are as desired.

We now give an example of an application of Theorem 5. For any sentence φ, let $e(\varphi,\Gamma) = \{\psi \in \Gamma: T \vdash \varphi \leftrightarrow \psi\}$. Let $E(\Gamma) = \{e(\varphi,\Gamma): \varphi \in \Gamma\}$. Elements of $E(\Gamma)$ will be written a, b etc. $a \leq b$ iff $T \vdash \varphi \rightarrow \psi$ where $\varphi \in a$ and $\psi \in b$. Clearly $\leq$ is a partial ordering which is a lattice with $a \cup b = e(\varphi \vee \psi,\Gamma)$ and $a \cap b = e(\varphi \wedge \psi,\Gamma)$ where again $\varphi \in a$ and $\psi \in b$. Moreover $E(\Gamma)$ has a minimal element $0 = e(0{=}1,\Gamma)$ and a maximal element $1 = e(0{=}0,\Gamma)$.

COROLLARY 2. The sentence

(*) $$\forall a_0 a_1 \exists b_0 b_1 (b_0 \le a_0 \wedge b_1 \le a_1 \wedge b_0 \cup b_1 = a_0 \cup a_1 \wedge b_0 \cap b_1 = 0)$$

is true in all the lattices $E(\Sigma^0_{n+1})$ and false in all the lattices $E(\Pi^0_{n+1})$.

PROOF. To show that (*) is false in $E(\Pi^0_{n+1})$ let θ_i be as in Theorem 5 and let $a_i = e(\neg\theta_i, \Pi^0_{n+1})$. Suppose (*) is true and let b_i be s. t. $b_i \le a_i$, $b_0 \cup b_1 = a_0 \cup a_1$, and $b_0 \cap b_1 = 0$. Let $\psi_i \in b_i$. Then $T \vdash \psi_i \to \neg\theta_i$, $T \vdash \neg\theta_0 \vee \neg\theta_1 \to \psi_0 \vee \psi_1$, and $T \vdash \neg(\psi_0 \wedge \psi_1)$. It follows that $T + \neg\theta_i + \theta_{1-i} \vdash \neg\psi_{1-i} \wedge (\psi_0 \vee \psi_1)$, whence $T + \neg\theta_i + \theta_{1-i} \vdash \psi_i$, whence $T + \neg\theta_i \vdash \psi_i$. But then $T \vdash \theta_0 \vee \theta_1$, contrary to hypothesis.

Our next two results are refinements of the following simple and certainly well-known observation: Suppose X is r.e. and bounded, i.e. $X \subseteq \Gamma$ for some Γ, and $T \cup X$ is consistent. Then there is a sentence θ s. t. $T + \theta \vdash X$ and $T + \theta$ is consistent.

THEOREM 6 ([10]). Let X be an r.e. set of Γ sentences. There is then a Γ sentence θ s. t. $T \cup X \dashv T + \theta \dashv_{\breve{\Gamma}} T \cup X$.

PROOF. By the proof of Craig's theorem, we may assume that X is primitive recursive. Let $\xi(x)$ be a PR binumeration of X. By Lemma 2, there is a Γ sentence θ s. t.

(1) $T + \theta \vdash \xi(\bar\varphi) \to \Gamma\text{-true}(\bar\varphi)$,

(2) $T + \theta \dashv_{\breve{\Gamma}} T \cup \{\xi(\bar q) \to \Gamma\text{-true}(\bar q) : q \in \omega\}$.

But then $T \cup X \dashv T + \theta$ follows at once from (1). Now suppose ψ is $\breve{\Gamma}$ and $T + \theta \vdash \psi$. Then, by (2),

$$T \cup \{\xi(\bar q) \to \Gamma\text{-true}(\bar q) : q \in \omega\} \vdash \psi.$$

But clearly $T \cup X \vdash \xi(\bar q) \to \Gamma\text{-true}(\bar q)$ for every q, whence $T \cup X \vdash \psi$. Thus $T + \theta \dashv_{\breve{\Gamma}} T \cup X$ as was to be shown.

The following result will be applied in §3.

THEOREM 7 ([10]). Suppose $X \subseteq \Gamma$ is r.e. and let Y be any r.e. set of sentences s. t. $T \cup X \nvdash \psi$ for every $\psi \in Y$. Then there is a Γ sentence θ s. t. $T \cup X \dashv T + \theta \nvdash \psi$ for every $\psi \in Y$.

PROOF. We may assume that X and Y are primitive recursive. Let $\xi(x)$ and $\eta(x)$ be PR binumerations of X and Y, respectively. If $\Gamma = \Pi^0_{n+1}$ let θ be s. t.

$$P \vdash \theta \leftrightarrow \forall y(\xi(y) \wedge \forall zu \le y(\eta(z) \to \neg Prf_{\tau(x)} \vee x = \bar\theta(z,u)) \to \Pi^0_{n+1}\text{-true}(y)).$$

If $\Gamma = \Sigma^0_{n+1}$ let θ be s. t.

$$P \vdash \theta \leftrightarrow \exists y(\exists zu \le y(\eta(z) \wedge Prf_{\tau(x)} \vee x = \bar\theta(z,u)) \wedge \forall z \le y(\xi(z) \to \Sigma^0_{n+1}\text{-true}(z)).$$

Next we observe that $Cons(\Gamma, T)$ is Π^0_2 and that Theorem 1 implies that $\Gamma \cap Cons(\breve{\Gamma}, T)$ is not r.e. Thus we are faced with the problem of classifying these sets. A very nearly complete solution follows from our next result. Let

$$Cons(\Gamma, Y, T) = \{\varphi : \text{for every } \psi \in \Gamma, \text{ if } T + \varphi \vdash \psi, \text{ then } \psi \in Y\}.$$

Thus $Cons(\Gamma, T) = Cons(\Gamma, Th(T), T)$.

THEOREM 8 ([10]). Suppose $\Gamma \ne \Pi^0_1$ and let Y be any r.e. set monoconsistent with P. Then to any Π^0_2 set X, there is a Γ formula $\xi(x)$ s. t.

(i) If $k \in X$, then $\xi(\bar k) \in Cons(\breve{\Gamma}, T)$,

(ii) if $k_r \notin X$ for $r \le q$, then $\bigvee_{r \le q} \xi(\bar k_r) \notin Cons(\Sigma^0_1, Y, T) \cup Cons(\Pi^0_1, Y, T)$.

To prove this we need the following

LEMMA 4. If X and Y are r.e. and Y is monoconsistent with P, then there is a Σ^0_1 formula $\xi_0(x)$ and a Π^0_1 formula $\xi_1(x)$ s. t.

(i) $P \vdash \xi_0(\bar{k}) \rightarrow \xi_1(\bar{k})$,

(ii) if $k \in X$, then $P \vdash \xi_0(\bar{k})$,

(iii) if $k_r \notin X$ for $r \leq q$, then $\bigvee_{r\leq q} \xi_1(\bar{k}_r) \notin Y$.

PROOF. Let $R(k,m)$ and $S(k,m)$ be primitive recursive relations s. t. $X = \{k: \exists m R(k,m)\}$ and $Y = \{k: \exists m S(k,m)\}$. Let $\rho(x,y)$ and $\sigma(x,y)$ be PR binumerations of $R(k,m)$ and $S(k,m)$. Let $S_0(\gamma,p)$ be the primitive recursive relation

$$\exists q k_r (S(\bigvee_{r\leq q} \gamma(\bar{k}_r), p) \ \& \ \forall m \leq p \ \forall s \leq q \sim R(k_s, m)).$$

Let $\sigma_0(x,y)$ be a PR binumeration of $S_0(\gamma,p)$ and let $\xi_1(x)$ be s. t.

$$P \vdash \xi_1(\bar{k}) \leftrightarrow \forall u(\sigma_0(\bar{\xi}_1, u) \rightarrow \exists v \leq u \ \rho(\bar{k}, v)).$$

Finally let $\xi_0(x)$ be s. t.

$$P \vdash \xi_0(\bar{k}) \leftrightarrow \exists z(\rho(\bar{k}, z) \wedge \forall u \leq z \neg \sigma_0(\bar{\xi}_1, u)).$$

Then (i) is obvious. To prove (ii) and (iii) first prove that $\sigma_0(\bar{\xi}_1, \bar{p})$ is false for every p.

Theorem 8 is now an easy consequence of Lemmas 2 and 4. It has the following (cf. [4,10,13,14,16])

COROLLARY 3. The sets $\Gamma \cap \mathrm{Cons}(\check{\Gamma},T)$, where $\Gamma \neq \Pi^0_1$, $\Sigma^0_2 \cap \mathrm{Cons}(\Sigma^0_1,T)$, and $\Pi^0_2 \cap \mathrm{Cons}(\Sigma^0_1,T)$ are complete Π^0_2 sets.

If T is Σ^0_1-sound, then $\Pi^0_1 \cap \mathrm{Cons}(\Sigma^0_1,T)$ is Π^0_1. But if T is not Σ^0_1-sound, then this set is complete Π^0_2 too [13].

Results similar to Theorem 8 and Corollary 3 can also be obtained from (the proofs of) Theorems 2,3,4,5.

Combining Theorem 8 and the proof of Theorem 6 we get the following corollary which will be applied in §3.

COROLLARY 4. If $\Gamma \neq \Pi^0_1$, then

$$\{\varphi: \exists \psi \in \Gamma \cap \mathrm{Cons}(\check{\Gamma},T)(T + \psi \vdash \varphi)\}$$

is a complete Σ^0_3 set.

Suppose X is r.e. and let $Y = \{\varphi: T + \varphi \vdash X\}$. Clearly Y is r.e. unless X is <u>infinite over</u> T in the sense that

there is no k s. t. $T \cup X{\restriction}k \vdash X$.

THEOREM 9 ([10]). Suppose X is r.e., bounded, and infinite over T. Then $Y = \{\varphi: T + \!\!/ \varphi \vdash X\}$ is a complete Π^0_2 set.

The proof is an application of Theorem 1. Suppose $X \subseteq \Gamma$. It is not known if it follows that $\Gamma \cap Y$ is complete Π^0_2. It can be shown, however, that $\Gamma \cap Y$ is not r.e.

<u>2. Interpretability.</u> We write $S \leq T$ to mean that S is interpretable in T. $S < T$ if $S \leq T \not\leq S$ and $S \equiv T$ if $S \leq T \leq S$. We assume that $P \dashv T$.

All proofs in what follows of the existence of interpretations are applications, directly or indirectly, of the following basic result established by Feferman [1].

LEMMA 5. If $\sigma(x)$ numerates S in T, then $S \leq T + \mathrm{Con}_\sigma$.

This is proved by showing that the denumerable case of the Henkin completeness proof can be carried out in P.

For any formula $\sigma(x)$, let $\sigma^*(x)$ be the formula

$$\sigma(x) \wedge \mathrm{Con}_{\sigma(y)} \wedge y \leq x.$$

The following lemma is again due to Feferman [1] (cf. also [11]).

LEMMA 6. (i) If $\sigma(x)$ binumerates S in T and for every n, $T \vdash \mathrm{Con}_{S{\restriction}n}$, then $\sigma^*(x)$ binumerates S in T.
(ii) $P \vdash \mathrm{Con}_{\sigma^*}$.

S is <u>reflexive</u> if $S \vdash \mathrm{Con}_{S{\restriction}n}$ for every n. S is <u>essentially reflexive</u> if all extensions of S in the language of S are reflexive. In the following A,B,C are essentially reflexive extensions of P.

If S_0 and S_1 are finite and $S_0 \leq S_1$, then $P \vdash \mathrm{Con}_{S_1} \rightarrow \mathrm{Con}_{S_0}$. Thus from Lemmas 5 and 6 we get the following key

LEMMA 7 ([3]). $S \leq A$ iff $A \vdash Con_{S\restriction n}$ for every n.

One immediate consequence of Lemma 7 is the following result known as Orey's compactness theorem [12].

LEMMA 8. $S \leq A$ iff $S\restriction n \leq A$ for every n.

Suppose φ is Π^0_1 and $Q + \varphi \leq A$, where Q is Robinson's arithmetic. Then there is an n s. t. $Q + \varphi \leq A\restriction n$. It follows that $P \vdash Con_{A\restriction n} \rightarrow Con_{Q+\varphi}$, whence $P \vdash Con_{A\restriction n} \rightarrow \neg Pr_Q(\overline{\neg\varphi})$. But $\neg\varphi$ is Σ^0_1, whence $P \vdash \neg\varphi \rightarrow Pr_Q(\overline{\neg\varphi})$. Since $A \vdash Con_{A\restriction n}$, it follows that $A \vdash \varphi$. This proves one half of the following lemma (cf. [2,3,9]). The other half follows at once from Lemma 7.

LEMMA 9. $A \leq B$ iff $A \dashv_{\Pi^0_1} B$.

In view of Lemma 9, results on partially conservative sentences can be applied to interpretability. Thus combining Corollary 1 and Lemma 9 we get (cf. [3,6,9,10,16])

THEOREM 10. If X is r.e. and monoconsistent with A, then there is a Σ^0_1 sentence $\varphi \notin X$ s. t. $A + \varphi \leq A$.

Similarly Theorem 2 yields the following

THEOREM 11 ([9,10]). If X is r.e. and monoconsistent with A, then there is a (Σ^0_2,Π^0_2) sentence θ s. t. $\theta^i \notin X$ and $A + \theta^i \leq A$, $i = 0,1$.

From our point of view we may regard ZF as an extension of P. Let P^2 be the finite conservative "second order" extension of P. Let $X = \{\varphi\colon GB + \varphi \leq GB\}$ and $Y = \{\varphi\colon P^2 + \varphi \leq P^2\}$. Then X and Y are r.e., since GB and P^2 are finite. Clearly X is monoconsistent with ZF and Y is monoconsistent with P. Hence, by Theorem 10 and recalling that P and ZF are essentially reflexive, we get the following

COROLLARY 5. There is a Σ^0_1 sentence φ s. t. $P + \varphi \leq P$ ($ZF + \varphi \leq ZF$) and $P^2 + \varphi \not\leq P^2$ ($GB + \varphi \not\leq GB$).

Similarly applying Theorem 11 we get

COROLLARY 6. There is a (Σ^0_2,Π^0_2) sentence θ s. t. $P + \theta^i \leq P$ ($ZF + \theta^i \leq ZF$) and $P^2 + \theta^i \not\leq P^2$ ($GB + \theta^i \not\leq GB$), $i = 0,1$.

From Theorem 4 we get the following (partial) answer to a question of Orey [12].

THEOREM 12 ([10]). There are sentences φ_i s. t. $A + \varphi_i \leq A$, $A + \varphi_0 \wedge \varphi_1 \not\leq A$, $A + \neg\varphi_i \not\leq A$, and $A + \neg\varphi_0 \vee \neg\varphi_1 \leq A$, $i = 0,1$.

By Theorem 6, we get the following

THEOREM 13 ([9,10]). (i) If $A \dashv B$, then there is a (Σ^0_2,Π^0_2) sentence θ s. t. $A + \theta \equiv B$.
(ii) If X is an r.e. set of Σ^0_1 sentences, then there is a Σ^0_1 sentence σ s. t. $A \cup X \equiv A + \sigma$.

Theorem 13 (ii) can be applied to answer a question raised by Švejdar [17] as follows. (For more information of the properties of Σ^0_1 and Π^0_1 sentences in terms of interpretability see Theorem 26 below.)

COROLLARY 7. If A is consistent, there is a Σ^0_1 sentence σ s. t. $A + \psi \not\equiv A + \sigma$ for every Π^0_1 sentence ψ.

PROOF. It can be shown without much difficulty (essentially by repeating the proof of Lemma 11 below) that there is a Σ^0_1 formula $\gamma(x)$ s. t. for every n, $A + \gamma(\bar{n}) \not\leq A \cup \{\gamma(\bar{m})\colon m < n\}$. Let $X = \{\gamma(\bar{n})\colon n \in \omega\}$. By Theorem 13 (ii), there is a Σ^0_1 sentence σ s. t. $A + \sigma \equiv A \cup X$. Let ψ be any Π^0_1 sentence and suppose $A + \psi \equiv A + \sigma$. Then, by Lemma 9, there is an n s. t. $A \cup X\restriction n \vdash \psi$. But then $A \cup X \leq A + \sigma \leq A + \psi \dashv A \cup X\restriction n$, whence $A \cup X \leq A \cup X\restriction n$, a contradiction.

By Lemma 7, $\{\varphi\colon A + \varphi \leq B\}$ is Π^0_2. Thus, by Theorem 8 with $Y = Th(B)$ (cf. [9,10,14,16])

THEOREM 14. If $A \leq B$ and B is consistent, then $\Sigma_1^0 \cap \{\varphi: A + \varphi \leq B\}$ is a complete Π_2^0 set.

Similarly, using Theorem 9, we get

THEOREM 15 ([9,10]). If there is no Π_1^0 sentence Θ s.t. $B \vdash \Theta$ and $B \leq A + \Theta$, then $\{\varphi: B \leq A + \varphi\}$ is a complete Π_2^0 set.

Corollary 5 leads to the question if there is a (Π_1^0) sentence φ s. t. $P + \varphi \not\leq P(ZF + \varphi \not\leq ZF)$ and $P^2 + \varphi \leq P^2(GB + \varphi \leq GB)$. Solovay [16] has developed a method, completely different from the one used here, by means of which these questions can be answered in the affirmative. Let A^* be P or ZF and let S^* be P^2 or GB, respectively. Then (cf also [5,18])

THEOREM 16. There is a Π_1^0 sentence Θ s. t. $S^* + \Theta \leq S^*$ and $A^* + \Theta \not\leq A^*$.

Note that, since A^* is (essentially) reflexive and φ is Π_1^0, $A^* + \varphi \not\leq A^*$ is equivalent to $A^* \nvdash \varphi$. Unfortunately it would take too long to explain Solovay's construction here. The reader is referred to [5,18].

We now introduce the following classification of sentences: φ is of type $\langle i, j_0, j_1\rangle$, where $i, j_0, j_1 \in \{0,1\}$, if $i = 0$ iff $A^* + \varphi \leq A^*$, $j_0 = 0$ iff $S^* + \varphi \leq S^*$, and $j_1 = 0$ iff $S^* + \neg\varphi \leq S^*$. Hájek [5] observed that $\neg\Theta$, where Θ is as in Theorem 16, is of type $\langle 1,0,0\rangle$ and went on to prove the following

THEOREM 17. There are independent Σ_1^0 sentences of all possible types.

By a similar but somewhat more complicated construction Švejdar [18] has proved the following remarkable

THEOREM 18. There is a Π_1^0 sentence ψ s. t. $A^* + \psi \not\leq A^*$ and $S^* + \psi \not\leq S^*$ but $A^* + \psi \leq S^*$.

3. Degrees of interpretability. Clearly $\equiv$ (mutual interpretability) is an equivalence relation. Its equivalence classes will be called degress (of interpretability). Now let T be a consistent essentially reflexive extension of P e.g. P of ZF. In the following A, B, etc. are extensions of T in the language of T. Thus A, B, etc. are essentially reflexive. Let D_T be the set of degrees of the form $d(A)$, where $d(A)$ is the degree of A, i.e. $A \in d(A)$. Elements of D_T will be written a, b, etc. D_T is partially ordered by the relation $\leq$ induced by interpretability, i.e. $a \leq b$ iff $A \leq B$, where $A \in a$ and $B \in b$.

We now define A^T, $\downarrow$, and $\uparrow$ as follows:

$$A^T = T \cup \{Con_{A\restriction n}: n \in \omega\},$$

$$A \downarrow B = \{\varphi \vee \psi: \varphi \in A \ \&\ \psi \in B\},$$

$$A \uparrow B = A^T \cup B^T.$$

Thus $Th(A \downarrow B) = Th(A) \cap Th(B)$. By Lemma 7, we get

LEMMA 10. (i) $A \leq B$ iff $A^T \dashv B$. Thus $A^T \equiv A$ and $A \leq B$ iff $A^T \dashv B^T$.

(ii) $A \leq B, C$ iff $A \leq B \downarrow C$,

(iii) $A, B \leq C$ iff $A \uparrow B \leq C$ iff $A \uparrow B \dashv C$.

For $A \in a$ and $B \in b$ let $a \cap b = d(A \downarrow B)$ and $a \cup b = d(A \uparrow B)$. By Lemma 10, $\cap$ and $\cup$ are well-defined, $a \cap b$ is the glb of a and b, and $a \cup b$ is the lub of a and b. Moreover it is easily verified that

$$A^T \downarrow (B \uparrow C) \dashv\vdash (A^T \downarrow B^T) \uparrow (A^T \downarrow C^T).$$

Thus we have the following

THEOREM 19 ([9,11]). D_T is a distributive lattice.

Švejdar [17] has defined the lattice V_T of degrees of the form $d(T + \varphi)$ and proved that V_T is distributive. By Theorem 13, $V_T = D_T$.

D_T has a minimal element $0 = d(T)$ and a maximal element 1, the common degree of all inconsistent theories. If T is Σ^0_1-sound, then $a, b < 1$ implies $a \cup b < 1$. In fact we have the following

THEOREM 20 ([9,11]). T is not Σ^0_1-sound iff there are $a_0, a_1 < 1$ s. t. $a_0 \cup a_1 = 1$ (and $a_0 \cap a_1 = 0$).

This may be proved in the following way. First prove the following simple but occasionally useful

LEMMA 11. If X is r.e. and monoconsistent with Q, then there is a true Π^0_1 sentence ψ s. t. $\psi, \neg\psi \notin X$.

Next applying Lemma 11 we can prove

LEMMA 12. Suppose X is r.e. and monoconsistent with P and let Θ be any true Π^0_1 sentence. There are then Π^0_1 sentences Θ_i s. t.

(i) $P \vdash \Theta_0 \vee \Theta_1$,

(ii) $P \vdash \Theta_0 \wedge \Theta_1 \rightarrow \Theta$,

(iii) $\Theta^j_i \notin X$, $i, j = 0, 1$.

PROOF OF THEOREM 20. Let Θ be a true Π^0_1 sentence s. t. $T \vdash \neg\Theta$. Let $X = Th(T)$. Let Θ_i be as in Lemma 12. Finally let $a_i = d(T + \Theta_i)$. Then $a_i < 1$, $a_0 \cap a_1 = 0$, and $a_0 \cup a_1 = 1$.

Suppose $a < 1$ and $A \in a$. By Lemma 11, there is a sentence ψ s. t. $Q + \psi^i \not\leq A$, $i = 0, 1$. Let $a_i = d(A + \psi^i)$. Then $a < a_i$ and $a_0 \cap a_1 = a$. Thus a is not meet-irreducible. Moreover no $a < 1$ is join-irreducible. In fact we have

THEOREM 21 ([9,11]). If $a < b < 1$, then there are degrees c_i s. t. $a < c_i < b$, $c_0 \cap c_1 = a$, and $c_0 \cup c_1 = b$.

PROOF. Let $A \in a$ and $B \in b$. By Orey's compactness theorem, there is a sentence ψ s. t. $B \vdash \psi$ and $\psi \not\leq A$. Moreover, by Lemma 7, there is an m s. t. $A \nvdash Con_{B\restriction m}$.

Let $\beta(x)$ be a PR binumeration of B and let

$$X = \{\varphi: \psi \leq A + \neg\varphi\} \cup \{\varphi: A \vdash \varphi \vee Con_{B\restriction m}\}.$$

Then X is r.e. and monoconsistent with P. Hence, by Lemma 12, there are Π^0_1 sentences Θ_i s. t.

$$P \vdash \Theta_0 \vee \Theta_1,$$

$$P \vdash \Theta_0 \wedge \Theta_1 \rightarrow Con_\beta,$$

$$\Theta^j_i \notin X, \; i, j = 0, 1.$$

Let $c_i = b \cap d(A + \Theta_i)$.

From Theorem 21 it follows, of course, that D_T is dense.

Suppose $a < b < 1$. Does it follow that there is a $c > a$ s. t. $b \cap c = a$? By our next result, the answers to this question and its dual are negative. We define $<<_j$ ("j" for "join") and $<<_m$ ("m" for "meet") as follows:

$a <<_j b$ iff $a < b$ and for every c, if $a \cup c \geq b$, then $c \geq b$,

$a <<_m b$ iff $a < b$ and for every c, if $b \cap c \leq a$, then $c \leq a$.

THEOREM 22 ([9,11]). (i) If $0 < a$, then there is a b s. t. $0 < b <<_j a$.

(ii) If $a < 1$, then there is a b s. t. $a <<_m b < 1$.

PROOF. (i) Let $A \in a$. By Lemma 9, there is a Π^0_1 sentence Θ s. t. $A \vdash \Theta$ and $T \nvdash \Theta$. Let $X = Th(T + \neg\Theta)$. Then X is r.e. and monoconsistent with $T + \neg\Theta$. Hence, by Corollary 1, there is a sentence $\psi \in \Pi^0_1 \cap Cons(\Sigma^0_1, T + \neg\Theta) - X$. Let $B = T + \psi \vee \Theta$ and $b = d(B)$. Then $0 < b \leq a$. Suppose $c \cup b \geq a$. Then there is an m s. t. $T + \psi + Con_{C\restriction m} \vdash \Theta$, whence $T + \neg\Theta + \psi \vdash \neg Con_{C\restriction m}$, whence $T + \neg\Theta \vdash \neg Con_{C\restriction m}$, whence $C \vdash \Theta$, whence $c \geq b$, whence $c = c \cup b \geq a$.

(ii) can rather easily be derived from the following simple

LEMMA 13. The following conditions are equivalent.

(i) For every c, if $b \cap c \leq a$, then $c \leq a$.

(ii) If σ is Σ^0_1, $A \in a$, and $b \leq d(A + \sigma)$, then $A \vdash \neg\sigma$.

We write $a <<^* b$ to mean that $a < b$ and there is a Π^0_1 sentence Θ s. t. $B \vdash \Theta$ and $A + \neg\Theta \leq A$, where $A \in a$ and $B \in b$. Then $a <<^* b$ implies $a <<_m b$. The converse of this, however, is false.

COROLLARY 7. To every $a < 1$, there is a b s. t. $a <<_m b$ but not $a <<^* b$.

PROOF. Let $A \in a$, $X = \{\varphi : a <<_m d(A + \varphi)\}$, and $Y = \{\varphi : a <<^* d(A + \varphi)\}$. By Lemma 7, $\{\varphi : A + \varphi \leq B\}$ is Π^0_2. Hence, by Lemma 13, X is Π^0_3. But, by Corollary 4, Y is a complete Σ^0_3 set and so is not Π^0_3. Finally $Y \subseteq X$. It follows that $X \nsubseteq Y$. Let $\psi \in X - Y$ and set $b = d(A + \psi)$.

Note that if $\alpha(x)$ is a PR binumeration of A, then $d(A) <<^* d(T + Con_\alpha)$ if A is consistent. This follows since $A + \neg Con_\alpha \leq A$ [1].

Theorem 22 (i) suggests the question if to each $a < 1$, there is a b s. t. $a <<_j b < 1$. The dual of this is obviously false. We now show that the answer is negative.

THEOREM 23 ([11]). There is a degree $a < 1$ s. t. if $a \leq b < 1$, then there is a degree $c < b$ s. t. $a \cup c = b$.

PROOF. If T is not Σ^0_1-sound, this is obvious, by Theorem 20. So suppose T is Σ^0_1-sound. Let $\tau(x)$ be a PR binumeration of T and let $a = d(T + Con_\tau)$. Then $a < 1$. Suppose now $a \leq b < 1$. Let $\beta(x)$ be a PR binumeration of a theory of degree b. Next let φ be s. t.

$$P \vdash \varphi \leftrightarrow \forall z(Prf_\tau(\bar{\varphi}, z) \rightarrow \exists u \leq z Prf_\beta(\overline{\neg 0=0}, u)).$$

Finally let $\hat{\varphi}$ be the sentence

$$\forall z(Prf_\beta(\overline{\neg 0=0}, z) \rightarrow \exists u < z Prf_\tau(\bar{\varphi}, u)).$$

Then, by standard arguments,

(1) $T \nvdash \varphi$,

(2) $P \vdash \varphi \vee \hat{\varphi}$,

(3) $P \vdash \varphi \wedge \hat{\varphi} \rightarrow Con_\beta$.

Clearly $P \vdash \neg\varphi \rightarrow Pr_\tau(\bar{\varphi})$. Since $\neg\varphi$ is Σ^0_1, it follows that $P \vdash \neg\varphi \rightarrow Pr_\tau(\overline{\neg\varphi})$. Thus

(4) $P \vdash Con_\tau \rightarrow \varphi$.

Let $d = d(T + \hat{\varphi})$. Then, since $\hat{\varphi}$ and Con_τ are Π^0_1, it follows from (3), (4), and Lemma 5 that $a \cup d \geq b$. Suppose $a \leq d$. Then $T + \hat{\varphi} \vdash Con_\tau$. Hence, by (2) and (4), $T \vdash \varphi$, contradicting (1). Thus $a \nleq d$. Now let $c = d \cap b$. Then $c < b$. Finally, by distributivity, $a \cup c = (a \cup d) \cap (a \cup b) = b$.

Let $\bigcup G$ ($\bigcap G$), where G is a set of degrees, be the supremum (infimum) of G if it exists. Somewhat surprisingly the following infinitary distributive laws hold.

THEOREM 24 ([11]). (i) If $\bigcup G$ exists, then $\bigcup G \cap b = \bigcup\{a \cap b : a \in G\}$.

(ii) If $\bigcap G$ exists, then $\bigcap G \cup b = \bigcap\{a \cup b : a \in G\}$.

(i) is an easy consequence of

LEMMA 14. $A{\downarrow}B \leq C$ iff $A \leq C + \neg Con_{B \restriction n}$ for every n.

To prove (ii) we need the following

LEMMA 15. $A{\uparrow}B \geq C$ iff for every (Σ^0_1) sentence Θ and every m, if $A^T + \neg Con_{C \restriction m} \dashv_{\Sigma^0_1} T + \Theta$, then $B \vdash \neg\Theta$.

PROOF. Suppose first $A{\uparrow}B \geq C$. Let Θ and m be s. t. $A^T + \neg Con_{C \restriction m} \dashv_{\Sigma^0_1} T + \Theta$. There is a k s. t. $A^T + Con_{B \restriction k} \vdash Con_{C \restriction m}$, whence $A^T + \neg Con_{C \restriction m} \vdash \neg Con_{B \restriction k}$. But then $T + \Theta \vdash \neg Con_{B \restriction k}$, whence $B \vdash \neg\Theta$. This proves "only if". To prove "if" suppose $A{\uparrow}B \ngeq C$. There is then an m s. t. for every k, $A^T + \neg Con_{C \restriction m} \nvdash \neg Con_{B \restriction k}$. But then, by Theorem 7, there is a Σ_1^0 sentence Θ s. t. $A^T + \neg Con_{C\restriction m} \dashv_{\Sigma_1^0} T + \Theta$ and $T + \Theta \not\vdash \neg Con_{B\restriction k}$ for every k. Since Θ is Σ_1^0, it follows that $B \not\vdash \neg\Theta$ and so the proof is complete.

Theorem 24 (ii) can be derived without much difficulty from Lemma 15.

By Theorem 20, if T is Σ_1^0-sound, no degree, except trivially 0 and 1, has a complement, whereas if T is not Σ_1^0-sound, some do. This leads to the question if all degrees have pseudocomplements. By (i) of the following result, the answer is negative.

THEOREM 25 ([11]). (i) If $c < 1$, then there is an $a > c$ s. t. $\{b: b \cap a = c\}$ has no supremum.
(ii) If $0 < c < 1$ and there is a Π_1^0 sentence Θ s. t. $c = d(T + \Theta)$, then there is a degree $a < c$ s.t. $\{b: b \cup a = c\}$ has no infimum.

Here (ii) is a partial dual of (i). The problem if the full dual is true remains open.

Next we consider degrees containing theories of a given form in analogy with e.g. the r.e. degrees of unsolvability. Let X be any set of sentences. Then $A[X] = \{d(A + \varphi): \varphi \in X\}$ and $a[X] = \bigcup\{A[X]: A \in a\}$. By Theorem 13, $A[\Sigma_2^0] = A[\Pi_2^0] = \{d(B): B \vdash A\}$ and so $a[\Sigma_2^0] = a[\Pi_2^0] = [a,1]$. (Here and in what follows $[a,b] = \{c: a \leq c \leq b\}$, $[a,b) = \{c: a \leq c < b\}$ and $(a,b]$ and (a,b) are defined in the obvious way.) If $A \leq B$ and σ is Σ_1^0, then, by Lemma 9, $A + \sigma \leq B + \sigma$. Hence $a[\Sigma_1^0] = A[\Sigma_1^0]$ for $A \in a$.

The following result contains some information on the sets $a[\Sigma_1^0]$, $A[\Pi_1^0]$, and $a[\Pi_1^0]$ and the relations between them.

THEOREM 26 ([11]). (i) If $a < b$, then there is a $c \in [a,b)$ s. t. $[c,b) \cap a[\Sigma_1^0] = \emptyset$.

(ii) To any $a < 1$, there is a b s. t. $a \leq b < 1$ and $[b,1] \subseteq a[\Pi_1^0]$.
(iii) If $d(A) \leq b < c$, then there are b', c' s. t. $b \leq b' < c' \leq c$ and $[b',c'] \cap A[\Pi_1^0] = \emptyset$.
(iv) If $a < b$, then there are c, c_0, c_1 s. t. $c \in a[\Sigma_1^0]$, $a \leq c_0 < c < c_1 \leq b$ and $[c_0,c_1] \cap a[\Pi_1^0] = \emptyset$.
(v) If $a < b$, there are c, d s. t. $a \leq c < d \leq b$ and $(a[\Pi_1^0] \cup a[\Sigma_1^0]) \cap [c,d] = \emptyset$.
(vi) If $A \in a < b$, then $A[\Sigma_1^0] \cap A[\Pi_1^0] \cap (a,b) \neq \emptyset$.

Our final result concerns the existence and nonexistence of infima of sets of the form A[X] where X is an r.e. set of (Σ_1^0) sentences. Let us say that the infimum $\bigcap G$ is trivial if there is a finite set $H \subseteq G$ s. t. $\bigcap G = \bigcap H$.

THEOREM 27 ([11]). Suppose A is consistent.
(i) There is a primitive recursive set X of Σ_1^0 sentences s. t. d(A) is the nontrivial infimum of A[X].
(ii) There is a primitive recursive set Y of Σ_1^0 sentences s. t. A[Y] has no infimum.

PROOF OF (i). By Corollary 7, there is a B s. t. $d(A) <<_m d(B)$ and not $d(A) <<^* d(B)$. Let $X = \{\neg Con_{B\restriction n}: n \in \omega\}$. If $C \leq A + \neg Con_{B\restriction n}$ for every n, then, by Lemma 14, $C \downarrow B \leq A$, whence $C \leq A$, since $A <<_m B$. Thus $d(A) = \bigcap A[X]$. This infimum cannot be trivial, since then there would be an m s. t. $A + \neg Con_{B\restriction m} \leq A$ contrary to the fact that not $d(A) <<^* d(B)$.

The proof of (ii) can be outlined as follows. First we observe that if Z is an r.e. set of Π_1^0 sentences, then A[Z] does not have a nontrivial infimum. Now let ψ_0 be any Π_1^0 sentence s. t. $A < A + \psi_0$. Next, applying the construction used to prove Theorem 26 (vi), we can effectively find Σ_1^0 sentences σ_n and Π_1^0 sentences ψ_{n+1} s. t. $A + \psi_n > A + \sigma_n \geq A + \psi_{n+1}$. Let $Y = \{\sigma_n : n \in \omega\}$. Then Y is as desired, since $A[\{\psi_n : n \in \omega\}]$ has no infimum.

We conclude by sketching a proof of Theorem 25 (i). Let $C \in c$. By Theorem 27 (ii), there is a primitive recursive set $Y = \{\sigma_n : n \in \omega\}$ of Σ_1^0 sentences s. t. C[Y] has no infimum. Let $A = C \cup \{\neg\sigma_n : n \in \omega\}$ and $a = d(A)$. Then, since the sentences $\neg\sigma_n$ are Π_1^0, by Lemma 14, $B \dotplus A \leq C$ iff $B \leq C + \sigma_n$ for every n. But then, by Theorem 24 (i), a supremum of $\{b : b \cap a = c\}$ would be an infimum of C[Y].

REFERENCES

[1] S.Feferman, Arithmetization of metamathematics in a general setting, Fund.Math. 49 (1960), 33-92.

[2] D.Guaspari, Partially conservative extensions of arithmetic, Trans.AMS 254 (1979), 47-68.

[3] P.Hájek, On interpretability in set theories, Comment.Math.Univ.Car. 12 (1971), 73-79.

[4] P.Hájek, On partially conservative extensions of arithmetic, Logic Colloquium 78, North-Holland, Amsterdam 1979, 225-234.

[5] P.Hájek, On interpretability in theories containing arithmetic II, Comment.Math.Univ.Car. 22 (1981), 667-688.

[6] M.Hájkova and P.Hájek, On interpretability in theories containing arithmetic, Fund.Math. 76 (1972), 131-137.

[7] D.Jensen and A.Ehrenfeucht, Some problems in elementary arithmetics, Fund.Math. 92 (1976), 223-245.

[8] G.Kreisel and A.Lévy, Reflection principles and their use for establishing the complexity of axiomatic systems, Z.Math.Logik Grundlag.Math. 14 (1968), 97-142.

[9] P.Lindström, Some results on interpretability, Proc.of the 5th Scand.Logic Symp. 1979, Aalborg 1979, 329-361.

[10] P.Lindström, On partially conservative sentences and interpretability, to appear in Proc.AMS.

[11] P.Lindström, On certain lattices of degrees of interpretability, Notre Dame JFL 25 (1984), 127-140.

[12] S.Orey, Relative interpretations, Z.Math.Logik Grundl.Math. 7 (1961), 146-153.

[13] J.Quinsey, Sets of Σ_k-conservative sentences are Π_2^0 complete, JSL 46 (1981), 442 (abstract).

[14] C.Smoryński, Calculating self-referential statements - Guaspari sentences of the first kind, JSL 46 (1981), 329-344.

[15] C.Smoryński, The incompleteness theorems, Handbook of mathematical logic (ed. J.Barwise), North-Holland 1977, 821-865.

[16] R.Solovay, On interpretability in set theories, to appear.

[17] V.Švejdar, Degrees of interpretability, Comment.Math.Univ.Car. 19 (1978), 789-813.

[18] V.Švejdar, A sentence that is difficult to interpret, Comment.Math.Univ.Car. 22 (1981), 661-666.