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HYPERABELIAN ALGEBRAS

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Commutator theory in Universal Algebra gives us the possibility of showing that seemingly different methods and concepts used by ring and group theorists are essentially similar. This, of course, is true of Universal Algebra, but the analogies thus discovered are far less evident than the ones which led to the basic concepts of substructure, congruence... As an example, a consequence of our results is that, in a certain sense, hyperabelian groups are analogues of commutative rings coinciding with their nilradical.

$\theta, \varphi, \psi \ldots$ denote (closed) elements of a given closure system $\mathcal{A}$ over A , a, b, x,... elements of A; $\langle\ \rangle$ means "generated by". We suppose that a monotone binary operation $[\ ,\ ]$ is defined on $\mathcal{A}$ in such a way that i) $[\theta,\theta] \subseteq \theta$; and ii) $[\theta,\varphi] \subseteq \psi$ implies $[\theta \vee \psi, \varphi \vee \psi] \subseteq \psi$. Examples are the congruence system of an algebra and the commutator ([4]); particularly the normal subgroups of a group and the commutator, or the bilateral ideals of an (associative) ring and $[I,J] = IJ + JI$.

$[a,b]$ stands for $[\langle a\rangle, \langle b\rangle]$; $a^{(0)} = \langle a \rangle$, and $a^{(n+1)} = [a^{(n)}, a^{(n)}]$. π is prime iff $\pi \supseteq [\theta,\varphi]$ implies $\pi \supseteq \theta$ or $\pi \supseteq \varphi$. This is the same of a prime ideal in ring theory ([2]): if $I \supseteq JH+HJ$ implies $I \supseteq J$ or $I \supseteq H$, then $I \supseteq KL$ implies $I \supseteq LKLK$, hence $I \supseteq LK$.

We define $\rho(\psi) = \bigvee_{[\theta,\theta]\subseteq\psi} \theta$; $\rho^0(\psi) = \psi$; $\rho^\alpha(\psi) = \bigvee_{\beta\in\alpha} \rho(\rho^\beta(\psi))$, and $\bar{\rho}(\psi) = \bigvee_{\alpha\in On} \rho^\alpha(\psi)$. The following theorem generalizes [3, Th. 2.15], the result of Ščukin quoted there and [2, Lemma 2.2.3]. Putting $[M,N]$ the commutator of M and G, we also obtain [3, Th. 2.19].

THEOREM. $\bar{\varrho}(\psi)=\bigcap\{\varphi\geq\psi \mid \varrho(\varphi)=\varphi\}=\bigcap\{\pi\supseteq\psi \mid \pi \text{ is prime}\}=\{x_0 \mid \text{if } (x_n)_{n\in\omega} \text{ is a sequence such that } x_{n+1}\in[x_n, x_n] \ (n\in\omega), \text{ then for some } m\in\omega: x_m\in\psi\}$. Moreover, if, for every $a\in A$ and $n\in\omega$, $a^{(n)}$ is finitely generated, then $\bar{\varrho}(\psi)=\{x \mid \text{for some } m \ x^{(m)}\subseteq\psi\}$. If $\mathscr{A}$ satisfies the a.c.c. and $[\,,\,]$ is distributive then for some m: $\bar{\varrho}(\psi)^{(m)}\subseteq\psi$.

COROLLARY. If R is a unitary ring, then $\bigcap\{P \mid P \text{ is a prime ideal of } R\}=\{x_0 \mid \text{if } (x_n)_{n\in\omega} \text{ is a sequence such that for every } n\in\omega \text{ there exist } y_n, z_n, w_n \text{ with } x_{n+1}=y_nx_nz_nx_nw_n, \text{ then for some } m: x_m=0\}$

If, in addition, R satisfies the a.c.c. on bilateral ideals, then the intersection of all primes is the largest nilpotent ideal of R.

In groups $\bar{\varrho}(0)$ coincides with the limit of the upper Fitting series, so it is natural to call an algebra hyperabelian iff in its congruence system $\bar{\varrho}(0)=1$. In this sense, hyperabelian commutative rings are rings consisting of nilpotent elements.

Bibliography.

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