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THE STRUCTURE OF FINITE ALGEBRAS

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A finite algebraic system, or algebra, is composed of a finite set of elements and a set (possibly infinite) of finitary operations acting on the set of elements. A locally finite variety is a class of similar algebras closed under the formation of homomorphic images, subalgebras, and Cartesian products, whose finitely generated algebras are finite. For example, the class of all groups satisfying the identity $x^3 = 1$ is a locally finite variety.

In these notes we present the skeleton (some definitions and results, with no proofs) of a theory of finite algebras, tame congruence theory, which effords deep insight into the structural possibilities inherent in finite algebras, as determined by their ordered systems of congruence relations. This theory has been applied to obtain new results over a broad spectrum of universal algebraic problem areas. It is particularly effective in classifying Mal'cev conditions for locally finite varieties, and in relating Mal'cev conditions to congruence properties.

The theory and results presented in this paper were obtained by myself and my student David Hobby. The results in section 3 were, for the most part, obtained by me during a seven month sabbatical spent at the University of Hawaii (from September 1983 into 1984). They owe much to a lively exchange of ideas in the Hawaii seminar, attended by Freese, Lampe, Nation, Tom Harrison, Doug Pickering, Jacob Shapiro , Stephen Tschantz and Matthew Valeriote.

1. TYPES.

Let $A = \langle A, f_i (i \in I) \rangle$ be a finite algebra. An <u>admissible relation</u> of A is any subset $S \subseteq A^n$ such that S is closed under the operations of the algebra A^n. A <u>congruence relation</u> of A is an admissible equivalence relation on A. The congruence lattice of A is denoted as Con A; its elements are the congruence relations of A and it is a sublattice of the lattice of all equivalence relations on A. By a quotient in Con A, or congruence quotient of A, we mean simply a pair $\langle\alpha,\beta\rangle$ where $\alpha,\beta \in$ Con A and $\alpha<\beta$. If $\alpha,\beta \in$ Con A and $\alpha<\beta$, we write $I[\alpha,\beta]$ for the <u>interval sublattice</u> of Con A consisting of all the δ with $\alpha\le\delta\le\beta$. We write $\alpha \prec \beta$ (or β covers α), and say that $\langle\alpha,\beta\rangle$ is a prime quotient, iff $|I[\alpha,\beta]| = 2$.

We are going to define five types of prime quotients such that every prime quotient in a finite algebra has precisely one of these types.

By the <u>clone</u> of A, written Clo A, we mean the set of all operations generated, under composition, by the basic operations $f_i (i\in I)$ of A and the trivial projection operations p_i^n $(p_i^n(x_0,\dots,x_{n-1})=x_i)$. The set of n-ary members of Clo A is written as $\mathrm{Clo}_n A$. By the <u>polynomial clone</u> of A, written Pol A, we mean the clone of the algebra $\langle A, f_i (i\in I), a (a\in A)\rangle$. Its set of n-ary members is denoted as $\mathrm{Pol}_n A$.

Now let $\langle\alpha,\beta\rangle$ be a prime congruence quotient in a finite algebra. We define the set $U_A(\alpha,\beta)$ of subsets of A (abbreviating $\mathrm{Pol}_1 A$ to $P_1 A$): $U_A(\alpha,\beta)$ consists of all sets $U = f(A)$ where $f\in P_1 A$ and $f(\beta)\not\subseteq\alpha$ (i.e. where there exist x,y in A such that $\langle x,y\rangle\in\beta$ and $\langle f(x),f(y)\rangle\notin\alpha$). Then we define $M_A(\alpha,\beta)$ to be the set of minimal members (under inclusion) of $U_A(\alpha,\beta)$. The sets $U\in M_A(\alpha,\beta)$ are called <u>$\langle\alpha,\beta\rangle$-minimal sets</u>.

To state the first theorem, we need the notions of polynomial isomorphism and induced algebra. Let $\alpha\in$ Con A, $f\in \mathrm{Pol}_n A$, and S be a nonvoid subset of an algebra A. By $\alpha|_S$ we mean $\alpha\cap(S\times S)$; by $f|_S$ we mean $f\cap(S^n\times A)$. By $A|_S$ we mean the non-indexed algebra $\langle S, (\mathrm{Pol}\ A)\mathring{|}_S\rangle$, where $(\mathrm{Pol}\ A)\mathring{|}_S = \bigcup_n \{f|_S : f\in \mathrm{Pol}_n A$ and $f(S^n)\subseteq S\}$. This algebra, $A|_S$, is called the <u>algebra induced by A on S</u>.

Let $f\in P_1 A$ and $S\cup T\subseteq A$. We write $f: S\simeq T$ to mean that $f(S) = T$ and for some $g\in P_1 A$ we have $g(T)=S$ and $gf|_S = id_S$ and $fg|_T = id_T$. We write $S\simeq T$ (or $S \overset{A}{\simeq} T$) iff there exists $f\in P_1 A$ with $f: S\simeq T$. The relation $\simeq$, of <u>polynomial isomorphism</u>, is an equivalence relation on the subsets of an algebra. Moreover, if $S \overset{A}{\simeq} T$ then the structures

$\langle S, (\mathrm{Pol}\ A)\mathring{|}_S, \alpha|_S (\alpha\in \mathrm{Con}\ A)\rangle$ and $\langle T, (\mathrm{Pol}\ A)\mathring{|}_T, \alpha|_T (\alpha\in \mathrm{Con}\ A)\rangle$

are isomorphic.

<u>THEOREM 1</u>. Let $\langle\alpha,\beta\rangle$ be a prime quotient in a finite algebra A.

(1) For each $U\in M_A(\alpha,\beta)$, we have $\{T: T\simeq U\} = M_A(\alpha,\beta)$.

(2) For each $U\in M_A(\alpha,\beta)$ there is $e\in P_1 A$ such that $e=e^2$ and $e(A)=U$.

(3) For $U\in M_A(\alpha,\beta)$ and $f\in P_1 A$ we have $f: U\simeq f(U)$ iff $f(\beta|_U)\not\subseteq\alpha$.

(4) If $f\in P_1 A$ then $f(\beta)\not\subseteq\alpha$ iff $f: U\simeq f(U)$ for some $U\in M_A(\alpha,\beta)$.

(5) if $\langle x,y\rangle\in\beta-\alpha$ and $U\in M_A(\alpha,\beta)$ then there exists $f\in P_1 A$ with $f(A) = U$ and $\langle f(x), f(y)\rangle\in\beta|_U-\alpha|_U$.

(6) If $U\in M_A(\alpha,\beta)$ and $\rho = \alpha\cup\bigcup\{f(\beta|_U): f\in P_1 A\}$, then the transitive closure of ρ is β.

Any quotient $\langle\alpha,\beta\rangle$ in a finite algebra A which satisfies Theorem 1 is called <u>tame</u>. According to the theorem, prime quotients are tame. By a <u>tight lattice</u> we mean a finite lattice L such that if ρ is any binary, reflexive, symmetric admissible relation of L (i.e. if ρ is a <u>tolerance</u>), and if $\rho\ne L^2$, then $\langle x,1\rangle\in\rho$ iff x=1 and $\langle x,0\rangle\in\rho$ iff x=0. Tight lattices include finite projective geometries, the full finite partition lattices, and many more.

THEOREM 2. Let $\langle\alpha,\beta\rangle$ be a quotient in a finite algebra. If $I[\alpha,\beta]$ is tight then $\langle\alpha,\beta\rangle$ is tame.

The five mutually exclusive types of tame quotients are discerned through an extensive study of the induced algebras associated with a quotient. Let $\langle\alpha,\beta\rangle$ be a tame quotient in a finite algebra A. By Theorem 2, all the structures $\langle U,(\mathrm{Pol}\,A)^{\circ}|_U, \alpha|_U,\beta|_U\rangle$ for $U\in M_A(\alpha,\beta)$, are isomorphic. We define the set $T_A(\alpha,\beta)$ of $\langle\alpha,\beta\rangle$-traces like this: $N\in T_A(\alpha,\beta)$ iff for some $U\in M_A(\alpha,\beta)$, and $x\in U$, we have $N = x/(\beta|_U) \neq x/(\alpha|_U)$. Thus N is an $\langle\alpha,\beta\rangle$-trace in A iff for some $\langle\alpha,\beta\rangle$-minimal set U, N is an $\langle\alpha|_U,\beta|_U\rangle$-trace in $A|_U$.

Let $\langle\alpha,\beta\rangle$ be tame in A and choose $U\in M_A(\alpha,\beta)$. Exactly one of the following five possibilities occurs.

(1) (Unary type). We write $\mathrm{typ}(\alpha,\beta) = 1$ iff for every $\langle\alpha|_U,\beta|_U\rangle$-trace N, $(A|_N)/(\alpha|_N)$ is essentially unary.

(2) (Affine type). We write $\mathrm{typ}(\alpha,\beta) = 2$ iff for some $\langle\alpha|_U,\beta|_U\rangle$-trace N, $(A|_N)/(\alpha|_N)$ is polynomially equivalent to a vector space. When this holds, all of these traces in U are $\simeq$, and their union B (the body of U) satisfies: $A|_B$ is Mal'cev and nilpotent

(3) (Boolean type). We write $\mathrm{typ}(\alpha,\beta) = 3$ iff there is a unique $\langle\alpha|_U,\beta|_U\rangle$-trace N and it satisfies: $|N| = 2$ and $A|_N$ is polynomially equivalent to a 2-element Boolean algebra.

(4) (Lattice type). We write $\mathrm{typ}(\alpha,\beta) = 4$ iff there is a unique $\langle\alpha|_U,\beta|_U\rangle$-trace N and it satisfies: $|N| = 2$ and $A|_N$ is polynomially equivalent to a 2-element lattice.

(5) (Semilattice type). We write $\mathrm{typ}(\alpha,\beta) = 5$ iff there is a unique $\langle\alpha|_U,\beta|_U\rangle$-trace N and $(A|_N)/(\alpha|_N)$ is polynomially equivalent to a 2-element semilattice. When this holds, N is the disjoint union of α_U-classes $I = \{u\}$ and O, and A has a binary polynomial $f(x,y)$ such that $f(u,z)=f(z,u)=z$ for all $z\in N$, and $f(O\times O)\subseteq O$.

THEOREM 3. Every tame quotient in a finite algebra has precisely one of the types 1,...,5.

Recall that a Mal'cev operation on a set is an operation $p(x,y,z)$ satisfying the equations $p(x,x,y) = y = p(y,x,x)$. An algebra A is called Mal'cev iff there is a Mal'cev operation $p\in \mathrm{Clo}_3 A$. From the above description of the types, it follows that if $\langle\alpha,\beta\rangle$ is a tame quotient of non-unary type in an algebra A, then $T_A(\alpha,\beta)$ (the set of traces) is an equivalence class of subsets of A under $\simeq$. The algebras A_N, $N\in T_A(\alpha,\beta)$, are Mal'cev if and only if $\mathrm{typ}(\alpha,\beta)\in\{2,3\}$. These algebras are 2-element algebras if $\mathrm{typ}(\alpha,\beta)\in\{3,4\}$.

Once the types are discerned, each type can be characterized in a more "natural" manner, without any mention of minimal sets and traces. To do this, we need new definitions.

DEFINITION 4. Let A be any algebra and $\alpha,\beta,\gamma\in\mathrm{Con}\,A$.

(1) We use the formula $C(\alpha,\beta;\gamma)$ (in words, α centralizes β modulo γ) as an abbreviation for the following property. For every $n>1$ and every $f\in\mathrm{Pol}_n A$ and for all $\langle u,v\rangle\in\alpha$ and $\langle x_1,y_1\rangle,\dots,\langle x_{n-1},y_{n-1}\rangle\in\beta$, this equivalence holds:

$$f(u,\bar{x}) \overset{\gamma}{\equiv} f(u,\bar{y}) \leftrightarrow f(v,\bar{x}) \overset{\gamma}{\equiv} f(v,\bar{y}).$$

(2) We use the formula $C^*(\alpha,\gamma)$ (in words, α is strongly Abelian modulo γ) to abbreviate this property. For every $n>1$ and every $f\in\mathrm{Pol}_n A$ and for all $\langle u,v\rangle\in\alpha$ and $x_1,y_1,z_1,\dots,x_{n-1},y_{n-1},z_{n-1}$ such that $x_i\equiv y_i\equiv z_i \pmod{\alpha}$ for all i, we have

$$f(u,\bar{x}) \overset{\gamma}{\equiv} f(v,\bar{y}) \rightarrow f(u,\bar{z}) \overset{\gamma}{\equiv} f(v,\bar{z}).$$

DEFINITION 5. Let α and β be congruences of an algebra A with $\alpha \le \beta$.

(1) β is Abelian over α iff $C(\beta,\beta;\alpha)$.

(2) β is strongly Abelian over α iff $C^*(\beta,\alpha)$.

(3) β is solvable (or strongly solvable) over α iff there exist congruences $\alpha = \alpha_0 \le \alpha_1 \le \dots \le \alpha_n = \beta$ for some n with α_{i+1} Abelian over α_i (or strongly Abelian over α_i) for each $i<n$.

(4) A quotient $\langle\delta,\gamma\rangle$ is Abelian, strongly Abelian, solvable, or strongly solvable iff γ has this property over δ.

(5) A congruence γ is said to have one of the four properties iff it has the property over 0_A (the least congruence).

(6) A has one of the four properties iff 1_A has the property over 0_A.

THEOREM 6. Let $\langle\alpha,\beta\rangle$ be tame in a finite algebra A.

(1) The following are equivalent.

(i) typ $(\alpha,\beta) = 1$.

(ii) $\langle\alpha,\beta\rangle$ is strongly Abelian.

(iii) There do not exist $\langle a,b\rangle \in \beta-\alpha$, $f \in \mathrm{Pol}_2 A$ such that $f(b,b)=b$ and $f(a,b)=f(b,a)=a$.

(2) The following are equivalent.

(i) typ $(\alpha,\beta) = 2$.

(ii) $\langle\alpha,\beta\rangle$ is Abelian and not strongly Abelian.

(3) The following are equivalent.

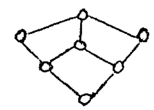
(i) typ $(\alpha,\beta) \in \{1,2\}$.

(ii) $\langle\alpha,\beta\rangle$ is Abelian.

(iii) There do not exist $\langle a,b\rangle \in \beta-\alpha$ and $f \in \mathrm{Pol}_2 A$ such that $f(b,b) = b$ and $f(a,b)=f(b,a)=f(a,a)=a$.

(4) We have typ$(\alpha,\beta) \in \{4,5\}$ iff $\langle\alpha,\beta\rangle$ is non-Abelian and A has an admissible pre-order ρ such that $\rho \cap \rho^{\cup} = \alpha$ and β = tr.clo.$(\rho \cup \rho^{\cup})$. (In this case, there are precisely two minimal such pre-orders).

(5) We have typ$(\alpha,\beta)=5$ iff typ $(\alpha,\beta) \in \{4,5\}$ and, where ρ and $\rho^{\cup}$ are the minimal pre-orders (from (4)) and $\tau = \rho \vee \rho^{\cup}$ (in the subalgebra lattice of A^2), the interval $I[(\alpha\times\alpha)|_\tau, (\beta\times\beta)|_\tau]$ is isomorphic to



We can often recognize immediately that a quotient $\langle\alpha,\beta\rangle$ is Abelian, or even strongly Abelian, just from the shape of the interval lattice $I[\alpha,\beta]$. A lattice homomorphism $\varphi: L \rightarrow L'$, where L has 0 and 1, is 0,1-separating iff $\varphi^{-1}\{\varphi(0)\} = \{0\}$ and $\varphi^{-1}\{\varphi(1)\} = \{1\}$

THEOREM 7. Let $\langle\alpha,\beta\rangle$ be any quotient in a finite algebra A.

(1) If $\langle\alpha,\beta\rangle$ is not strongly Abelian, the following are equivalent.

(i) $\langle\alpha,\beta\rangle$ is tame.

(ii) $I[\alpha,\beta]$ is tight.

(iii) $I[\alpha,\beta]$ admits an (essentially unique) 0,1-separating homomorphism onto the congruence lattice of a vector space (implying that $I[\alpha,\beta]$ is complemented and every non-constant lattice homomorphism from it is 0,1-separating).

(2) If $\langle\alpha,\beta\rangle$ is not Abelian then $I[\alpha,\beta]$ is tight iff $\alpha \prec \beta$.

From this theorem, we conclude that for every tight interval $I[\alpha,\beta]$ in the congruence lattice of a finite algebra, if $|I[\alpha,\beta]|>2$ then $\langle\alpha,\beta\rangle$ is Abelian, and if in addition, 7.1.(iii) fails, then $\langle\alpha,\beta\rangle$ is strongly Abelian. Consult [7] for some nice qpplications of these results.

2. LABELLED CONGRUENCE LATTICES.

The congruence lattice of a finite algebra becomes a labelled graph when we label each prime quotient with its type. We define <u>type sets</u> as follows.

<u>DEFINITION 8</u>. Let A be a finite algebra, $\mathcal{V}$ be a locally finite variety, and let $\delta < \gamma$ in Con A.

(1) $\operatorname{typ}\{\delta,\gamma\} = \{\operatorname{typ}(\alpha,\beta) : \delta \leq \alpha \prec \beta \leq \gamma\}$.

(2) $\operatorname{typ}\{A\} = \operatorname{typ}\{0_A, 1_A\}$.

(3) $\operatorname{typ}\{\mathcal{V}\} = \bigcup \{\operatorname{typ}\{A\} : A \in \mathcal{V}_{fin}\}$ where $\mathcal{V}_{fin}$ is the class of finite algebras in $\mathcal{V}$.

We shall see that the labelling of a congruence lattice is not entirely arbitrary. The type set of a locally finite variety is very strongly related to the Mal'cev conditions it satisfies. The solvability congruences on Con A are a very useful tool for all further results.

We define a 1-<u>snag</u> of an algebra A to be a pair $\langle a,b\rangle \in A^2$ such that $a \neq b$ and there is $f \in \operatorname{Pol}_2 A$ with $f(b,b) = b$, $f(a,b) = f(b,a) = a$, (See Theorem 6 (1-(iii)) and (3(iii)).) A 2-<u>snag</u> is a 1-snag with the added requirement that $f(a,a) = a$. (Equivalently, "$\langle a,b\rangle$ is a 2-snag or $\langle b,a\rangle$ is a 2-snag" just in case $A|_{\{a,b\}}$ is non-Abelian).

<u>THEOREM 9</u>. Let A be finite and $\delta < \gamma$ be congruences of A. The following are equivalent.

(1) $\langle\delta,\gamma\rangle$ is solvable (strongly solvable).

(2) $\operatorname{typ}\{\delta,\gamma\} \subseteq \{1,2\}$ ($\operatorname{typ}\{\delta,\gamma\} = \{1\}$).

(3) For all $\delta \leq \alpha \prec \beta \leq \gamma$, $\langle\alpha,\beta\rangle$ is Abelian (strongly Abelian).

(4) There does not exist a 2-snag $\langle a,b\rangle$ (a 1-snag $\langle a,b\rangle$) with $\langle a,b\rangle \in \gamma - \delta$.

Motivated by Theorem 9, we define for any pair of congruences γ_0, γ_1 in any algebra A, $\gamma_0 \overset{s}{\sim} \gamma_1$ (or $\gamma_0 \overset{ss}{\sim} \gamma_1$) iff there does not exist a 2-snag (or a 1-snag) $\langle a,b\rangle \in (\gamma_0 \vee \gamma_1) - (\gamma_0 \wedge \gamma_1)$.

A lattice L is said be <u>meet-semi-distributive</u>, or to satisfy SD($\wedge$), iff for all $x,y,z,u \in L$, $x \wedge y = x \wedge z = u \rightarrow x \wedge (y \vee z) = u$. The <u>join-semi-distributivity</u> condition, SD($\vee$), is dual to SD($\wedge$).

If the algebra A is locally finite and $\delta \leq \gamma$ are congruences of A, then by Theorem 9 it follows that $\delta \overset{s}{\sim} \gamma$ iff for all finitely generated $B \subseteq A$, we have $\gamma|_B$ solvable over $\delta|_B$. We say that A is <u>locally solvable</u> iff $0_A \overset{s}{\sim} 1_A$.

<u>THEOREM 10</u>. Let A be any locally finite algebra.

(1) $\overset{s}{\sim}$ and $\overset{ss}{\sim}$ are congruences on $L = \mathrm{Con}\,A$. $L/\overset{s}{\sim}$ is an algebraic lattice and $L \twoheadrightarrow L/\overset{s}{\sim}$ is a complete lattice homomorphism. The some facts hold for $\overset{ss}{\sim}$.

(2) $L/\overset{s}{\sim}$ satisfies SD($\wedge$).

(3) If A is finite and $5 \notin \mathrm{typ}\{A\}$ then $L/\overset{s}{\sim}$ satisfies SD($\vee$).

(4) For every $\theta \in L$, $\{\Psi : \theta \overset{s}{\sim} \Psi\}/\overset{ss}{\sim}$ is a modular lattice.

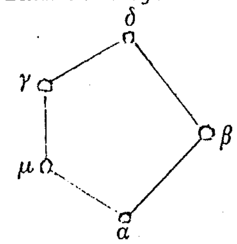
The proof of this theorem involves some work. It follows easily from the special case when A is finite. Some of the facts used are in the next lemmas.

<u>LEMMA 11</u>. Let $\langle\alpha,\beta\rangle$ be a non-Abelian prime quotient of an algebra A. There exists a unique congruence δ such that for all congruences μ
$\mu \wedge \beta = \alpha \leftrightarrow \alpha \leq \mu \leq \delta$.

<u>LEMMA 12</u>. Let $\langle\alpha,\beta\rangle$ be a prime quotient of type 3 or 4 in a finite algebra A. There exists a congruence δ such that for all congruences μ, $\mu \vee \alpha = \beta$ iff $\delta \leq \mu \leq \beta$.

<u>LEMMA 13</u>. Let $\langle\alpha_i, \beta_i\rangle$ $(i=0,1)$ be transposed prime quotients in a finite algebra A (i.e. $\alpha_0 \wedge \beta_1 = \alpha_1$ and $\alpha_0 \vee \beta_1 = \beta_0$). Then $\mathrm{typ}(\alpha_0,\beta_0) = \mathrm{typ}(\alpha_1,\beta_1)$ and $M_A(\alpha_0,\beta_0) = M_A(\alpha_1,\beta_1)$.

<u>LEMMA 14</u>. Let $(\alpha,\beta,\delta,\gamma,\mu)$ be a pentagon in the congruence lattice of a finite algebra A.



If $\langle\alpha,\beta\rangle$ is solvable, then $\langle\mu,\gamma\rangle$ is strongly solvable.

Consider the lattices

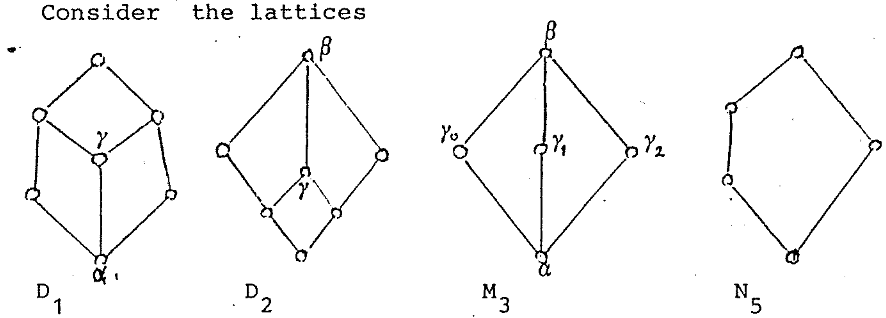


Fig. 15

It follows from Theorem 10.(2) that <u>if</u> D_1 is a sublattice of Con A, with A locally finite, <u>then</u> γ is locally solvable over α. From Theorem 10.(3), <u>if</u> D_2 occurs, A is finite, and $5 \notin \mathrm{typ}\{A\}$, <u>then</u> $\langle\gamma,\beta\rangle$ is solvable. If M_3 occurs then β is locally solvable over α. (In fact, one can show directly that in an M_3, the quotients $\langle\alpha,\gamma_i\rangle$ must be Abelian.)

The presence of a copy of N_5 in Con A is, of course, equivalent to the non-modularity of this lattice. At the level of (locally finite) varieties, the absence of copies of D_1 in congruence lattices defines a large new class of moderately well-behaved varieties. The same is true for each of the lattices D_2 and M_3. These results are mostly presented in the next section. Part of the background facts used in their proofs will now be given.

LEMMA 16. Let A be a finite algebra and suppose that Con A has a sublattice isomorphic to D_1 as in Fig. 15. For each congruence δ satisfying $\alpha \prec \delta \leq \gamma$ we have $\mathrm{typ}(\alpha,\delta) = 1$.

LEMMA 17. Let A be a finite algebra and suppose that Con A has a sublattice isomorphic to D_2 as in Fig.15. For each congruence δ satisfying $\gamma \leq \delta \prec \beta$ we have $\mathrm{typ}\,(\delta,\beta) \in \{1,5\}$.

THEOREM 18. Let $\mathcal{V}$ be any locally finite variety. The following are equivalent.

(1) $1 \notin \mathrm{typ}\,\{\mathcal{V}\}$.

(2) D_1 is not a sublattice of the congruence lattice of any algebra in $\mathcal{V}$.

(3) Con A is in the class SD($\wedge$)/(Modular) for all $A \in \mathcal{V}$.

(4) If $\alpha, \beta \in$ Con A, $A \in \mathcal{V}$, and $\alpha \overset{s}{\sim} \beta$ then $\alpha \circ \beta = \beta \circ \alpha$.

(5) $\mathcal{V}$ has a term operation $p(x,y,z)$ satisfying $p(x,x,x) = x$ (as an identity in $\mathcal{V}$) and such that for every locally solvable congruence β on an algebra $A \in \mathcal{V}$, and for every β-class $T = x/_{\beta}$, the operation $p|_T$ is Mal'cev.

We record in passing the following result.

THEOREM 19. Let $\mathcal{V}$ be any locally finite variety. The class $\mathcal{V}^s$ of locally solvable algebras in $\mathcal{V}$, and the class $\mathcal{V}^{ss}$ of locally strongly solvable algebras, are varieties. If $1 \notin \mathrm{typ}\,\{\mathcal{V}\}$ then $\mathcal{V}^s$ is a congruence permutable variety (and $\mathcal{V}^{ss}$ is trivial).

3. MAL'CEV CLASSIFICATION AND OMITTING TYPES.

The clone of a variety $\mathcal{V}$, denoted Clo $\mathcal{V}$, is equal to the clone of term operations of the free algebra in $\mathcal{V}$ on denumerably many generators: Clo $\mathcal{V}$ = Clo $F_{\mathcal{V}}(\omega)$.
We can consider clones as abstract multi-sorted algebras with operations of composition. An algebra in $\mathcal{V}$ with base set A is basically just a clone homomorphism of Clo $\mathcal{V}$ into the clone of all operations on A.

W. Taylor and W. Neumann have defined a lattice of varieties in which they put $\mathcal{V} \leq \mathcal{W}$ iff hom (Clo $\mathcal{V}$, Clo $\mathcal{W}$) $\neq \emptyset$. (They say $\mathcal{V}$ is interpretable into $\mathcal{W}$ when this holds. The relation $\leq$ is a pre-order on varieties. The members of the lattice of interpretability are equivalence classes of varieties. See [4]).

Filters in the lattice of interpretability tend to have an algebraic significance. Principal filters of the form $\{\mathcal{V} : \mathcal{W} \leq \mathcal{V}\}$ where $\mathcal{W}$ is a finitely presented variety, have been called strong Mal'cev conditions. For an example, a variety $\mathcal{V}$ has permuting congruences (we have called such a variety Mal'cev) iff $\mathcal{W} \leq \mathcal{V}$ where $\mathcal{W}$ is the variety with one basic operation $m(x,y,z)$ defined by the identities $m(x,x,y)=y$ and $m(x,y,y)=x$. (Mal'cev proved this.) Jónsson proved that a variety $\mathcal{V}$ has distributive congruence lattices iff $\mathcal{D}_n \leq \mathcal{V}$ for some n, where $\mathcal{D}_n$ is defined by identities:

$$x = f_o(x,y,z) = f_i(x,y,x) \qquad (0 \leq i \leq n)$$
$$z = f_n(x,y,z)$$
$$f_{2i}(x,y,y) = f_{2i+1}(x,y,y) \qquad (0 \leq i \leq [(n-1)/2])$$
$$f_{2i-1}(x,x,y) = f_{2i}(x,x,y) \qquad (1 \leq i \leq [n/2]).$$

Since $\mathcal{D}_1 \geq \mathcal{D}_2 \geq \ldots$, congruence distributive varieties constitute another filter in the lattice of interpretability; this filter has a countable base of finitely presented varieties. Such filters are called, simply, Mal'cev conditions. Countable intersections of Mal'cev conditions are called weak Mal'cev conditions. For every pure lattice identity ε, the class of all varieties $\mathcal{V}$ such that $\mathcal{V} \models_{con} \varepsilon$ (i.e. the congruence lattice of every algebra in $\mathcal{V}$ satisfies ε) is a weak Mal'cev condition (Pixley [11], Wille [12]).

Another important strong Mal'cev condition is that of n-permutability (where $n \geq 2$). A variety is called n-permutable iff it satisfies $\alpha \circ_n \beta = \beta \circ_n \alpha \ldots$ where $\alpha \circ_n \beta = \alpha \circ \beta \circ \ldots$ (n alternating α's and β's) for all congruences α, β on any algebra. $\mathcal{V}$ is n-permutable iff there are $f_1, \ldots f_{n-1} \in Clo_3 \mathcal{V}$ satisfying

$$f_1(x,y,y) = x$$
$$f_i(x,x,y) = f_{i+1}(x,y,y) \qquad (1 \leq i < n-1)$$
$$f_{n-1}(x,x,y) = y$$

Our theorems in this section impose a well defined order on the diversity of Mal'cev conditions, at least for locally finite varieties. Before we present these theorems, we require new definitions.

A variety $\mathcal{W}$ will be called special if it is finitely presented and its operations are idempotent (satisfy $f(x,\ldots,x) = x$). A variety $\mathcal{W}$ will be called simple if it is special, has basic operations $h_1, \ldots, h_n$ for some n, and is defined by simple identities, i.e. identities of the form $h_a(x_{i_o}, \ldots, x_{i_{k-1}}) = x_j$ and $h_a(x_{i_o}, \ldots, x_{i_{k-1}}) = h_b(y_{j_o}, \ldots, y_{j_{l-1}})$ in which there is no superposition of operations.

The following lemma is involved in part of the proof of each result of this section.

LEMMA 20. Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$, $e \in Pol_1 A$, $\theta \in$ Con A, and suppose that $e = e^2$ and $N = e(A) \wedge x/\theta \neq \emptyset$ for some $x \in A$. If $\mathcal{W}$ is any simple variety such that $\mathcal{W} \leq \mathcal{V}$ then $\mathcal{W} \leq HSP(A|_N)$.

COROLLARY 21. Let $\mathcal{V}$ be a locally finite variety and $\mathcal{M} = \bigcap_n \bigcup_m \{\mathcal{X} : \mathcal{W}_{nm} \leq \mathcal{X}\}$ be a weak Mal'cev condition where each $\mathcal{W}_{nm}$ is simple. Suppose that $\mathcal{V} \in \mathcal{M}$. For i = 1,2,...,5 in typ$\{\mathcal{V}\}$, $\mathcal{M}$ contains, respectively, the variety of sets, of vector spaces over some finite field, of Boolean algebras, of lattices, of semilattices.

We remark that every weak Mal'cev condition we have mentioned so far can be defined using only simple varieties, and so the corollary applies for it.

The types of prime quotients can be ordered by the richness of the set of essentially binary polynomial operations in a 2-element algebra whose prime quotient has that type. We get this picture (a lattice):

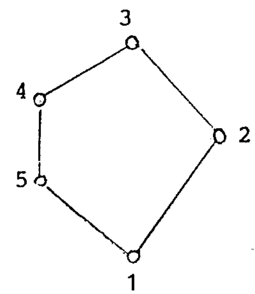


Fig.22

For each nonvoid order ideal I(with $3 \notin I$) in this lattice of types, we shall give a nice characterization of the class of locally finite varieties $\mathcal{V}$ satisfying typ$\{\mathcal{V}\} \cap I = \emptyset$.

The class of varieties omitting 1 has already been characterized in several ways in Theorem 18. Here are additional characterizations. In the next several theorems, β_m and γ_m denote lattice terms built from lattice variables α, β, γ as follows:

$$\beta_o = \beta, \gamma_o = \gamma$$

$$\beta_{m+1} = \beta \vee (\alpha \wedge \gamma_m), \gamma_{m+1} = \gamma \vee (\alpha \wedge \beta_m).$$

THEOREM 23. For a locally finite variety $\mathcal{V}$ the following are equivalent.

(1) $1 \notin \text{typ}\{\mathcal{V}\}$.

(2) There exists a simple $\mathcal{W}$ such that $\mathcal{W} \leq \mathcal{V}$ and $\mathcal{W} \nleq$ SETS (the variety of sets without operations).

(3) There exists a special $\mathcal{W} \leq \mathcal{V}$ with $\mathcal{W} \nleq$

(4) For some $m \geq 1$, $\mathcal{V} \models_{con} \alpha \wedge (\beta \circ \gamma) \subseteq \gamma_m \circ \beta_m$.

The property (4) of this theorem can be expressed as a Mal'cev condition, $\bigvee_{m,n} (\mathcal{W}_{mn} \leq \mathcal{V})$, where the $\mathcal{W}_{mn}$ are simple. As far as locally finite varieties can tell, this is the largest non-trivial Mal'cev condition defined by special varieties.

In the next theorem, $\mathcal{L}$ is the variety of semilattices and D_2 is the lattice of Fig. 15.

THEOREM 24. For a locally finite $\mathcal{V}$ the following are equivalent

(1) $1, 5 \notin \text{typ}\{\mathcal{V}\}$.

(2) There exists a special $\mathcal{W} \leq \mathcal{V}$ such that $\mathcal{W} \nleq \mathcal{L}$.

(3) $\mathcal{V} \models_{con} \alpha \wedge (\beta \circ \gamma) \subseteq (\gamma \vee (\alpha \wedge \beta)) \circ (\beta \vee (\alpha \wedge \gamma))$.

(4) D_2 is not a sublattice of the congruence lattice of any finite algebra in $\mathcal{V}$.

(5) For every finite $A \in \mathcal{V}$, $L = \text{Con}\, A$ has a congruence Θ such that $L/\Theta \models SD(\vee)$ (and $SD(\wedge)$) and each α/Θ is a modular lattice.

Notice that Theorem 24 (3) can be written out quite easily as a Mal'cev condition. Namely

(6) There exist $f_o, \ldots, f_{2n}, g_o, \ldots, g_{2n} \in \text{Clo}_3 \mathcal{V}$ for some n, satisfying

$$x = f_o(x,y,z),\ z = g_{2n}(x,y,z),\ f_{2n}(x,y,z) = g_o(x,y,z)$$

$$f_{2i}(x,y,x) = f_{2i+1}(x,y,x),\ f_{2i}(x,x,y) = f_{2i+1}(x,x,y)$$

$$g_{2i}(x,y,x) = g_{2i+1}(x,y,x),\ g_{2i}(x,y,y) = g_{2i+1}(x,y,y)$$

$$f_{2i+1}(x,y,y) = f_{2i+2}(x,y,y),\ g_{2i+1}(x,x,y) = g_{2i+2}(x,x,y)$$

(for $0 \leq i < n$).

Here is a sketch of the proof that 24(1) $\Rightarrow$ 24(3). Let $F = F_{\mathcal{V}}(x,y,z)$ and let a,b,g be endomorphisms of F defined by

t	a(t)	b(t)	g(t)
x	x	x	x
y	y	x	z
z	x	z	z

Let $\alpha = \text{Ker}(a)$, $\beta = \text{Ker}(b)$, $\gamma = \text{Ker}(g)$. Thus $\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma$. By Theorem 10 (especially (3)), $\gamma \vee (\alpha \wedge \beta) \overset{s}{\sim} \gamma \vee \beta \overset{s}{\sim} \beta \vee (\alpha \wedge \gamma)$. Then by Theorem 18(4), $\gamma \vee (\alpha \wedge \beta) = \gamma_1$ and $\beta \vee (\alpha \wedge \gamma) = \beta_1$ commute. Thus $\langle x,z \rangle \in \beta_1 \circ \gamma_1 = \gamma_1 \circ \beta_1$. This gives (6), which easily implies (3). The proof that 23(1) $\Rightarrow$ 23(4) is similar.

For every field k, ${}_k\mathcal{M}$ denotes the variety of vector spaces over k. The lattice terms $\alpha, \beta, \gamma, \beta_m, \gamma_m$ are as in Theorem 23.

THEOREM 25. For any locally finite variety $\mathcal{V}$ the following are equivalent.

(1) $1, 2 \notin \text{typ}\{\mathcal{V}\}$.

(2) There exists a simple $\mathcal{W} \leq \mathcal{V}$ such that $\mathcal{W} \nleq {}_k\mathcal{M}$ for every finite field k.

(3) $\mathcal{V} \models_{con} \alpha \wedge (\beta \circ \gamma) \subseteq \beta_m$ for some m.

(4) M_3 is not a sublattice of Con A for any $A \in \mathcal{V}$.

(5) $\mathcal{V} \models_{con}$ SD($\wedge$).

(6) $\mathcal{V}$ has no non-trivial Abelian congruences.

THEOREM 26. The following are equivalent, for any locally finite variety $\mathcal{V}$.

(1) typ $\{\mathcal{V}\} \subseteq \{3,4\}$.

(2) There exists a simple $\mathcal{W} \leq \mathcal{V}$ such that:
$\mathcal{W} \nleq \mathcal{L}$ (semilattices) and $\mathcal{W} \nleq {}_k\mathcal{M}$ for every finite field k.

(3) $\mathcal{V} \models_{con} \alpha \wedge (\beta \circ \gamma) \subseteq \beta \vee (\alpha \wedge \gamma)$.

(4) Neither M_3 nor D_2 is a sublattice of Con A for any finite $A \in \mathcal{V}$.

(5) $\mathcal{V}_{fin} \models_{con}$ SD($\vee$) (and SD($\wedge$)).

Let $\mathcal{D}$ denote the variety of distributive lattices.

THEOREM 27. The following are equivalent, for any locally finite variety $\mathcal{V}$.

(1) typ $\{\mathcal{V}\} \subseteq \{2,3\}$.

(2) There exists a simple $\mathcal{W} \leq \mathcal{V}$ such that $\mathcal{W} \nleq \mathcal{D}$.

(3) $\mathcal{V}$ is congruence n-permutable for some n.

Proof. That (3) $\to$ (2) is trivial. That (2) $\to$ (1) is by Corollary 21. (If $\mathcal{W} \nleq \mathcal{D}$, then $\mathcal{W} \nleq \mathcal{L}$ and $\mathcal{W} \nleq$ SETS). To prove (1) $\to$ (3), suppose (3) is false. Let $F = F_{\mathcal{V}}(x,y)$, a finite algebra. Let ρ be the admissible binary relation on F generated by <x,x>, <y,y>, <y,x>, and let ρ^t be the transitive closure of ρ, and $\theta = \rho^t \cap \rho^{t\cup}$. Since $\mathcal{V}$ is not n-permutable for any n, $\langle x,y\rangle \notin \theta$. Now θ is a congruence. We can find congruences α, β with $\theta \leq \alpha \prec \beta$ and $\langle x,y\rangle \in \beta - \alpha$. By Theorem 1(5), there is an $\langle\alpha,\beta\rangle$-trace N and $f \in \mathrm{Pol}_1 F$ with $\langle f(x), f(y)\rangle \in \beta|_N - \alpha|_N$.

If typ$(\alpha,\beta) \in \{2,3\}$ then $F|_N$ has a Mal'cev operation, and thus there exists $g \in \mathrm{Pol}_1 F$ with $gf(x) = f(y)$, $gf(y) = f(x)$. But ρ is closed under all polynomials (it is reflexive). Thus $\langle f(x), f(y)\rangle \in$

$\in \rho \cap \rho^{\cup} \leq \theta \leq \alpha$ if typ$(\alpha,\beta) \in \{2,3\}$. This contradiction proves that typ $\{\mathcal{V}\} \nsubseteq \{2,3\}$.

THEOREM 28. The following are equivalent, for any locally finite variety $\mathcal{V}$.

(1) typ $\{\mathcal{V}\} \subseteq \{3\}$.

(2) There exists a simple $\mathcal{W} \leq \mathcal{V}$ such that: $\mathcal{W} \nleq \mathcal{D}$ and $\mathcal{W} \nleq {}_k\mathcal{M}$ for every finite field k.

(3) For some n there exist $f_0, \dots, f_n \in \mathrm{Clo}_4 \mathcal{V}$ satisfying

$$x = f_0(x,y,y,z)$$
$$f_i(x,x,y,x) = f_{i+1}(x,y,y,x)$$
$$f_i(x,x,y,y) = f_{i+1}(x,y,y,y) \quad \text{(for } i<n)$$
$$f_n(x,x,y,z) = z.$$

(4) $\mathcal{V}$ is n-permutable for some n and $\mathcal{V} \models_{con}$ SD($\wedge$).

(5) $\mathcal{V}$ is n-permutable for some n and $\mathcal{V} \models_{con} \alpha \wedge (\beta \vee \gamma) \leq \beta_m$ for some m.

We remark that G. Czedli [2] proved the equivalence of (4) and (5) for every variety, not just locally finite ones.

We have now seen that the type set of a locally finite variety is an important determinant of the structural and Mal'cev properties of the variety. Of the six broad classes of varieties defined in Theorems 23-28, only the last two have been much studied, and the special facts we have revealed in the locally finite case were unknown.

Curiously, the most studied families of varieties, namely, the classes of congruence distributive varieties, do not fall quite so naturally into our scheme. In order to characterize them, we need a lemma.

LEMMA 29. Let A be any finite algebra, $\langle\alpha_1,\beta_1\rangle,\ldots,\langle\alpha_n,\beta_n\rangle$ be all the prime quotients of A, and $U_i \in M_A(\alpha_i,\beta_i)$ for $i=1,\ldots n$. We have a subdirect representation

$$\mathrm{Con}\ A \overset{sd}{\hookrightarrow} \prod_{1,n} \mathrm{Con}\ A|_{U_i}$$

given by $\theta \to \langle \theta|_{U_i} : 1 \leq i \leq n \rangle$.

THEOREM 30. Let $\mathcal{V}$ be a locally finite variety.

(1) $\mathcal{V}$ is congruence modular iff typ $\{\mathcal{V}\} \subseteq \{2,3,4\}$ and (*) for every prime quotient $\langle\alpha,\beta\rangle$ in a finite algebra of $\mathcal{V}$, each $U \in M_A(\alpha,\beta)$ is the union of its traces.

(2) $\mathcal{V}$ is congruence distributive iff typ $\{\mathcal{V}\} \subseteq \{3,4\}$ and (*) holds.

The representation of Con A defined in Lemma 29 produces interesting results when A lies in a congruence modular variety.

THEOREM 31. Let A be a finite algebra belonging to a congruence modular variety. There exist finite algebras $B, B_1,\ldots,B_n$, each a loop with operators (and therefore Mal'cev), such that $B_1,\ldots,B_n$ are nilpotent and

$$\mathrm{Con}\ A \cong \mathrm{Con}\ B \overset{sd}{\hookrightarrow} \prod_{1,n} \mathrm{Con}\ B_i.$$

The next result is perhaps only amusing. If $\mathcal{V}$ is locally finite and $\mathcal{V} \models_{con} \varepsilon$ for a non-trivial pure lattice identity ε, then typ $\{\mathcal{V}\} \subseteq \{2,3,4\}$. If typ $\{\mathcal{V}\} \subseteq \{2,3\}$, we can deduce a pure lattice identity.

THEOREM 32. Let $\mathcal{V}$ be a locally finite variety, n-permutable for some n. Then for some m, congruences in $\mathcal{V}$ satisfy the following identity (with β_m as in Theorem 23):

$$\lambda \wedge [(\lambda \wedge \tau) \vee (\alpha \wedge (\beta \vee \gamma))] \leq \tau \vee (\alpha \wedge (\lambda \vee (\alpha \wedge \beta_m))).$$

4. OTHER APPLICATIONS.

The results presented in section 3 cannot be proved, so far as we know, without the use of the theory (tame congruence theory) presented in sections 1 and 2.These results were quite unexpected. Perhaps they open up new problem areas for universal algebraists to investigate.

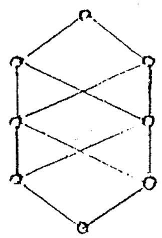
The tame congruence theory was actually developed in the hope of solving some old and difficult problems. We shall now describe the progress made on those problems with its help.

Conjecture 33. Let A be a finite algebra and $\mathcal{V} = V(A)$ be the variety generated by A. If there exists an infinite cardinal λ such that $\mathcal{V}$ has no subdirectly irreducible algebra of cardinality λ, then there exists an integer n such that every subdirectly irreducible algebra in $\mathcal{V}$ has cardinality < n. (In otherwords, if $\mathcal{V}$ is residually small, then it is residually < n for an integer n.)

This conjecture was first intertained in a weaker form by R.W. Quackenbush. It has been proved for finite algebras in congruence modular varieties (Freese and McKenzie [3]) and for finite semigroups (by Golubov and Sapir, and by McKenzie; see [8,9]). Now we have the following result, proving the conjecture in case $\mathcal{V}$ = = V(A) is n-permutable, or satisfies any non-trivial lattice identity for congruences, or in fact, any special Mal'cev condition not satisfied by semilattices.

THEOREM 34. Every residually small, locally finite variety $\mathcal{V}$ such that $1,5 \notin \text{typ}\{\mathcal{V}\}$ is congruence modular.

REMARK 35. It has been conjectured by McNulty ([10]) that if A is a finite algebra such that Clo A is a maximal proper subclone of the clone of all operations on A (a pre-primal algebra of I. Rosenberg), then $V(A)$ must be residually $< n$ for a finite n. The 8-element algebra A whose operations are all the monotone operations on the partially ordered set



is a counter-example to McNulty's conjecture. This algebra is known to be pre-primal. One can prove that $V(A)$ is not congruence modular, using Gumm's 3-ary definition of the Mal'cev condition for congruence modularity. But one can find 3-ary operations of A to satisfy Theorem 24(6), with n=3. (A difficult exercise.) If follows from Theorem 34 that $V(A)$ is not residually small.

Conjecture 36. This is like Conjecture 33, with "simple algebra" everywhere replacing "subdirectly irreducible algebra".

For a finite algebra A, it is much more likely that the simple algebras in V(A) are finite and bounded, than that the subdirectly irriducible algebras are. We should observe that a finite simple algebra S has the unique prime quotient $\langle 0_s, 1_s \rangle$, and its type can be called the type of S. A simple algebra S of type 5 (or 4) is orderable, that is, there exists a connected, admissible partial ordering of the universe of S. (See Theorem 7(4).)

THEOREM 37. Let A be a finite algebra.

(1) There exists n such that every finite simple algebra in $V(A)$ with more than n elements is of type 5.

(2) If $5 \notin \text{typ}\{V(A)\}$ then $V(A)$ has only finitely many simple algebras, and they are finite.

Our theory gives a lot of information about finite simple algebras, but it would require too much space to go further into that topic.

Conjecture 38. If a locally finite variety has a decidable first order theory, then it has an "extremely well-behaved structure theory".

This is, of course, a vague conjecture, which was somewhat substantiated in Burris and McKenzie [1]. There it was proved that a locally finite, congruence modular, and decidable variety $\mathcal{V}$ must decompose structurally as $\mathcal{V} = \mathcal{D} \otimes \mathcal{A}$ where $\mathcal{D}$ is a discriminator variety and $\mathcal{A}$ is an Abelian variety. Here, the algebras of $\mathcal{A}$ are polynomially equivalent to modules over a certain finite ring R. If $\mathcal{V} = \mathcal{D} \otimes \mathcal{A}$ is generated by a finite algebra, then $\mathcal{D}$ is decidable, furthermore $\mathcal{V}$ is decidable iff ${}_R\mathcal{M}$ has a decidable theory. Because of these results, some authors have begun to investigate the question of which finite rings have a decidable theory of modules; but this question remains open in general. On Conjecture 38, tame congruence theory produces two very satisfactory results.

THEOREM 39. Every locally finite, decidable variety $\mathcal{V}$, with $1 \notin \text{typ}\{\mathcal{V}\}$, is congruence modular (and thus the above analysis applies to $\mathcal{V}$).

THEOREM 40. (McKenzie and Valeriote). Let $\mathcal{V}$ be any locally finite and decidable variety. $\mathcal{V}$ has a subvariety $\mathcal{D}$ which is a discriminator variety, such that for every $A \in \mathcal{V}$, $A/Z(A) \in \mathcal{D}$. (We say that $\mathcal{V}$ is discriminator-by-central).

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