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THE BEGINNINGS OF TOPOLOGY FROM 1850 TO 1914

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To people who are not professional mathematicians, there are "common sense" notions which they would not consider as belonging to mathematics, such as proximity, neighborhood, limit, dimension, position, orientation, shape, deformation, etc. One may say that one of the primary aims of Topology is to find mathematical concepts corresponding to these notions, on the model of what the Greeks had been able to do with notions such as straightness, rigidity, superposition, movement, in their axiomatization of Euclidean geometry. Efforts towards a similar goal for "topological" notions started much later, becoming systematic only in the XIXth century, and it was not reached before the first third of the XXth century. Since then, the tools invented for that purpose have become more and more useful in almost all parts of mathematics, and I think that, in the history of mathematics, the XXth century will remain as "the century of Topology".

1. Limits.

The concept of "variable" objects such as ratios of magnitudes, lines, areas, volumes, having an object of the same kind as "limit", was correctly conceived by the Greeks, and used by mathematicians such as Eudoxus and Archimedes. We know by a treatise of Archimedes (which was only discovered around 1900) that he used "intuitive" methods of approximation (and even arguments drawn from statics) to guess the value L of a magnitude (for instance the area of a segment of parabola); to prove that L was indeed the correct value L', he argued by contradiction, showing that L' could neither greater nor smaller than L, by comparing L with magnitudes which certainly were greater or smaller than L', but arbitrary close to L' (an example of such a comparison, going back at least to Eudoxus, is the proof of the formula for the volume of a pyramid, by comparing it with a piling up of prisms of arbitrarily small height).

The modern definition of the limit of a sequence of real numbers emerged with difficulty in the XVIIIth century, after the long controversy over the undefined notion of "infinitesimals"; it was finally realized that one could dispense with that notion altogether and replace it by the clear concept of limit which we still use.

Once that concept had been universally accepted in the XIXth century, the concept of continuous function (at first a real function f of a real variable x) could be explicitly defined by reference to limits (for every sequence (x_n) having a limit a, the sequence $(f(x_n))$ should have the limit f (a)), whereas in the previous period, its properties had been freely used without even giving a name to that notion; these properties were gradually stated and proved from 1810 to 1880.

During that process, it was realized that fundamental properties of real numbers (such as Cauchy's principle of "nested intervals") had to be repeatedly applied. We now consider that they can be taken as an axiom system for real numbers (we say that the set of real numbers is a complete, ordered, archimedean field); but at that time there was a growing uneasiness at taking for granted properties for which the only justification was the "intuition" we think we have of the "continuum" (recall that the Pythagoreans had thought at first that all real numbers were rational, on the basis of their "intuition"!). This gave rise to the various methods of "construction" of the real numbers, starting from the integers and the Peano axioms (Weierstrass, Méray, Cantor, Dedekind). At present most mathematicians consider that these constructions really belong to mathematical logic; their only use is to show that the axioms of real numbers are not

contradictory if one assumes that the Zermelo-Fraenkel axioms of set theory are not contradictory.

The notion of limit was easily extended to sequences of points in a space R^m for any m: one may consider, either the m sequences of coordinates of the points of the given sequence, or the sequence of distances of these points to their limit, thus in both approaches reducing the problem to limits in R. But as Analysis progressed, one needed the notions of approximation and limit for functions (in particular in order to expand a function as sum of a series); and there new phenomena began to appear. The "natural" definition was to extend the way limits were defined in R^m by the use of coordinates: one would say the sequence (f_n) of functions had limit f if, for each value of the variable x, $f(x) = \lim_n f_n(x)$ (this is what is called simple convergence). However, when it came to properties involving limits of functions, this notion soon appeared inadequate: one would naturally have thought that a simple limit of continuous functions would be continuous, and Cauchy thought he could prove that result; but Abel found a counterexample. Similarly, when (f_n) had f as simple limit, $\int_a^b f_n(x)dx$ did not always tend to $\int_a^b f(x)dx$. A deeper analysis of these phenomena led to the notion of uniform convergence; so there were at least two different ways for a function to be a "limit" of a sequence of functions. Before the end of the XIXth century, still other notions of "limit" appeared in other problems. For instance, in the Calculus of variations, one had to distinguish the case in which the sequence (f_n) tends uniformly to f, from the case in which both the sequence (f_n) and the sequences (f'_n) of the derivatives tend uniformly to f and f' respectively. In the theory of approximation and in the Calculus of probability, one met sequences (f_n) for which one only knew that $\int_a^b |f(x) - f_n(x)|^2 dx$ tends to 0; should then one say that f is the "limit" of the sequence (f_n)?

II. Distances and metric spaces.

The big step forward was to transcend all these particular questions, and to reach for an underlying general structure. Even if, in 1900, the idea of "structure" in mathematics had not yet been formulated in a general way, it clearly was behind the progress of algebra since 1840, in the long process from which gradually emerged the concepts of group, field and module on unspecified underlying sets, in contrast with the "classical" type of mathematics, which dealt with explicitly defined sets on which such structures were considered (think of number theory, of groups of permutations or of algebraic geometry).

A similar widening of perspective had not yet occurred in Analysis before 1900; it was first conceived in Fréchet's thesis of 1906, which marks the birth of what we now call General Topology. His best idea was to take his cue from elementary geometry, and define on any set E a "distance" d(x,y) of any pair elements of E, for which only the most rudimentary properties of euclidean distance were postulated:

1) $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$;
2) $d(y,x) = d(x,y)$;
3) $d(x,z) \leq d(x,y) + d(y,z)$ for any 3 elements of E ("triangle inequality").

We now say that a set E equipped with such a "distance" is a metric space, and its elements are called points; it is then immediate to transfer in E the definition of limit given in the spaces R^m with the help of the euclidean distance: a sequence (x_n) of points of E has a point $x \in E$ as limit (or converges to x) if the sequence $(d(x,x_n))$ of real numbers tends to 0. Similarly, if $f : E \to E'$ is a map of a metric space E (with distance d) into a metric space E' (with distance d'), f is a continuous at a point $x \in E$ if for every sequence (x_n) in E with limit x, the sequence $(f(x_n))$ converges in E' to f(x). Finally the notion of homeomorphism (which had appeared with Möbius and Jordan in the 1860's) could be defined for metric spaces: a map $f : E \to E'$ is a homeomorphism of a metric space E onto a metric space E' if f is bijective and both f and its inverse $f^{-1} : E' \to E$ are continuous.

With the exception of simple convergence, all other types of "limits" for functions, described above, can be defined by means of an appropriate "distance" on the set E of all continuous (or C^1) functions in an interval $[a,b]$

$$d_\infty(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|,$$
$$d^{(1)}(f,g) = d_\infty(f,g) + d_\infty(f',g')$$
$$d_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2 dx \right)^{1/2}.$$

Fréchet already knew that some types of convergence (for instance, simple convergence for bounded functions (continuous or not) on an interval) cannot be defined by a "distance". In the most useful cases for applications to Analysis (including simple convergence), one can generalize the notion of metric space to uniform spaces: to define such a space on a set E, one considers on E a family (finite or infinite) of "pseudodistances $d_i(x,y)$: they satisfy contitions 2) and 3) above and one has $d_i(x,y) \geq 0$; but one may have $d_i(x,y) = 0$ for some pairs with $x \neq y$. Let E, E' be two uniform spaces, with families of pseudo-distances (d_i), (d'_j); to say that a map $f : E \to E'$ is continuous at $x \in E$ then means that, given any finite number of pseudo-distances d'_{j_k} $(1 \leq k \leq m)$ on E' and a number $\varepsilon > 0$, there are a finite number of pseudodistances d_{i_h} $(1 \leq h \leq n)$ on E and a number $\delta > 0$ such that, for all $y \in E$ for which $d_{i_h}(x,y) \leq \delta$ for $1 \leq h \leq n$ one has $d'_{j_k}(f(x), f(y)) \leq \varepsilon$ for $1 \leq k \leq m$.

III. Coarse classification of subsets.

Until 1850, most "sets" considered by mathematicians were intervals of R, "curves" or "surfaces" in R^2 or R^3, portions of surfaces "limited" by curves, portions of space "limited" by surfaces. These concepts were taken for granted, by reference to "parametrizations" by 1, 2, or 3 "real variables". With Bolzano, Weierstrass, and above all Cantor, began the study of arbitrary subsets, first of R, then of the R^m, which was made necessary by problems from Analysis (e. g. convergence of trigonometrical series for Cantor).

Cantor first considered the set of cluster points of a subset E of R, which he called the derived set E' of E; then he iterated the process, and it is well known that this is what gave him the idea of transfinite ordinals. This is a typical example of a question where problems of pure set theory and topological notions are closely related; it is Cantor who taught mathematicians to dissociate, on the "continuum" R, the notions of cardinality, order, topology and measure, which, by their simultaneous presence, had created much confusion; and certainly nobody before him had ever conceived the possibility of a set such as the "Cantor discontinuum" which has the power of the continuum, is totally disconnected and has measure 0!

Between 1875 and 1914, the notion of derived set brought to light several families of subsets of R which proved very important in applications: closed sets (such that $E' \subset E$), perfect sets ($E' = E$), dense sets ($E' = R$). Closed sets in R could be described as complements of a finite or denumerable family of non overlapping open intervals; a remarkable result was the Cantor - Bendixson theorem, which described any subset of R as the disjoint union of a perfect set and a finite or denumerable set.

The notion of derived set, as well as all those we have just enumerated, could immediately be defined in all R^m for $m \geq 1$, and more generally in all metric spaces; but even in R^2, trying to classify up to homeomorphism all closed subsets was very soon seen to be completely unrealistic (see below). What turned out

to have much greater importance in applications were notions which in the space R were quite trivial; they were chiefly emphasized by Fréchet and F. Riesz in general metric spaces: connected sets (in R only intervals are connected), compact sets (in R^m they are just the bounded closed sets), separable spaces, i.e. those in which there is a denumerable dense set (a trivial property of all R^m).

Another property of metric spaces is not purely topological, but linked to the choice of distance: a metric space is complete if any Cauchy sequence (x_n) (i.e. such that $d(x_m, x_n)$ tends to 0 when both m and n tend to $+\infty$) is convergent. As we have seen, this is one of the basic properties of real numbers. In Analysis, complete metric spaces are very useful for several reasons; one is the so-called contraction principle, which enables one to establish, under rather general circumstances, the existence of a fixed point for a continuous map f of a complete space E into itself; it is assumed that $d(f(x), f(y)) \leq k \cdot d(x,y)$ for all points x,y in E, with $0 \leq k < 1$; the sequence (x_n) defined by $x_{n+1} = f(x_n)$ then converges to a point a such that $a = f(a)$. Many existence proofs for solutions of functional equations can be obtained by applications of that principle. Another fundamental property of complete metric spaces is the theorem proved around 1899 by Osgood and Baire: in such a space, the intersection of a denumerable family of dense open sets is not empty.

All these "abstract" results gave a tremendous boost to problems concerning function theory and functional equations by applying them to metric spaces whose elements are functions, so that general Topology was called "General Analysis" by Fréchet, and "Functional Analysis" was meant to be the study of applications of Topology to spaces of functions. As outstanding examples, we may mention the use of compactness in Hilbert's proof of the "Dirichlet principle" in 1900, or in his work on the spectrum of a self-adjoint operator (under the name "principle of choice").

IV. Dimension, boundary, position.

Before 1870, nobody thought that the concept of "dimension" needed elucidation, being linked to the "natural" idea of parametrization. This naive conception was rudely shattered when Cantor defined a bijection of R on any R^m and when Peano constructed his famous "curve" filling a square. The only possibility which remained to transform the intuitive idea of dimension into a genuine mathematical notion was to show that an open set in R^m could not be homeomorphic to an open set in R^n if $m \neq n$. After some partial results for small values of m and n, this general result was only established by Brouwer in 1911, using his notion of degree of a map and the technique of simplicial approximation (see below).

Other "naive" notions concerned "positions" and "boundaries" of curves and surfaces in R^2 or R^3: a portion of curve would have a boundary consisting of 2 points, a portion of surface a boundary consisting of curves or portions of curves, a volume a boundary consisting of surfaces or portions of surfaces. After the discoveries of Cantor and Peano, a clarification of these concepts was urgently needed; it began in 1893 with the famous Jordan theorem on plane curves: one defines a Jordan curve as homeomorphic to a circle; what Jordan stated and almost proved is that for a plane Jordan curve J, its complement $R^2 - J$ has exactely two components (usually called the "interior" and the "exterior" of J), another that J is the common boundary of both; this somewhat restored, at least partially, the intuitive concept of "plane curve".

The theorem immediately attracted universal attention, and many proofs were proposed. Furthermore, Schoenflies, in a series of papers around 1900, added to Jordan's result several properties equally expected by "intuition", for instance the fact that any point $x_0 \in J$ can be joined to an arbitrary point of the "interior" I (resp. the "exterior" E) by a segment of curve (continuous image of the interval $[0,1]$) all of whose points except x_0 are in I (resp. E); another property is that I is simply connected (as well as the union of E and the point at infinity when R^2 is embedded in the sphere S_2).

Schoenflies also thought that the property of a Jordan curve to be the common boundary of the components of its complement in R^2 was characteristic of Jordan curves. However, when Brouwer, in 1909, made a thorough examination of Schoenflies' proofs, he found that many were defective, and showed by counterexamples that some of this assertions were wrong. The most remarkable of these counterexamples was a closed set W in R^2 with empty interior, and whose complement $R^2 - W$ has $k \geqslant 3$ components, each one of which has W as boundary.

Since then, a very large literature has been devoted to the properties of closed sets in R^2, bringing to light a great variety of "monsters" for the intuition.

V. Shape, deformation, orientation.

Before the arrival of these unusual objects, mathematicians very seldom considered "curves" in the plane which were not differentiable, or piecewise differentiable. The idea of "deformation" of a curve was not absent from mathematical thought, for instance in the Calculus of variations, where one has a family of curves

$$C_\lambda : x = f(t,\lambda) \ , \ y = g(t,\lambda)$$

depending on a parameter λ; usually f and g are differentiable with respect to both parameters t, λ, and all curves C_λ are homeomorphic.

If one allows f and g to be merely continuous, one has a more general idea of "deformation"; already in the middle of the

XIX^{th} century, some mathematicians thought of classifying curves "up to deformation" so that, for instance, a circle, an ellipse and the perimeter of a triangle or a square would be in the same class; more generally, all Jordan curves are in the same class.

One may define in the same manner "deformation" of surfaces, and more generally of "manifolds" of arbitrary dimension. But already for compact surfaces in R^3 without boundary, it is intuitive that there are many different classes, exemplified by spheres with k handles; the problem is to prove that these objects belong to different classes. It is advisable to replace "deformation" (which is only meaningful within an "ambient" space) by homeomorphism, which is an intrinsic notion. The general strategy is to associate to a space a number, or a system of numbers, or algebraic objects (groups, rings, etc.) in such a way that to homeomorphic spaces correspond the same objects (up to isomorphism); if different numbers, or algebraic objects, are associated to two spaces, they certainly are not homeomorphic, the converse being not true in general and raising much deeper problems. This general method is called algebraic Topology.

Its birth may also be dated to the middle of the XIX^{th} century. It did not come into being out of mere curiosity, but because it was needed by Riemann in his great work on holomorphic functions and abelian integrals: he constructed the famous compact Riemann surfaces and had to classify them, which he did by assigning to each one what was called its genus g, an integer $g \geqslant 0$ which is invariant by homeomorphism. Developing an idea which von Staudt had essentially used in 1847 to characterize the sphere, Riemann considers, for a compact surface S without boundary, the "Jordan curves" (homeomorphicto the circle S_1) on S; on the sphere, any such curve disconnects the surface (before Jordan this was of course a statement only supported by "intuition"); but on an arbitrary surface S there may be several

Jordan curves $J_1, \ldots, J_m$ without common point which, taken together, still do not disconnect S ; the maximum number g of these curves is Riemann's genus (for a sphere with k handles, g = k). An equivalent definition is that 2g is the maximum number of Jordan curves having finite intersections and which, taken together, do not disconnect S.

In the hands of Riemann and of his followers, in particular Klein and Poincaré, these new notions opened up an unsuspected new world for the theory of analytic functions of a complex variable and for algebraic geometry. But, especially after Cantor and Peano, a curious rift took place between analysts during 30 years: one one side were those who obeyed the strict Weierstrassian rules of "rigor" in all their arguments, and on the other the mathematicians who felt that, although the topological foundations of Riemann's work were far from conforming to these rules, one might use them with confidence; it was only after the work of Brouwer and H. Weil in 1911-1913 that the rift could be closed.

Meanwhile, the further progress of Analysis called for an extension of Riemann's definition of the genus to higher dimensions. Already, for 2 dimensions, Riemann's results were not complete, for the surfaces he was considering were orientable: this is a concept which only emerged after Riemann's work, with the surprising discovery by Listing and Möbius of the famous "onesided surfaces". This notion is linked to the position of a surface in R^3, but one can define for a connected surface a notion of orientability, independently of any imbedding: take at a point M two non colinear tangent vectors e_1, e_2, and move continuously M, e_1, e_2 along a closed curve in such a way that the bivector $e_1 \wedge e_2$ never vanishes; when M has come back to its initial position, it may happen $e_1 \wedge e_2$ has changed sign (this happens on the Möbius band); if it never happens, one says the surface is orientable (most usual surfaces are orientable), and the choice of a bivector at one point determines an orientation, so that there are two "opposite" orientations of the surface. The complete classification of C^1 surfaces, with or without boundary, had to take into account orientability; the results were already stated at the end of the XIXth century, but only proved after 1915.

For higher dimension, Riemann had realized that an n-dimensional C^1 manifold (a notion he had essentially introduced in his famous work on Riemannian spaces) should be attached not one number but a system of n-1 numbers, which his disciple Betti endeavored to define on the basis of some fragmentary notes of Riemann. But the theory only got started when Poincaré imagined regular methods of computation of these "Betti numbers" as he called them. Although he did not use our algebraic language, we may say that Poincaré has been the first mathematician to use an extraordinary algebraic device (which has become standard in modern mathematics) in the study of a set M of objects: one considers the module C of formal linear combinations of objects of M with coefficients in Z or in another ring; using properties of M, one then defines submodules of C, and these, or their quotients, turn out to be intrinsically attached to M. In a first attempt, Poincaré, following Betti, considered, for an n-dimensional manifold V and any p such that $1 \leq p \leq n-1$, the set M of oriented "Jordan p-dimensional manifolds" in V, i.e. submanifolds of V homeomorphic to the sphere S_p; to define a submodule B of C which would have yielded a significant quotient module C/B attached to V and to p, he tried, like Betti, to consider the boundaries in V of all (p + 1)-dimensional submanifolds, but he was unable to find a technique which would have given a regular way of computing B (50 years later, R. Thom, in a closely related question, invented such a technique in his theory of cobordism).

So, in 1899, Poincaré changed his tactics by turning his attention to a different set M of objects attached to V; this time, the machinery he invented turned out to be amenable to a purely algebraic treatment; after dropping superflous assumptions, it has remained the backbone of algebraic Topology. One of the difficulties encountered by Poincaré in working with "Jordan submanifolds" of V is that there are too many of them, and they may have a very nasty behavior, for instance when taking intersections; hence the idea of considering only some of these submanifolds, having "good" properties, but in sufficiently large numbers to "represent" all the others.

The starting point seems very artificial and remote from the problem to be solved: it is the study of very special compact subsets of the spaces R^m, the euclidean simplicial complexes, which generalize the classical "polyeders". Such a "complex" X is the disjoint union of a finite family S of open simplices of all dimensions $\leq m$, satisfying the following condition: all open simplices (of dimension p) whose disjoint union is the boundary of a p-simplex of S, also belong to S. For each $p \geq 0$, one then applies the general method to the set M_p of p-simplices of S: the formal linear combinations of elements of M_p with integer coefficients form a Z-module C_p, whose elements are called the p-chains of (X,S).

However, if we look for substitutes to the "Jordan submanifolds", we should take into account the fact that they are manifolds without boundary; so not all p-chains should be considered. To single out those which matter, Poincaré introduces purely algebraic notions which serve as substitutes for the topological notions of boundary and orientation; the relevance of these definitions is justified by the "intuitive" case p = 2.

By definition, an orientation of a p-simplex is an ordering

$(a_0, a_1, \ldots, a_p)$ of its vertices, two orderings giving the same orientation if and only if they are deduced from one another by an even permutation. The boundary operator is then the homomorphism b_p of C_p into C_{p-1} defined on p-simplices by

$$b_p(a_0, a_1, \ldots, a_p) = \sum_{j=0}^{p} (-1)^j (a_0, a_1, \ldots, a_{j-1}, a_{j+1}, a_p)$$

and then by linearity in C_p. What is to take the place of the "Jordan submanifolds" are the p-cycles, i.e. the p-chains forming the kernel Z_p of b_p; and what takes the place of the boundaries of (p+1)-submanifolds are the p-boundaries, i.e. the p-cycles forming the image B_p of b_{p+1} (one verifies that $b_p b_{p+1} = 0$, i.e. a p-boundary is a p-cycle). And then the original idea of Poincaré (in our language) is to take the quotient $H_p(X) = Z_p / B_p$, the p-th homology module of (X,S), whose rank is the p-th Betti number.

What justifies a posteriori the consideration of these seemingly very special "euclidean simplicial complexes" is that most "usual" compact spaces are homeomorphic to such a "complex", for instance C^1 manifolds (with or without boundary) and complex algebraic varieties (singular or not); one says such spaces are triangulable and the images of the simplices of S constitute a triangulation. So it appears that the Poincaré method gives indeed a regular method of computation; however, to be sure that the "homology groups" thus obtained are invariant under homeomorphism, one has to prove that two euclidean simplicial complexes (X,S) , (X',S') which are homeomorphic have isomorphic homology groups $H_p(X)$, $H_p(X')$. Poincaré tried to give such a proof, but did not succeed, and the invariance theorem was only proved in 1915 by J. Alexander, with the technique of simplicial approximation.

This was invented in 1911 by Brouwer in order to handle the properties of continuous maps: the Peano curve showed that it was hopeless to try to analyze their structure directly, and

Brouwer's idea was to replace such a map by one which is arbitrarily close to it, but has "regularity" properties which allow computations to be made. The simplest case is the classical approximation of a continuous function of one variable in an interval by a "piecewise linear" function coinciding with the given one at the points of a fine subdivision of the interval. In general, what replace "piecewise linear" functions are simplicial maps: if (X,S) , (X',S') are two euclidean simplicial complexes, a simplicial map $g : X \to X'$ is such that:

1) the image by g of any vertex of a simplex of S is a vertex of a simplex of S';

2) for every p-simplex σ of S, the images by g of the vertices of σ are not necessarily distinct vertices of a p-simplex τ of S' and the restriction of g to σ is an affine map of σ into τ.

To obtain a simplicial approximation for an arbitrary continuous map $f : X \to X'$ one first subdivides the simplices of S into simplices small enough for the image by f of such a simplex to be contained in the star of some vertex of S'. It is then easily seen that for any vertex a_k of S there is a vertex b_k of S' such that the line segment joining $f(a_k)$ and b_k is contained in X', and that if j + 1 vertices a_k are the (distinct) vertices of a j-simplex of S, the corresponding b_k are (not necessarily distinct) vertices of a j-simplex of S'. Taking $g(a_k) = b_k$ for each k defines unambiguously a simplicial map $g : X \to X'$ and g approximates f in the sense that $|f(x) - g(x)|$ is at most the maximum diameter of the simplices of S', for all $x \in X$.

The virtue of a simplicial map $g : X \to X'$ is that it determines a homomorphism $g_p : H_p(X) \to H_p(X')$ of the homology groups: to each oriented p-simplex $(a_0, a_1, \ldots, a_p) = \sigma$ one assigns the oriented p-symplex $(g(a_0), g(a_1), \ldots, g(a_p)) = \tilde{g}(\sigma)$

if all the $g(a_j)$ are distinct, and 0 otherwise: this defines a homomorphism $g : C_p(X) \longrightarrow C_p(X')$ of p-chains by linearity, and g maps cycles into cycles and boundaries into boundaries, hence, by passage to quotients, one gets a homomorphism
$g_p : H_p(X) \longrightarrow H_p(X')$.

One of Alexander's proofs consists in showing that if f is a homeomorphism of X onto X', a sufficiently close simplicial approximation g to f yields an isomorphism of $H_p(X)$ onto $H_p(X')$ for every p.

Once the intrinsic character of homology groups had been established for triangulable spaces, several methods were devised between 1900 and 1930 to compute them explicitly in many cases (Mayer-Vietoris sequences, Künneth theorem, Poincaré duality, etc.). Using these devices, it was possible to give very simple proofs for theorems which Brouwer had proved in 1911 by extremely complicated "simplicial approximations". Two key results are that in $R^n (n \geqslant 2)$ a compact subset homeomorphic to a cube $I^r (r \leqslant n)$ does not disconnect the space whereas the complement of a "Jordan hypersurface", homeomorphic to S_{n-1}, has two connected components of which the hypersurface is the common boundary ("Jordan-Brouwer theorem"); but it is not true any more for $n \geqslant 3$ that these components are always simply connected. From these results, easy arguments lead to the Brouwer theorems on the invariance of dimension and invariance of domain.