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FINITE DISTRIBUTIVE LATTICES AS CONCEPT LATTICES

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In [5], ideas of restructuring lattice theory are developed as *an attempt to reinvigorate connections with our general culture* and *to promote better communication between lattice theorists and potential users of lattice theory*; an approach to lattice theory is described where lattices are interpreted as hierarchies of concepts. This approach is carried out further in [6], [7], [8], [9], [10], [3]. Since distributive lattices occur in many lattice-theoretical developments and applications, it seems appropriate to study (finite) distributive lattices as hierarchies of concepts. In this paper we shall represent finite distributive lattices as iterated sum of glued atlases of concept lattices which are created by $\vee$-and $\wedge$-irreducible elements of the distributive lattices.

First let us recall some notions concerning concept lattices. A *context* is a triple (G,M,I) where G and M are sets and I is a binary relation between G and M (i.e. $I \subseteq G\times M$); the elements of G and M are called *objects* and *attributes*, respectively. If gIm holds, we say: the object g *has* the attribute m. For $A\subseteq G$ and $B\subseteq M$ we define:

$$A':=\{m\in M \mid gIm \text{ for all } g\in A\},\quad B':=\{g\in G \mid gIm \text{ for all } m\in B\}.$$

This definition establishes a *Galois connection* characterized by the properties:

(1) $A_1 \subseteq A_2$ implies $A_1' \supseteq A_2'$ for $A_1, A_2 \subseteq G$,
(1') $B_1 \subseteq B_2$ implies $B_1' \supseteq B_2'$ for $B_1, B_2 \subseteq M$,
(2) $A \subseteq A''$ and $A' = A'''$ for $A \subseteq G$,
(2') $B \subseteq B''$ and $B' = B'''$ for $B \subseteq M$.

Now, a *concept* of a context (G,M,I) is a pair (A,B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$; A and B are called the *extent* and the *intent* of the concept (A,B), respectively. The set of all concepts of (G,M,I) is denoted by $\mathfrak{B}(G,M,I)$. The hierarchy of concepts is expressed by the definition:

$$(A_1,B_1) \leq (A_2,B_2) :\Longleftrightarrow A_1 \subseteq A_2 \quad (\Longleftrightarrow B_1 \supseteq B_2).$$

The hierarchical structure of all concepts of (G,M,I) is denoted by $\underline{\mathfrak{B}}(G,M,I) := (\mathfrak{B}(G,M,I), \leq)$.

Basic Theorem for Concept Lattices [5]: Let (G,M,I) be a context. Then $\underline{\mathfrak{B}}(G,M,I)$ is a complete lattice, called the *concept lattice* of (G,M,I), for which the infimum and the supremum can be described as follows:

$$\bigwedge_{t\in T}(A_t,B_t) = \Big(\bigcap_{t\in T} A_t, \big(\bigcap_{t\in T} A_t\big)'\Big), \quad \bigvee_{t\in T}(A_t,B_t) = \Big(\big(\bigcap_{t\in T} B_t\big)', \bigcap_{t\in T} B_t\Big).$$

In general, a complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G,M,I)$ if and only if there exist mappings $\gamma: G \to L$ and $\mu: M \to L$ such that γG is *supremum-dense* in L (i.e. $L = \{\bigvee X \mid X \subseteq \gamma G\}$), μM is *infimum-dense* in L (i.e. $L = \{\bigwedge X \mid X \subseteq \mu M\}$, and gIm is equivalent to $\gamma g \leq \mu m$; in particular $L \cong \underline{\mathfrak{B}}(L,L,\leq)$

By Birkhoff [1], a finite distributive lattice is isomorphic to the lattice of all order ideals of the ordered set of its $\vee$-irreducible elements. We recall that an *order ideal (order filter)* of an orderd set P is a

subset A of P which contains all elements x of P with $x \leq a$ $(x \geq a)$ for some $a \in A$; a *principal ideal (principal filter)* of P is defined by $(a] := \{x \in P \mid x \leq a\}$ $([a) := \{x \in P \mid x \geq a\})$ for $a \in P$. The lattice of all order ideals of P ordered by set inclusion is denoted by $\mathfrak{J}(P)$.

Proposition 1: Let P be an ordered set. The concepts of the context $(P,P,\ngeq)$ are exactly the pairs $(A, P\setminus A)$ where A is an order ideal of P; especially, $\underline{\mathfrak{B}}(P,P,\ngeq) \cong \mathfrak{J}(P)$.

Proof: The assertion follows immediately from the equivalence $x \ngeq y \Longleftrightarrow (x] \cap [y) = \emptyset$.

For the structural analysis of finite distributive lattices we shall use further lattices derived from an ordered set P. In general, for subsets A and B of P, we define $A \leq B$ if for every $a \in A$ there exists $b \in B$ with $a \leq b$. We recall that a subset A of P is *convex* if $a,b \in A$ and $a \leq x \leq b$ in P always imply $x \in A$. A *k-subset* of P is a subset of P which does not contain a chain of length k. Derived relations on P are obtained by the definition: $x \geq_k y$ if $x \geq y$ in P and $l[y,x] \geq k$ (i.e. the length of the interval $[y,x]$ is at least k). As abreviations we define $(a]_k := \{x \in P \mid a \geq_k x\}$ and $[a)_k := \{x \in P \mid x \geq_k a\}$ for $a \in P$.

Proposition 2: Let P be an ordered set of finite length and let k be a positive integer. Then $(A,B) \mapsto A \cap B$ describes an isomorphism from $\underline{\mathfrak{B}}(P,P,\ngeq_k)$ onto the lattice of all maximal convex k-subsets of P.

Proof: For $(A,B)\in\mathfrak{B}(P,P,\nparallel_k)$, $A\cap B$ is a k-subset of P which is convex because A is an order ideal and B an order filter of P; furthermore, $A\cup B = P$. Let $x\in B$ with $l[y,x]<k$ for all $y\in A\cap B\cap(x]$. Suppose $x\notin A$. Then $B\cap(x]_k \neq \emptyset$; but A contains all minimal elements y of $B\cap(x]_k$ what contradicts $l[y,x]<k$ for $y\in A\cap B\cap(x]$. Therefore $x\in A\cap B$. Together with the dual argument, this shows that $A\cap B$ is a maximal convex k-subset of P. Conversely, if C is a maximal convex k-subset of P, then $(\{x\in P \mid x\leq y$ for some $y\in C\}, \{x\in P \mid x\geq y$ for some $y\in C\})$ is a concept of $(P,P,\nparallel_k)$. Furthermore, for $(A_1,B_1),(A_2,B_2)\in\mathfrak{B}(P,P,\nparallel_k)$, $(A_1,B_1)\leq(A_2,B_2)$ is equivalent to $A_1\cap B_1\leq A_2\cap B_2$. All together yield the assertion of the proposition.

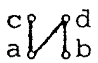
It is often natural to understand a (complete) lattice as a union of intervals which itself form a (complete) lattice. For a finite lattice L, such a decomposition can be given by a *tolerance relation* Θ of L, i.e. a reflexive symmetric relation on L compatible with the join and meet operation of L. The *blocks* of Θ are the intervals $[a_\Theta,(a_\Theta)^\Theta]$ where $a_\Theta := \bigwedge\{x\in L \mid a\Theta x\}$ and $b^\Theta := \bigvee\{x\in L \mid b\Theta x\}$. Since the maps $a\mapsto a_\Theta$ and $b\mapsto b^\Theta$ form a Galois connection, the set L/Θ of all blocks of Θ together with the order relation defined by $[a_\Theta,(a_\Theta)^\Theta]\leq[b_\Theta,(b_\Theta)^\Theta] :\Longleftrightarrow a_\Theta\leq b_\Theta$ is again a lattice, called the *factor lattice* of L by Θ. The tolerance relation Θ of L is said to be *glued* if blocks which form a covering pair in the factor lattice L/Θ have always non-empty intersection. There is a smallest glued tolerance relation of L, denoted by $\Sigma(L)$, which is generated by all covering pairs of L; $S(L):=L/\Sigma(L)$ is called the *skeleton* of L (cf. [4], [8]).

The structural analysis by the blocks of tolerance relations can be turned into a construction method for finite lattices; then a finite lattice is understood as the sum of a *(glued) atlas* consisting of all blocks of a (glued) tolerance relation [8]. We aim at this construction when we analyse finite distributive lattices by tolerance relations. In decomposing the factor lattices by tolerance relations again and again, the construction may be iteratively applied.

Tolerance relations of a finite concept lattice $\mathfrak{B}(G,M,I)$ can be discovered directly within the context (G,M,I): A tolerance relation Θ of $\mathfrak{B}(G,M,I)$ corresponds to a *block relation* J of (G,M,I), i.e. a relation between G and M extending I for which $g^J:=\{m\in M \mid gJm\}$ is an intent of (G,M,I) for all $g\in G$ and $m^J:=\{g\in G \mid gIm\}$ is an extent of (G,M,I) for all $m\in M$; the correspondence is given by $(A,B)\Theta(C,D) \Longleftrightarrow A\cup C\subseteq H$ and $B\cup D\subseteq N$ for some concept (H,N) of (G,M,J) (see [8]). The sets $\{(A,B)\in\mathfrak{B}(G,M,I) \mid A\subseteq H$ and $B\subseteq N\}$ with $(H,N)\in\mathfrak{B}(G,M,J)$ are exactly the blocks of Θ and so $\mathfrak{B}(G,M,I)/\Theta \cong \mathfrak{B}(G,M,J)$. Let us denote by $\Sigma(I)$ the block relation corresponding to $\Sigma(\mathfrak{B}(G,M,I))$; thus, a block relation J corresponds to a glued tolerance relation if and only if $\Sigma(I)\subseteq J$. The next proposition indicates that there is a natural chain of glued tolerance relations for each finite distributive lattice.

<u>Proposition 3:</u> Let P be an ordered set of finite length and let k be a positive integer. Then $\updownarrow_k$ is a block relation of $(P,P,\updownarrow_{k-1})$ with $\Sigma(\updownarrow_{k-1})\subseteq\updownarrow_k$; in particular $\updownarrow_1 = \nearrow = \Sigma(\updownarrow)$.

Proof: Notice that the intent of $(P,P,\updownarrow_k)$ consisting of all $y\epsilon P$ with $x\updownarrow_k y$ equals $P\setminus(x]_k$. Now, we show that $P\setminus(x]_k = \bigcap_{y\prec x} P\setminus(y]_{k-1}$ ($y\prec x$ means that y is covered by x). Obviously, $y\prec x$ implies $(y]_{k-1}\subseteq(x]_k$ and so $P\setminus(x]_k\subseteq P\setminus(y]_{k-1}$; hence $P\setminus(x]_k\subseteq\bigcap_{y\prec x} P\setminus(y]_{k-1}$. Let $z\epsilon(x]_k$. Then there exists $y\prec x$ with $z\epsilon(y]_{k-1}$ and so $z\notin\bigcap_{y\prec x} P\setminus(y]_{k-1}$. Hence $P\setminus(x]_k = \bigcap_{y\prec x} P\setminus(y]_{k-1}$. This together with the dual identity yields that $\updownarrow_k$ is a block relation of $(P,P,\updownarrow_{k-1})$. By [8; Theorem 10], $\Sigma(\updownarrow_{k-1})$ is the smallest block relation of $(P,P,\updownarrow_{k-1})$ containing all pairs $(x,y)\epsilon P^2$ such that $(x]_{k-1}$ is minimal in $\{(z]_{k-1}\,|\,z\epsilon[y)_{k-1}\}$ or $[y)_{k-1}$ is minimal in $\{[z)_{k-1}\,|\,z\epsilon(x]_{k-1}\}$. For such a pair (x,y), we have $l[y,x] = k-1$ and so $x\updownarrow_k y$. Therefore $\Sigma(\updownarrow_{k-1})\subseteq\updownarrow_k$. Furthermore, it follows that $\Sigma(\updownarrow)$ is the smallest block relation of $(P,P,\updownarrow)$ containing all pairs $(x,x)\epsilon P^2$; hence $\updownarrow_1 = \nearrow = \Sigma(\updownarrow)$.

In general, $\updownarrow_k$ does not equal $\Sigma(\updownarrow_{k-1})$; for instance, if P is the fence  , then $\updownarrow_2 = P^2$ but $\Sigma(\updownarrow_1) = P^2\setminus\{(d,a)\}$.

Proposition 3 yields a scheme how to construct finite distributive lattices as iterated sum of glued atlases of certain concept lattices. It should be mentioned that

the derived lattices need not to be distributive; C. Herrmann even showed in [4] that every finite lattice L is isomorphic to the skeleton of some finite distributive lattice. This can be seen by defining $P:=L\times\{0,1\}$ and $(x,1)>(y,0):\Leftrightarrow x\updownarrow y$ in L because then we obtain $L\cong S(\mathfrak{B}(P,P,\updownarrow))$ using Proposition 3.

In the following, the iterative construction method is worked out further for the case that the $\vee$-irreducible elements form a ranked ordered set. We make the <u>general assumption</u> that P is an ordered set in which every maximal chain has the same finite length n, i.e. P is a *ranked ordered set of length* n. Let P_k be the set of all elements of P of rank k. We define $P(r,s):=\mathfrak{B}(P_{n-r},P_s,\updownarrow)$ for $r+s\leq n$. $A\mapsto A'$ and $B\mapsto B'$ denotes the Galois connection in $(P,P,\updownarrow)$.

<u>Lemma 4:</u> For $r+s<n$, an injective $\bigvee$-morphism $\phi_{rs}:P(r,s)\to P(r+1,s)$ is given by $\phi_{rs}(A,B):=(B'\cap P_{n-r-1},B)$ and an injective $\bigwedge$-morphism $\psi_{rs}:P(r,s)\to P(r,s+1)$ is given by $\psi_{rs}(A,B):=(A,A'\cap P_{s+1})$.

Proof: By the Basic Theorem for Concept Lattices, we have only to show that $(B'\cap P_{n-r-1},B)$ is a concept of $(P_{n-r-1},P_s,\updownarrow)$. This is an immediate consequence of $B'\cap P_{n-r-1}=\{y\epsilon P\,|\,y\prec x$ for some $x\epsilon A\}$. Analogously, we see that $(A,A'\cap P_{s+1})$ is a concept of $(P_{n-r},P_{s+1},\updownarrow)$.

Next we define the lattice $P(m):=P(m,0)\times P(m-1,1)\times\ldots\times P(0,m)$, the $\bigvee$-morphism $\phi_m:P(m)\to P(m+1)$ by

$$\phi_m(a_{mo},a_{(m-1)1},\ldots,a_{om}):=$$
$$(\phi_{mo}a_{mo},\phi_{(m-1)1}a_{(m-1)1},\ldots,\phi_{om}a_{om},0_{o(m+1)}),$$

and the $\wedge$-morphism $\psi_m : P(m) \to P(m+1)$ by

$$\psi(a_{mo}, a_{(m-1)1}, \ldots, a_{om}) =: (1_{(m+1)o}, \psi_{mo} a_{mo}, \psi_{(m-1)1} a_{(m-1)1}, \ldots, \psi_{om} a_{om})$$

for $0 \leq m < n$.

<u>Lemma 5:</u> $\phi_m \psi_{m-1} \leq \psi_m \phi_{m-1}$ for $1 \leq m < n$.

Proof: We have to show that $\phi_{r(s+1)} \psi_{rs}(A,B) \leq \psi_{(r+1)s} \phi_{rs}(A,B)$ for $s+r = m-1$ and $(A,B) \in P(r,s)$. By definition, the inequality is equivalent to $(A' \cap P_{s+1})' \cap P_{n-r-1} \subseteq B' \cap P_{n-r-1}$. Let $x \in P_{n-r-1}$ with $x \not\geq y$ for all $y \in A' \cap P_{s+1}$. Since $A' \cap P_{s+1} = \{v \in P \mid v \succ u$ for some $u \in B\}$, it follows that $x \in B'$. This proves the desired inclusion.

Now, let $L_0(P) := P(0,0)$ and $L_{m+1}(P) := \bigcup_{x \in L_m(P)} [\phi_m x, \psi_m x]$ for $0 \leq m < n$; notice that $\phi_m x \leq \psi_m x$ for $x \in [\phi_{m-1} y, \psi_{m-1} y]$ because $\phi_m x \leq \phi_m \psi_{m-1} y \leq \psi_m \phi_{m-1} y \leq \psi_m x$ by Lemma 4 and 5.

<u>Theorem 6:</u> $L_{n-k}(P) \cong \mathfrak{B}(P,P,\not\geq_k)$ for $0 \leq k \leq n$.

Proof: For $m := n-k$, an injective (complete) homomorphism ι_m from $\mathfrak{B}(P,P,\not\geq_k)$ into $P(m)$ is defined by

$$\iota_m(A,B) := ((A \cap P_k, B \cap P_o), (A \cap P_{k+1}, B \cap P_1), \ldots, (A \cap P_n, B \cap P_m))$$

for $(A,B) \in \mathfrak{B}(P,P,\not\geq_k)$, because $(A \cap P_{n-r}, B \cap P_s) \in P(r,s)$ for $r+s = m$. Clearly, ι_o is an isomorphism from $\mathfrak{B}(P,P,\not\geq_n)$ onto $P(0,0) (= L_o(P))$. Let us assume that ι_m is an isomorphism from $\mathfrak{B}(P,P,\not\geq_k)$ onto $L_{n-k}(P)$ for some $m := n-k < n$. For $(A,B) \in \mathfrak{B}(P,P,\not\geq_k)$, let $((C_{k-1}, D_o), (C_k, D_1), \ldots, ((C_k, D_m)) \in [\phi_m \iota_m(A,B), \psi_m \iota_m(A,B)]$.

It follows that $((B \cap P_i)' \cap P_{k+i-1}, B \cap P_i) \leq (C_{k+i-1}, D_i) \leq (A \cap P_{k+i-1}, (A \cap P_{k+i-1})' \cap P_i)$ for $1 \leq i \leq m$; therefore $\{y \in P \mid y \prec x$ for some $x \in A \cap P_{k+i}\} \subseteq C_{k+i-1} \subseteq A \cap P_{k+i-1}$ and $B \cap P_i \supseteq D_i \supseteq \{y \in P \mid y \succ x$ for some $x \in B \cap P_{i-1}\}$ for $1 \leq i \leq m$. From this we obtain that $C := C_{k-1} \cup C_k \cup \ldots \cup C_n$ is an order ideal of P and $D := D_o \cup D_1 \cup \ldots \cup D_m$ is an order filter of P. For $y \in P_i$ we have the equivalence $y \in D_i \Leftrightarrow x \not\geq_{k-1} y$ for all $x \in C$; hence $D = \{y \in P \mid x \not\geq_{k-1} y$ for all $x \in C\}$ and dually $C = \{x \in P \mid x \not\geq_{k-1} y$ for all $y \in D\}$. Thus, (C,D) is a concept of $(P,P,\not\geq_{k-1})$ and so $L_{m+1}(P) \subseteq \iota_{m+1} \mathfrak{B}(P,P,\not\geq_{k-1})$. Now, let $(E,F) \in \mathfrak{B}(P,P,\not\geq_{k-1})$. We choose the concept $(\overline{E},\overline{F})$ of $(P,P,\not\geq_k)$ with $\overline{E} = \{x \in P \mid x \not\geq_k y$ for all $y \in F\}$. Because of $E \subseteq \overline{E}$, we have $E \cap P_{k+i-1} \subseteq \overline{E} \cap P_{k+i-1}$ for $1 \leq i \leq m+1$ and so $\iota_{m+1}(E,F) \leq \psi_m \iota_m(\overline{E},\overline{F})$; furthermore, $F \subseteq \overline{F}$ implies $F \cap P_i \subseteq \overline{F} \cap P_i$ for $0 \leq i \leq m$ and so $\iota_{m+1}(E,F) \geq \phi_m \iota_m(\overline{E},\overline{F})$. This proves that $\iota_{m+1} \mathfrak{B}(P,P,\not\geq_{k-1}) = L_{m+1}(P)$. Thus the theorem is proved by induction.

By an example, we demonstrate how the iterative construction method may be applied. We choose the free bounded distributive lattice FBD(4), the $\vee$-irreducible elements of which form an ordered set P isomorphic to the power set $\mathfrak{P}(\{1,2,3,4\})$ (see [1]). Thus, $FBD(4) \cong \mathfrak{B}(\mathfrak{P}(\{1,2,3,4\}), \mathfrak{P}(\{1,2,3,4\}), \not\supseteq)$ by Proposition 1. The underlying context may be described by the table in fig. 1. The lattices $P(r,s)$ for $0 \leq r+s \leq 4$ and the morphisms ϕ_{rs} and ψ_{rs} for $0 \leq r+s \leq 3$ are shown in fig. 2. From this figure we may deduce the diagrams of $L_{n-k}(P) (\cong \mathfrak{B}(P,P,\not\geq_k))$ for $0 \leq k \leq 4$ which are presented in fig. 3 and 4. The diagram of $FBD(4) (\cong L_4(P))$ in fig. 4

is divided into subdiagrams which are linked by perforated lines; to obtain the common Hasse diagram, we have to translate the subdiagrams along the perforated lines such that their end points coincide (cf. [10]). The subdiagrams in fig. 4 represent the blocks of $\Sigma(FBD(4))$ which are the maximal Boolean subintervals of FBD(4) (this is a fact for every finite distributive lattice [4]). We conclude with the remark that the described construction method has not only led to an iterative representation of FBD(4), it has also suggested a better understandable diagram (cf. [2]).

	∅	1	2	3	4	12	13	23	14	24	34	123	124	134	234	1234
∅		×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
1			×	×	×	×	×	×	×	×	×	×	×	×	×	×
2		×		×	×	×	×	×	×	×	×	×	×	×	×	×
3		×	×		×	×	×	×	×	×	×	×	×	×	×	×
4		×	×	×		×	×	×	×	×	×	×	×	×	×	×
12				×	×		×	×	×	×	×	×	×	×	×	×
13			×		×	×		×	×	×	×	×	×	×	×	×
23		×			×	×	×		×	×	×	×	×	×	×	×
14			×	×		×	×	×		×	×	×	×	×	×	×
24		×		×		×	×	×	×		×	×	×	×	×	×
34		×	×			×	×	×	×	×		×	×	×	×	×
123					×				×	×	×		×	×	×	×
124				×			×	×			×	×		×	×	×
134			×			×		×		×		×	×		×	×
234		×				×	×		×			×	×	×		×
1234																

fig. 1

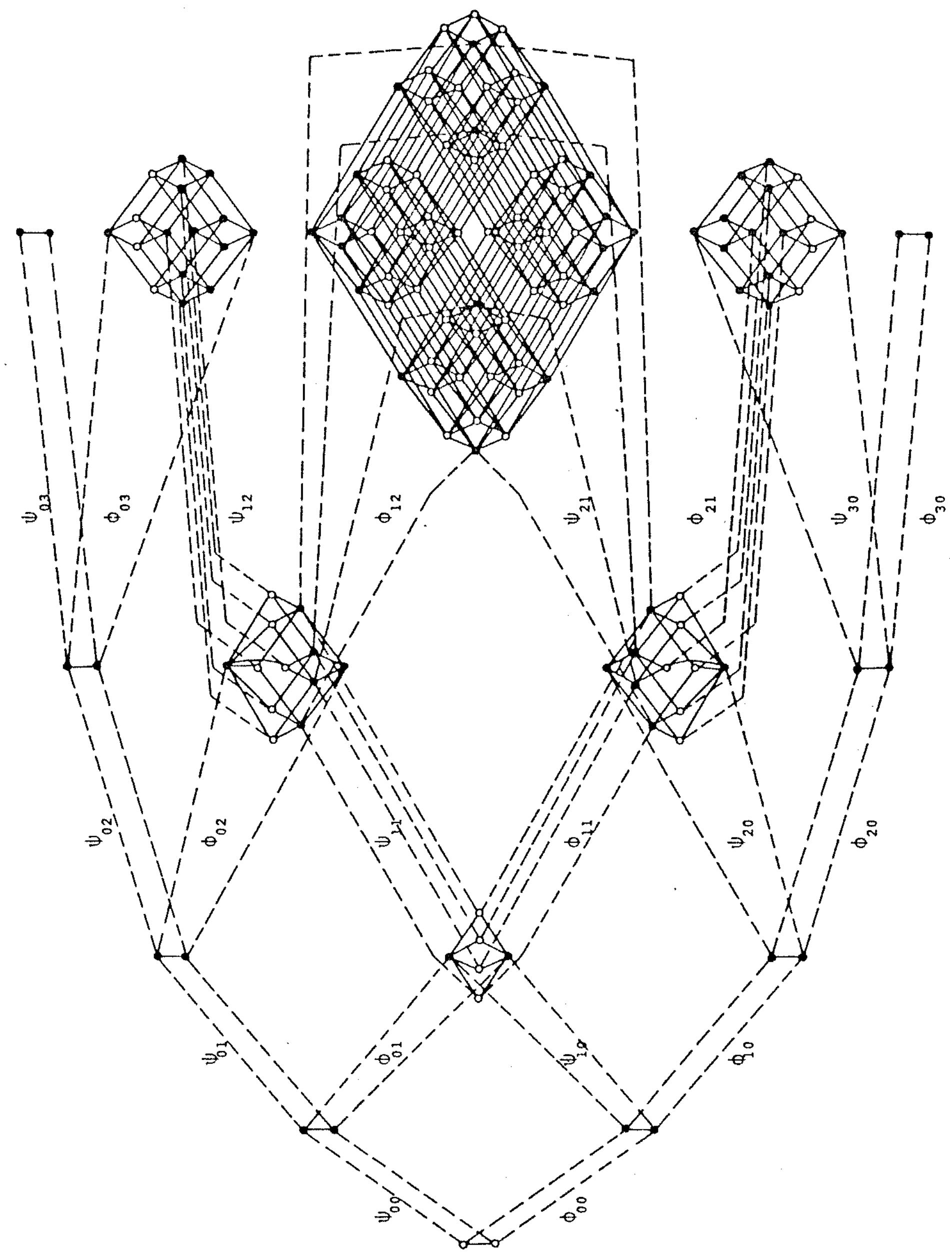
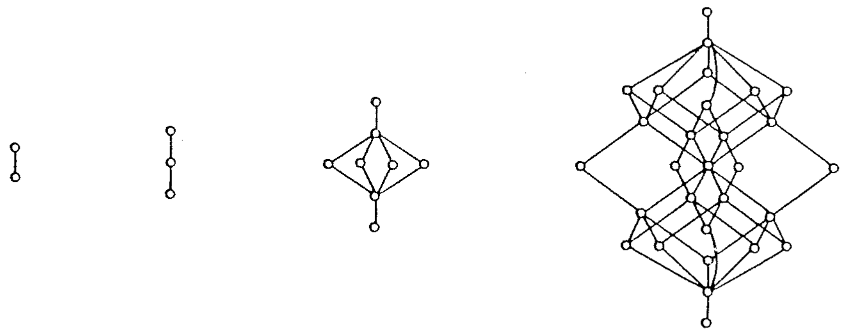


fig.2



$L_o(P)$ $L_1(P)$ $L_2(P)$ $L_3(P)$

fig. 3

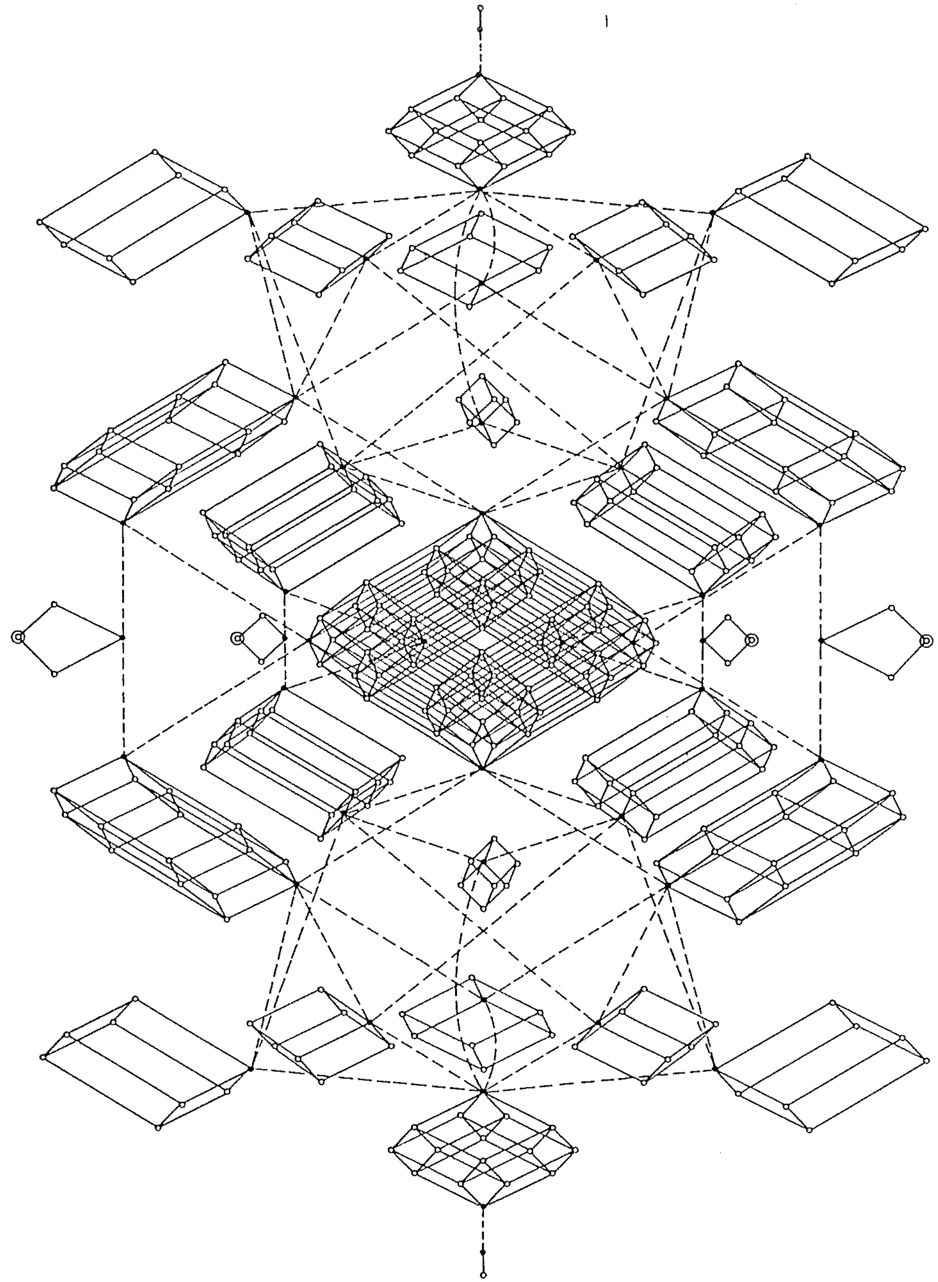


fig. 4: FBD(4) ($\cong L_4(P)$)

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