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Π^1_2 - LOGIC AND RELATED TOPICS

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This series of lectures tries to advocate the use of category-theoretic constructions in logic and computer science. The first lecture details this possibility in the case of ordinals, making explicit the hidden functoriality in the Cantor Normal Form theorem. This study leads to dilators, whose various aspects are sketched in the second lecture. The same spirit, but with a slightly different technical apparatus has been developed for denotational semantics (third lecture). First a mere simplification of Scott semantics, the approach lead to a redefinition of the very principles of logic (linear logic).

Further readings could be : the author's paper "Π^1_2-logic, part I : dilators" Ann. Math. Log. vol 21 (1981), pp. 75-219.

Abrusci's expositary paper on Goodstein's sequences, to appear in the Proceedings of the AMS congress held in Arcata, 1985.

The author's texts on denotational semantics for F, and on linear logic, both to appear in Theoretical Computer Science.: "The system F of variable types, fifteen years later", "Linear logic".

Unfortunately, no published material on the relation between dilators and descriptive set theory is available.

I THE NORMAL FORM THEOREM

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I.1. ordinals

We often deal with order relations R ; we use the notation $x \leq_R y$ to say that x is less or equal than y w.r.t. R, and $|R|$ will denote the domain of R, i.e. $|R| = \{x; x \leq_R x\}$.

I.1.1. R is a well-order when any non-void subset of $|R|$ has a smallest element :

$$\forall X\ (X \subset |R| \wedge X \neq \emptyset \ \rightarrow \exists x \in X\ \ \forall y \in X\ x \leq_R y\ )$$

A well-order is always linear :

$$\forall x\ \forall y\ (x \in |R| \wedge y \in |R| \ \rightarrow\ x \leq_R y \vee y \leq_R x\ )$$

I.1.2. R is an ordinal when R is a well-order induced by the relation $\in$ on some transitive set α, i.e. $|R| = \alpha$, $<_R = \ \in / \alpha^2$:

$$\forall x\ \forall y\ (x \in \alpha \wedge y \in \alpha \rightarrow x \in y \vee x = y \vee y \in x\ ) \wedge$$

$$\forall x\ \forall y\ (x \in \alpha \wedge y \in x \rightarrow y \in \alpha\ )$$

(the axiom of foundation ensures the smallest element condition).

I.1.3. If R is a well-order, there is a unique ordinal α and a unique isomorphism φ between R and α. Notations $\alpha = \|R\|$, $\varphi = h_R$.

I.1.4. ordinals are ordered by the relation :

$$\alpha \leq \beta \quad \text{iff} \quad \alpha \in \beta \vee \alpha = \beta$$

This order is a well-order. But the class On of all ordinals is not an ordinal, since it is not a set (paradox of Burali-Forti, 1898). Apart from this, On is a very reasonable ordinal.

I.1.5. If α is an ordinal, then α is the set of all its predecessors (w.r.t. the ordering of On). In particular the following are ordinals : $\emptyset$ (= 0), $\{\emptyset\}$ (= 1), $\{\emptyset,\{\emptyset\}\}$ (= 2), $\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ (=3) etc.

in general the intger n will be identified with the finite set $\{0,1,\ldots,n-1\}$; the first infinite ordinal, ω, is the set of all integers : $\omega = \{0,1,2,\ldots\ \}$.

I.1.6. If $X \subset$ On is a set of ordinals, then this set has a l.u.b. in On, defined by

$$\sup(X) = \bigcup X$$

I.1.7. Let α be an ordinal ; then

i) either $\alpha = \emptyset$

ii) or α has a greatest element β ; in that case $\alpha = \beta \cup \{\beta\}$, and we use the notation $\alpha = \beta + 1$.

iii) or $\alpha \neq \emptyset$ and $\alpha = \sup(\alpha) = \bigcup \alpha$.

These three cases are mutually exclusive ; according to the case, α will be said to be zero, successor (of β), limit.

I.2. ordinal functions

I.2.1. The successor function

$$\alpha + 1 = \alpha \cup \{\alpha\}$$

is the most immediate of all ordinal functions.

I.2.2. the sum function

$$\alpha + 0 = \alpha$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$

$$\alpha + \lambda = \sup_{\beta < \lambda} \alpha + \beta \qquad (\lambda \text{ limit})$$

generalizes the sum of integers.

I.2.3. the product function

$$\alpha . 0 = 0$$

$$\alpha . (\beta + 1) = \alpha . \beta + \alpha$$

$$\alpha . \lambda = \sup_{\beta < \lambda} \alpha . \beta \qquad (\lambda \text{ limit})$$

generalizes the product of integers.

I.2.4. the exponential function

$$\alpha^0 = 1$$
$$\alpha^{\beta+1} = \alpha^\beta . \alpha$$
$$\alpha^\lambda = \sup_{\beta<\lambda} \alpha^\beta \qquad (\lambda \text{ limit})$$

generalizes the exponentiation of integers.

I.2.5. There are many other interesting ordinal functions ; let us mention the Veblen hierarchy (1908) :

$V_0(\alpha) = \omega^\alpha$

$V_{\beta+1}(\alpha)$ = the α^{th} fixed point (in increasing order) of the function V_β.

$V_\lambda(\alpha)$ = the α^{th} point in the range of all V_β's for $\beta<\lambda$. (λ limit).

Generalizations of recursion theory (e.g. set recursion of Dag Normann) give wide schemes for generating ordinal functions. The cardinal functions $\alpha \rightsquigarrow \beth_\alpha$ are also ordinal functions, but belong to a completely different universe of thought.

I.2.6. Sum, product, exponential are generalizations of familiar operations between integers ; two questions arise :

i) in which sense are these generalizations "natural" ?

ii) what is the reason why some equations between integers persist for ordinal values (e.g. (a + b) + c = a + (b + c)), whereas others don't (e.g. a + b = b + a) ?

The approach to ordinals and ordinal functions through topology cannot help : typically the functions we are dealing with are not even continuous, i.e. do not commute with limits. For instance $\omega+\omega \neq \sup_{n<\omega} n+\omega = \omega$. The category-theoretic approach (dilators) will make explicit a deeper notion of continuity, and will

clarify these two points in a quite satisfactory way.

I.3. the Cantor Normal Form Theorem

I.3.1. take a point $z<\alpha+\beta$; then two cases

- either $z<\alpha$
- or z can be (uniquely) written $z=\alpha+y$ for some $y<\beta$.

I.3.2. take a point $z<\alpha.\beta$; then there are unique $x<\alpha$ and $y<\beta$ such that $z=\alpha.y+x$

I.3.3. (Cantor Normal Form) Take a point $z<(1+\alpha)^\beta$; it is possible to find an integer n, together with $y_{n-1}<\dots y_0<\beta$ and $x_{n-1},\dots,x_0<\alpha$ such that

$$z = (1+\alpha)^{y_{n-1}}.(1+x_{n-1}) + \dots + (1+\alpha)^{y_0}.(1+x_0)$$

The integer n, the sequences y_i and x_i are unique.

I.3.4. the normal forms I.3.1.-I.3.4. have the additional property that, if we take any expression satisfying the requirements, then it is the normal form of something : typically, given any $x<\alpha$, any $y<\beta$ then $z=\alpha.y+x$ is the normal form of a point of $\alpha.\beta$.

I.3.5. The main problem is to generalize the normal form to general ordinal functions ; this is surely impossible to do this in general. Hence our task will be to find a suitable class of functions for which something like a normal form theorem holds. The answer is given through dilators.

I.4. the category ON

I.4.1. a category (here the category ON of ordinals) is defined by specifying :

i) a class of objects : here On, the class of all ordinals

ii) given any two objects (here ordinals) α and β, the set of all morphisms from α to β: in ON, we take the set of all strictly increasing maps from α to β

$$I(\alpha,\beta) = \{f : \alpha \to \beta \ ; \ \forall x,y \ (x < y \ \to \ f(x) < f(y))\}$$

iii) define an associative <u>composition</u> of morphisms : given f from α to β, g from β to γ, then gf will be a morphism from α to γ. In the case of ON, we take the usual composition of functions.

iv) define, for any object α, a distinguished <u>identity</u> morphism from α to itself, which must be neutral w.r.t. composition. In ON, we take, as expected, as identities, the maps

$$E_\alpha(z) = z \quad \text{for all } z \in \alpha .$$

I.4.2. ON has the very useful property :

given $\alpha \in$ On and $X \subset \alpha$, there are unique $\beta \in$ On and $f \in I(\beta,\alpha)$ such that $X = rg(f)$.

In particular, ON is a very "rigid" category : when α and β are isomorphic, i.e. when there are $f \in I(\alpha,\beta)$, $g \in I(\beta,\alpha)$ such that $fg = E_\beta$, $gf = E_\alpha$, then $\beta = \alpha$, and $f = g = E_\alpha$.

I.4.3. When $\mathcal{C}$ and $\mathcal{D}$ are categories, a <u>functor</u> F from $\mathcal{C}$ to $\mathcal{D}$ consists in the following data :

i) a function F sending objects of $\mathcal{C}$ to objects of $\mathcal{D}$

ii) given any two objects α and β of $\mathcal{C}$, a function (still denoted by F) sending morphisms from α to β in $\mathcal{C}$ to morphisms from $F(\alpha)$ to $F(\beta)$ in $\mathcal{D}$.

iii) F must be compatible with the compositions in $\mathcal{C}$ and $\mathcal{D}$:

$$F(gf) = F(g)F(f)$$

iv) F must preserve the identities

$$F(E_\alpha) = E_{F(\alpha)}$$

I.4.4. It is possible to define functors in several variables ; for instance a functor from ON^2 to ON will associate :

i) to any pair (α,β) of ordinals, an ordinal $F(\alpha,\beta)$

ii) to any pair (f,g), $f \in I(\alpha,\alpha')$, $g \in I(\beta,\beta')$ of morphisms, a morphism $F(f,g) \in I(F(\alpha,\beta),F(\alpha',\beta'))$.

iii) one must have

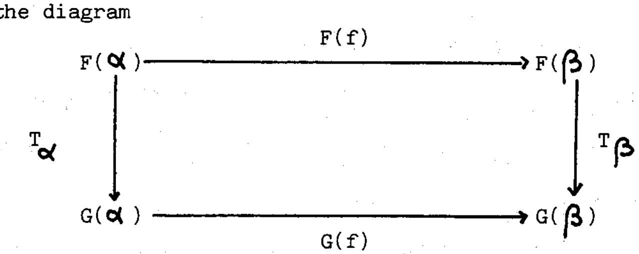
$$F(f'f,g'g) = F(f',g')F(f,g)$$

iv) and also $\quad F(E_\alpha,E_\beta) = E_{F(\alpha,\beta)}$

I.4.5. when F and G are functors from $\mathcal{C}$ to $\mathcal{D}$, then a <u>natural transformation</u> T from F to G is a family (T_α) such that :

i) for any object α of $\mathcal{C}$, T_α is a $\mathcal{D}$-morphism from $F(\alpha)$ to $G(\alpha)$

ii) if α,β are objects of $\mathcal{C}$, if f is a $\mathcal{C}$-morphism from α to β, then the diagram



is commutative, i.e. $T_\beta F(f) = G(f)T_\alpha$. It is possible to define, when f is a $\mathcal{C}$-morphism from α to β, a $\mathcal{D}$-morphism T(f) from $F(\alpha)$ to $G(\beta)$, by $\quad T(f) = T_\beta F(f) = G(f)T_\alpha$.

I.4.6. If T and U are natural transformations respectively from F to G and from G to H, then one can define another one from F to H by :

$$(UT)_\alpha = U_\alpha T_\alpha$$

and it is easily checked that $(UT)(gf) = U(g)T(f)$.

I.4.7. Sum, product and exponential are indeed functors from ON^2 to ON : we have already defined functions $\alpha+\beta$, $\alpha.\beta$, $(1+\alpha)^\beta$; so it remains to define F(f,g) when $f \in I(\alpha,\alpha')$, $g \in I(\beta,\beta')$. This is done by using the normal form theorems :

i) $f + g \in I(\alpha+\beta,\alpha'+\beta')$ is defined by

$(f+g)(z) = f(z) \qquad \text{if } z < \alpha$

$(f+g)(\alpha + z) = \alpha' + g(z) \qquad \text{if } z < \beta$

ii) $f \,.\, g \quad I(\alpha.\beta, \alpha'+\beta')$ is defined by

$(f \,.\, g)(\alpha.y + x) = \alpha'.g(y) + f(x) \quad \text{if } y<\beta, x<\alpha$

iii) $(1+f)^g \in I((1+\alpha)^{\beta}, (1+\alpha')^{\beta'})$ is defined by

$$(1+f)^g((1+\alpha)^{y_{n-1}}.(1+x_{n-1}) + \ldots + (1+\alpha)^{y_0}.(1+x_0)) =$$
$$(1+\alpha')^{g(y_{n-1})}.(1+f(x_{n-1})) + \ldots + (1+\alpha')^{g(y_0)}.(1+f(x_0))$$

Observe that, in case iii) the fact that f and g preserve strict inequality is essential : this ensures that $(1+f)^g$ applied to a normal form is still a normal form. Another important property is used, namely, that, in order to compare two Cantor Normal forms, one with $n,(x_i),(y_i)$, another with $m,(x'_j),(y'_j)$, then only matters the mutual relations between the y_i's and the y'_j's on one hand, and between the x_i's and the x'_j's on the other hand.

I.4.8. We have therefore discovered an important feature of the ordinal functions sum, product, exponential, namely that the normal form theorems enable us to consider them as functors. We shall try to prove the converse, namely to recover a normal form theorem from a functor. In fact, most of ordinal functors are pathological, hence it will be necessary to require some properties from our functors. To simplify matters, we shall restrict our attention to unary functors.

I.4.9. examples of unary functors, i.e. functors from ON to ON :

i) constant functors : if $\alpha \in$ On, define

$\underline{\alpha}(\beta) = \alpha \qquad \underline{\alpha}(f) = E_{\alpha}$

ii) the identity functor

$Id(\alpha) = \alpha \qquad Id(f) = f$

iii) using these basic unary functors, then one can define other ones

by composition with sum, product and exponential :

. $(\underline{1}+Id)(\alpha) = 1 + \alpha \qquad (\underline{1}+Id)(f) = E_1 + f$

. $(Id+\underline{1})(\alpha) = \alpha + 1 \qquad (Id+\underline{1})(f) = f + E_1$

. $(Id+Id)(\alpha) = \alpha + \alpha \qquad (Id+Id)(f) = f + f$

. $(Id^2)(\alpha) = \alpha^2 \qquad (Id^2)(f) = f^2$

. $(1+Id)^{Id}(\alpha) = (1+\alpha)^{\alpha} \qquad (1+Id)^{Id}(f) = (1+f)^f$

Let us for instance detail the functor $\underline{10}^{Id} = (1+\underline{9})^{Id}$:

$10^{Id}(\alpha) = 10^{\alpha}$; when $f \in I(\alpha,\beta)$, then

$$10^{Id}(f)(10^{x_{n-1}}.a_{n-1}+\ldots+10^{x_0}.a_0) = 10^{f(x_{n-1})}.a_{n-1}+\ldots+10^{f(x_0)}.a_0$$

when $a_0,\ldots,a_n < 10$ and $x_0 < \ldots. < x_{n-1} < \alpha$.

In particular, if α and β are integers, then 10^f consists in adding zeros in the decimal expansions. For instance

$10^f(51003401300991) = 500100304001300900091000)$, if $\alpha = 20$, $\beta = 36$ and $f(0) = 3$, $f(1) = 4$, $f(2) = 8$, $f(5) = 11$, $f(6) = 12$, $f(8) = 15$, $f(9) = 17$, $f(12) = 20$, $f(13) = 23$; the other values of f are irrelevant to the result.

I.4.10. the condition that will be required on a functor will be so strong that, as soon as we know it on integers (and on morphisms of integers), then we know it everywhere. This answers the questions raised in I.2.6. :

i) from the definition of the sum of integers n + m and the sum of morphisms of integers f + g, there is only one possible extension to ON enjoying our strong requirement.

ii) if we consider the associativity of sum $(n + m) + p = n + (m + p)$ observe that we have also $(f + g) + h = f + (g + h)$ when f,g,h are morphisms of integers. The unicity of the extension into a ternary functor enjoying the strong condition forces the functors

$(\alpha+\beta)+\gamma$ and $\alpha+(\beta+\gamma)$ to be equal. On the other hand the commutativity of addition $n + m = m + n$ holds for integers, but fails for morphisms (in general $f + g \neq g + f$), hence there is no way to conclude that $\alpha+\beta=\beta+\alpha$. In other terms a sufficient condition for an equation to persist for ordinal values, is that it also holds for morphisms of integers. (The condition is not necessary, as shown by $1+\alpha+\alpha=\alpha+1+\alpha$, which holds for any $\alpha \in$ On, but is not a functorial equality, since $\underline{1} + Id + Id \neq Id + \underline{1} + Id$.)

I.4.11. exercise : prove the distributivity of product w.r.t. sum by checking it on integers and morphisms of integers.

I.5. direct limits

I.5.1. assume that I is a non void ordered set, and that I is directed : $\forall i,j \in I \ \exists \ k \in I \quad i,j \leq_I k$.

A direct system indexed by I is a family (α_i, f_{ij}) such that :

i) for all $i \in I$, α_i is an ordinal

ii) for all $i,j \in I$ such that $i \leq_I j$, $f_{ij} \in I(\alpha_i, \alpha_j)$

iii) for all $i,j,k \in I$ such that $i \leq j \leq k$,

$$f_{ik} = f_{jk} f_{ij}$$

iv) for all $i \in I$, $f_{ii} = E_{\alpha_i}$.

I.5.2. examples

i) $I = \mathbb{N}$, $\alpha_n = n$, $f_{nm} = E_{nm}$ where $E_{\alpha\beta} \in I(\alpha,\beta)$ is defined by

$E_{\alpha\beta}(z) = z$ for all $z < \alpha$.

ii) $I = \mathbb{N}$ $\alpha_n = n+1$, $f_{nm} = E_{nm} + E_1$, i.e.

$f_{nm}(p) = p$ for $p < n$, $f_{nm}(n) = m$.

iii) $I = \mathbb{N}$ $\alpha_n = n+n$, $f_{nm} = E_{nm} + E_{nm}$, i.e.

$f_{nm}(p) = p$ for $p < n$, $f_{nm}(n+p) = m+p$ for $p < n$.

iv) let α be a limit ordinal, and let $I = \alpha$. For $i < \alpha$, let

$\alpha_i = i$, and, for $i \leq j < \alpha$ let $f_{ij} = E_{ij}$.

v) let α be an ordinal, and let I be the set of all finite subsets of α . When $i \in I$, define $\alpha_i = \|i\|$ (the order type of the subset i of α). When $i \subset j$, then the inclusion map from i to j induces, using the isomorphisms h_i, h_j of I.1.3., a function $f_{ij} \in I(\alpha_i, \alpha_j)$:

$$f_{ij}(h_i(z)) = h_j(z) \quad \text{for } z \in i.$$

I.5.3. assume that (α_i, f_{ij}) is a direct system indexed by I ; then a direct limit for (α_i, f_{ij}) is a pair (α, f_i) such that :

i) $\alpha \in$ On

ii) for all $i \in I$, $f_i \in I(\alpha_i, \alpha)$

iii) for all $i,j \in I$ such that $i \leq j$ $\quad f_j f_{ij} = f_i$.

iv) if (β, g_i) is any other solution to i)-iii) then there is a unique $h \in I(\alpha,\beta)$ such that $g_i = hf_i$ for all $i \in I$.

This last condition is extremely important ; observe that it implies the unicity of the direct limit (when it exists).

I.5.4. Definition I.5.3. is a general category-theoretic definition ; in the special case of ON, we have a more manageable equivalent of iv) :

iv)' $$\alpha = \bigcup_{i \in I} rg(f_i).$$

I.5.5. examples : in the cases considered in I.5.2., we have direct limits, namely

i) $(\omega, E_{n\omega})$

ii) $(\omega+1, E_{n\omega}+E_1)$, i.e. $f_n(p) = p$ for $p < n$, $f_n(n) = \omega$.

iii) $(\omega+\omega, E_{n\omega}+E_{n\omega})$, i.e. $f_n(p) = p$, $f_n(n+p) = m+p$ for $p < n$.

iv) $(\alpha, E_{i\alpha})$: generalization of i).

v) (α, h_i^{-1}) . Let us check iv)' in that case : $z \in \alpha$ is in the range of $f_{\{z\}}$.

I.5.5. notation : $(\alpha, f_i) = \varinjlim(\alpha_i, f_{ij})$ when (α, f_i) is the direct limit of (α_i, f_{ij}).

I.5.6. In examples i) and iv) above, a direct limit was nothing more than a supremum : this is when the morphisms are of the form $E_{\dots}$. In particular the categorical notion of limit contains the topological one. But we have also seen many examples of direct limits, where the limit was not a supremum. For instance example v) shows that any ordinal is a direct limit of integers. In fact, what is the important property is not exactly this, but that <u>the class of "functions" compatible with that kind of approximation is extremely rich.</u>

<u>I.6. existence of direct limits</u>

I.6.1. there are many systems in ON without a direct limit. For instance, if $\alpha_n = n$, and, when $n \leq m$, $f_{nm} = E_{0m-n} + E_n$, i.e. $f_{nm}(p) = p+m-n$ for $p < n$: consider a direct limit (α, f_n) : then $f_{n+1}(0) < f_{n+1}(1) = f_{n+1}(f_{nn+1}(0)) = f_n(0)$. Hence the set $\{f_n(0); n \in \mathbb{N}\}$ has no smallest element, contradiction.

I.6.2. it is convenient to introduce OL, a category whose objects are linear orders, and whose morphisms are strictly increasing maps. In OL, it is easily checked that all direct systems have direct limits, for instance, the system considered in I.6.1. admits as direct limit in OL, the order opposite to .

I.6.3. A direct system (α_i, f_{ij}) in ON, can be viewed as a system in OL, and so has a direct limit in OL, (x, f_i). Then (α_i, f_{ij}) has a direct limit in ON exactly when x is a well-order, in which case the direct limit in ON is $(\|x\|, h_x f_i)$.

I.6.4. a <u>sufficient</u> condition for the existence of a direct limit in ON for (α_i, f_{ij}) is the existence of (β, g_i) enjoying conditions i)-iii) of direct limits.

<u>I.7. preservation of direct limits</u>

I.7.1. A functor F from ON to ON <u>preserves direct limits</u> iff, given any direct system (α_i, f_{ij}) in ON admitting the direct limit (α, f_i), then the system $(F(\alpha_i), F(f_{ij}))$ admits the direct limit $(F(\alpha), F(f_i))$.

I.7.2. The following condition is easily shown to be equivalent to preservation of direct limits :

given any $\alpha \in$ On and any $z \in F(\alpha)$, one can find an integer n and a morphism $f \in I(n, \alpha)$ such that $z \in rg(F(f))$.

I.7.3. The functors considered in I.4.7.,I.4.9. preserve direct limits :

i) the identity preserves everything, in particular direct limits.

ii) the constant functors preserve direct limits

iii) the binary functors sum, product and exponential, preserve direct limits : this is a consequence of the <u>finiteness</u> of the normal forms. For instance, let us consider the exponential : the obvious analogue of I.7.2. for binary functors, says that, given $\alpha, \beta \in$ On, $z \in (1+\alpha)^{\beta}$, there are integers n,m and morphisms $f \in I(n, \alpha)$, $g \in I(m, \beta)$ such that $z \in rg((1+f)^g)$. Now write $z = (1+\alpha)^{y_{p-1}}.(1+x_{p-1}) + \dots + (1+\alpha)^{y_0}.(1+x_0)$. Let m = p and define $g \in I(m, \beta)$ by $g(i) = y_i$; define $f \in I(n, \alpha)$ by the condition $rg(f) = \{x_0, \dots, x_{p-1}\}$ (hence $n = card(\{x_0, \dots, x_{p-1}\})$). If $z_0 = (1+n)^{m-1}.(1+f^{-1}(x_{m-1})) + \dots + (1+n)^0.(1+f^{-1}(x_0))$, it is plain that $z = (1+f)^g(z_0)$.

I.7.4. If F and G are two functors from ON to ON preserving direct limits, and such that F and G coincide on integers and morphisms of integers, then F = G. For instance, since every ordinal is a direct limit of integers, if follows that F and G take the same ordinal values.

I.7.5. The provisional conclusion of the work so far done, is that we msut restrict our attention to functors preserving direct limits.

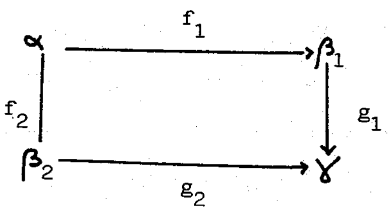
I.8. Pull-backs

I.8.1. Let β_1, β_2, γ be ordinals, let $g_1 \in I(\beta_1, \gamma)$, $g_2 \in I(\beta_2, \gamma)$. A 3-tuple (α, f_1, f_2) is said to be a pull-back of (g_1, g_2) when the following holds :

i) $\alpha \in On$

ii) $f_1 \in I(\alpha, \beta_1)$ $\quad f_2 \in I(\alpha, \beta_2)$

iii) $\quad g_1 f_1 = g_2 f_2$



iv) given any other solution (α', f_1', f_2') to i)-iii), there is a unique $h \in I(\alpha', \alpha)$ such that $f_1' = f_1 h$, $f_2' = f_2 h$.

I.8.2. Definition I.8.1. is a general category-theoretic definition ; in the case of ON, iv) can be replaced by

iv)' $rg(g_1 f_1) = rg(g_2 f_2) = rg(g_1) \cap rg(g_2)$.

(Hence direct limits correspond to directed unions, and pull-backs to finite intersections.)

In the category ON, f_1 and f_2 can be recovered from $h = g_1 f_1 = g_2 f_2$, and it is enough to concentrate on h ; notation $h = g_1 \& g_2$.

I.8.3. in ON, pull-backs always exist : this is immediate from condition iv)'.

I.8.4. a functor F from ON to ON preserves pull-backs, iff for any two morphisms f and g with the same target, one has

$$F(f \& g) = F(f) \& F(g).$$

I.8.5. the functors considered in I.4.7.,I.4.9. preserve pull-backs :

i) the identity functor Id preserves everything

ii) the constant functors preserve pull-backs because $E_\alpha \& E_\alpha = E_\alpha$

iii) the binary functors sum product, exponential, preserve pull-backs : this is a consequence of the unicity of normal forms. For instance, w.r.t. the exponential, write the Cantor Normal form of $z < (1+\alpha)^\beta$ which involves coefficients $x_0, \ldots, x_{n-1} < \alpha$, $y_0, \ldots, y_{n-1} < \beta$. Then $z \in rg(F(f,g))$ exactly when the x_i's belong to rg(f) and the y_i's belong to rg(g). From this one gets

$$rg(F(f \& f', g \& g') = rg(F(f,g)) \cap rg(F(f',g')).$$

I.9. dilators

I.9.1. a dilator is a functor F from ON to ON preserving direct limits and pull-backs

I.9.2. all functors considered in I.4.9. are dilators. There is no link between preservation of direct limits and preservation of pull-backs ; however, one can prove that, from a functor F from ON to ON preserving only direct limits, one can obtain a functor G preserving also pull-backs (i.e. a dilator) : $G = F \circ \underline{\omega}.(\underline{1}+Id)$

I.9.3. The Normal Form Theorem

Let F be a dilator, let α be an ordinal, and let $z \in F(\alpha)$; then it is possible to find an integer n, a morphism $f \in I(n, \alpha)$, and a point $z_0 \in F(n)$ such that

$$z = F(f)(z_0) \qquad (1)$$

Moreover, if (n, f, z_0) enjoying (1) have been chosen with n minimum, then there is only one solution, i.e. this determines f and z_0.

(The proof of the normal form theorem is an immediate consequence of the preservation properties : the existence of a solution to (1) is preseravation of direct limits, the unicity of a solution with n minimal, is preservation of pull-backs.)

I.9.4. notation : Assume that n, f, z_0 are the data uniquely associated with F, α, z by I.9.3. ; then we shall indicate the situation by

$$z = (z_0; x_0, \dots, x_{n-1}; \alpha)_F$$

where $x_i = f(i)$.

I.9.5. the <u>trace</u> of F is the set of all pairs (z_0, n) such that

i) $z_0 \in F(n)$

ii) if $m \leq n$, if $f \in I(m,n)$ is such that $z_0 \in rg(F(f))$, then $m = n$.

The trace Tr(F) of F is just the set of invariants of normal forms w.r.t. F : if $(z_0, n) \in Tr(F)$ and $x_0 < \dots < x_{n-1} < \alpha$, then

$$(z_0; x_0, \dots, x_{n-1};\ )_F \text{ is a normal form w.r.t. F.}$$

I.9.6. a dilator acts as follows on its normal forms : given $f \in I(\alpha, \beta)$, then

$F(f)(z_0; x_0, \dots, x_{n-1}; \alpha)_F = (z_0; f(x_0), \dots, f(x_{n-1}); \beta)_F$.

I.9.7. a natural transformation T from a dilator F to a dilator G acts as follows on their respective normal forms :

$T(\alpha)(z_0; x_0, \dots, x_{n-1}; \alpha)_F = (T(n)(z_0); x_0, \dots, x_{n-1}; \alpha)_G$

(The result is non-trivial). It is possible to define a function Tr(T) from Tr(F) to Tr(G) by

$$Tr(T)(z_0, n) = (T(n)(z_0), n).$$

I.9.8. Let us compute the normal forms w.r.t. the functor Id^2 : start with $z < \alpha^2$; we know that $z = \alpha . u + v$, for some $u, v < \alpha$; then three cases occur

i) $u < v$ then $z = (1; u, v; \alpha)_{Id^2}$

ii) $u = v$ then $z = (0; u; \alpha)_{Id^2}$

iii) $u > v$ then $z = (2; v, u; \alpha)_{Id^2}$

so $Tr(Id^2) = \{(1,2), (0,1), (2,2)\}$.

<u>I.10. effectivity of dilators</u>

I.10.1. a functor F from $ON{<}\omega$(integers) to ON, and preserving pull-backs, can be extended into a functor F' from ON to OL preserving direct limits and pull-backs. Moreover, if $F'(\alpha)$ happens to be a well-order for all $\alpha \in On$, then F' can be chosen to be a dilator, and this extension is of course unique.

I.10.2. a dilator is not only completely determined by its restriction to $ON < \omega$, but there is an effective way to compute its values from the integer values :

i) the trace of F can be obtained from the restriction of F to $ON{<}\omega$.

ii) we define $|F(\alpha)|$ as the set of all formal expressions

$$(z_0; x_0, \dots, x_{n-1}; \alpha)_F$$

with $(z_0, n) \in Tr(F)$, $x_0 < \dots < x_{n-1} < \alpha$.

iii) the functions F(f) from $|F(\alpha)|$ to $|F(\beta)|$ are defined (when $f \in I(\alpha, \beta)$) by

$F(f)(z_0; x_0, \dots, x_{n-1}; \alpha)_F = (z_0; f(x_0), \dots, f(x_{n-1}); \beta)_F$.

iv) it remains to define the order relation on $|F(\alpha)|$: in order to compare $(z_0; x_0, \dots, x_{n-1}; \alpha)_F$ and $(z_1; y_0, \dots, y_{m-1}; \alpha)_F$, let p be the cardinal of the set $\{x_0, \dots, x_{n-1}, y_0, \dots, y_{m-1}\} = X$, and define $h \in I(p, \alpha)$ by $rg(h) = X$. Then our two formal expressions are ordered as $(z_0; h^{-1}(x_0), \dots, h^{-1}(x_{n-1}); p)_F$ and $(z_1; h^{-1}(y_0), \dots, h^{-1}(y_{m-1}); p)_F$ (actual normal forms) were ordered in F(p).

This completes the effective description of F ; of course the formal normal forms correspond to real ones w.r.t. F.

I.10.3. In fact there is no reason why the trace should be effective from the restriction of F to $ON^{<\omega}$; but we can of course take as data :

i) the trace Tr(F)

ii) for any (z_0,n), $(z_1,m) \in Tr(F)$, for any p and $f \in I(n,p), g \in I(m,p)$ such that $p = rg(f) \cup rg(g)$, the order between $F(f)(z_0)$ and $F(g)(z_1)$.

In particular, it makes perfectly good sense to speak of a recursive dilator.

II DILATORS : A SURVEY

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II.1. Dilators and proof-theory

II.1.1. A ß-language $\mathbf{L}$ is a language including a specific type On for ordinals, together with a specific binary predicate $<$ taking arguments of type On. ß-languages are always supposed to be denumerable. A ß-theory T is a theory within a ß-language. If T is ß-theory, a ß-model for T is a model for T in the usual sense, such that the interpretation of On and $<$ are an ordinal : $M(On) = \alpha$, $M(<) = \in \upharpoonright \alpha^2$. The value $\alpha = 0$ is not excluded. We define $\|M\| = \alpha$.

II.1.2. Mostowski raised the problem of finding a syntactical notion of ß-proof which should be complete w.r.t. validity in ß-models ; in 1974, Apt claimed to have disproved the existence of any reasonable ß-rule. In fact his work did not prove anything of that kind, because he sticked too much to the familiar idea of infinitary rule.

II.1.3. A (trivial) approximation to the solution is as follows : fix an ordinal α ; if A is valid in all ß-models of T, then A is valid in all ß-models of T such that $\|M\| = \alpha$. Then, a straightforward generalization of the familiar ω-completeness theorem, shows that A will be provable by means of the "α-rule", in the theory $T[\alpha]$, obtained by adding to $\mathbf{L}$ constants $\bar{i}$ for all $i < \alpha$, and the axioms $\bar{i} < \bar{j}$ when $i < j$, $\neg(\bar{i} < \bar{j})$ when $j \leq i$. The "α-rule" is just the generalization of the familiar ω-rule :

$$\frac{\ldots B[\bar{i}] \ldots \quad (\text{all } i < \alpha)}{\forall x^{On}.B[x]}$$

Now, A will provable by means of the α-rule in all $T[\alpha]$'s. This means that we can take as a ß-proof of A in T, a family (P_α), such that P_α is an α-proof of A for all $\alpha \in$ On.

However, the situation is ridiculous : a family of infinite proofs, indexed by On, is not quite a syntactical object !

II.1.4. We therefore try to find a way to express that a family (P_α) of proofs of various sizes, like in II.1.3., can be generated from its subfamily (P_n), in an effective way :

i) if P' is an α'-proof in $T[\alpha']$, if $f \in I(\alpha, \alpha')$, then we shall try to define an α-proof $f^{-1}(P)$ in $T[\alpha]$:

. in P delete all formulas which are premises of index $i \notin rg(f)$ of some α'-rule ; delete also premises of deleted premises etc., so that to get a subtree P' of P

. it may happen that in P', we have still parameters not belonging to rg(f). Then $f^{-1}(P)$ is not defined.

. otherwise, $f^{-1}(P)$ is defined by replacing in P' all parameters $\overline{f(i)}$ by $\bar{i}$.

ii) a ß-proof of A in T in a family $(P_\alpha)_{\alpha \in On}$ such that, for all α, P_α is a proof of A in $T[\alpha]$; furthermore, it is required that

$$\forall \alpha, \alpha' \in On\ \forall f \in I(\alpha, \alpha')\ f^{-1}(P_{\alpha'}) = P_\alpha .$$

iii) it is possible to build a category whose objects are pairs (α, P), where $\alpha \in$ On, and P is an α-proof ; morphisms from (α, P) to (α', P') are just functions $f \in I(\alpha, \alpha')$ such that $P = f^{-1}(P')$.

Then a ß-proof appears as a functor from ON to this category of proofs.

Such a functor necessarily preserves direct limits (and also pull-backs). As a consequence, the family (P_α) is effectively computable from its subfamily (P_n), just as a dilator is computable from its behaviour on ON $< \omega$.

II.1.5. ß-completeness theorem

A is valid in all ß-models of T iff there is a ß-proof of A in T. Furthermore, when T is recursive, the ß-proof can in turn be chosen recursive : the function $n \rightsquigarrow P_n$ (recall that P_n is a finite proof) is recursive.

II.1.6. There are more than superficial analogies between dilators and ß-proofs. The combination of both ingredients in the framework of sequent calculus, leads to a proof-theoretic analysis of inductive definitions. The methods used are too sophisticated to be explained here ; let us say that the essential output has been the majorization results in generalized recursion.

II.2. Dilators and generalized recursion

II.2.1. Which kind of ordinal functions are likely to be bounded by dilators ? There are obviously functions which grow too fast, typically $\alpha \rightsquigarrow \beth_\alpha$. Using (AC) + (CH), one easily finds a function from $\beth_1$ to itself, which is not bounded by any denumerable dilator, i.e. by any dilator with a denumerable trace.

II.2.2. Hence some kind of definability condition is needed in order to ensure that the function f from On to itself should be bounded by a dilator. The first result in that direction is the following :

II.2.3. Let f be a ω_1^{CK}-recursive function, sending ω_1^{CK} (the first non-recursive ordinal) into itself. Then there is a recursive dilator F such that

$$\forall \alpha < \omega_1^{CK} \qquad f(\alpha) \leq F(\alpha).$$

The result was first obtained as a corollary to the proof-theoretic analysis of ID_1, the theory of one inductive definition.

II.2.4. This result has been generalized to many other kinds of generalized recursion. The value of those results is that, to some extent, the infinitary features that are used in the computation of generalized recursive functions, can be, as far as we are only concerned with their growth, be eliminated in favor of a purely finitary approach.

II.3. Dilators and descriptive set-theory

II.3.1. Π^1_2-completeness of dilators : assume that A is a Π^1_2 formula $\forall f \exists g \forall n\ R\ \bar{f}(n), \bar{g}(n)$; then there is a recursive functor D from $ON^{<\omega}$ to itself such that

$$A \longleftrightarrow D \text{ can be extended into a dilator}$$

This means that the values $D(\alpha)$ computed by direct limits, are well-orders, for $\alpha \in On$. This generalizes the familiar Π^1_1-completeness of ordinals.

II.3.2. The use of ordinals in descriptive set-theory is linked to the following features

i) Π^1_1-completeness of ordinals : a Π^1_1 formula A can be represented by "R is a well-order", for an appropriate recursive linear order R.

ii) the existence of a simple well-ordering of ordinals, enabling us to select an ordinal in any non-void set of ordinals.

The problem with the use os dilators is that, although they are Π^1_2-complete, there is no reasonable well-ordering od dilators ; for instance, how should we compare Id and 2^{Id} ?

II.3.3. In fact, it is eanough to restrict the comparizon to the denumérable case. The idea will be to find a dilator $\mathring{\Omega}$ such that

$$F \circ \mathring{\Omega} \text{ and } G \circ \mathring{\Omega}$$

are comparable in a very straightforward sense, for all denumerable F and G. Here comparable means w.r.t. the inclusion relation :

$H \sqsubseteq H'$ means $\exists H''. (H' = H + H'')$, in other terms, H' is an end extension of H.

II.3.4. A dilator F is said to be a flower when the following holds :

$$F(E_{\alpha\beta}) = E_{F(\alpha)F(\beta)} \qquad \text{for all } \alpha \leq \beta .$$

If F is a flower, then the function $\alpha \rightsquigarrow F(\alpha)$ is topologically continuous (and conversely). α, Id, α + Id, 10^{Id} are flowers. A denumerable dilator F is a flower iff $F(\aleph_1) \leq \aleph_1$.

A flower F is regular when it enjoys the following :

$\forall \alpha, \beta \forall f \in I(\alpha, \beta)\ \forall z < \alpha\ (F(z) < F(\alpha) \wedge F(f)(F(z)) = F(f(z)))$

In other terms, the functions $T_\alpha(z) = F(z)$ $(T_\alpha \in I(\alpha, F(\alpha)))$ define a natural transformation from μ_F from Id to F.

The most remarkable feature of regular flowers, is the possibility of defining infinitary products : if $(F_i)_{i<\alpha}$ is a family of regular flowers, then one can define a new one $\prod_{i<\alpha} F_i$, which coincides, in case $\alpha = n$, with $F_0 \circ \ldots \circ F_{n-1}$. The product enjoys the expected generalized associativity property. In fact, the generalized product is obtained by a direct limit of finite products.

II.3.5. let us go back to descriptive set theory. It is easily shown that the condition "$F \circ \mathring{\Omega}$, $G \circ \mathring{\Omega}$ comparable w.r.t. $\sqsubseteq$ for all denumerable F and G", follows from the special case

$$F \circ \mathring{\Omega} = \mathring{\Omega} \quad \text{for any denumerable regular flower F.}$$

This suggests the following definition :

II.3.6. axiom of equalization

If F and G are two regular flowers, then one can find a regular flower H such that $\qquad F \circ H = G \circ H$.

II.3.7. From the axiom of equalization, one can prove the following : if X is any family of regular dilators, then one can find a regular dilator H such that

$\forall F, F' \in X \quad F \circ H = F' \circ H$

Furthermore, if there is such an equalizer for the family X, there is a smallest one, smallest in the sense of a category-theoretic universal problem that we shall not explain here.

II.3.8. In particular, the smallest equalizer $\overset{+}{\Omega}$ of the set X of all regular denumerable flowers is such that

$$F \circ \overset{+}{\Omega} = \overset{+}{\Omega} \quad \text{for all } F \in X.$$

II.3.9. There are deep connections between "sharps" and the axiom of equalization. In fact, the axiom of equalization (for denumerable regular flowers) is equivalent to the determinacy of $\underline{\Pi}^1_1$ games. Another interesting equivalent to $\underline{\Pi}^1_1$-determinacy is the intuitionistic analogue of ß-completeness.

II.4. dilators and combinatorics

II.4.1. It is easy to define the sum $\sum_{i<\alpha} F_i$ of any family of dilators. The first decomposition theorem states that a dilator can be uniquely written as a sum $\sum_{i<\alpha} F_i$, where the F_i's are <u>connected</u>, i.e. are non zero, and cannot in turn be written as sums. Typically the ordering $F \sqsubseteq G$ means that, with $F = \sum_{i<\alpha} F_i$, $G = \sum_{j<\beta} G_j$: $\alpha \leq \beta$ and $F_i = G_i$ for all $i < \alpha$.

II.4.2. A connected dilator may be equal to the constant $\underline{1}$; it may also be something like Id, that cannot be broken into a sum. Dilators are classified into four kinds :

F is of kind	0	when $F = \underline{0}$
F is of kind	1	when F is of the form $G + \underline{1}$
F is of kind	ω	when F has a first decomposition $\sum_{i<\alpha} F_i$, with α limit.
F is of kind	Ω	when $F = F' + F''$, F'' connected and $F'' \neq \underline{1}$.

The classification is linked to the values taken by F on the ordinal class On : F(On) can be defined by direct limit and is a class

F is of kind	iff	F(On) is
0		0
1		successor
ω		limit of small cofinality
Ω		limit of cofinality On

For instance, (Id + Id)(On) = On + On, hence Id + Id is of kind .

II.4.3. The second decomposition theorem breaks a dilator F of kind Ω into a family (F_α) of dilators such that :

i) $F = F_0 + G$ for some connected G

ii) $\alpha \leq \beta \rightarrow F_\alpha \sqsubseteq F_\beta$

iii) $F(On) = \sup_{\alpha \in On} (F_\alpha(On))$

The theorem is obtained by means of a sophisticated analysis of the behaviour of normal forms. For instance, if F = Id + Id, then $F_\alpha = Id + \alpha$.

II.4.4. If we define the strict order $\ll$ between dilators by :

i) $F \ll F + G$ when $G \neq \underline{0}$

ii) if F is of kind Ω and (F_α) is the second decomposition of F, if $\alpha_0 \in On$, then $F_{\alpha_0} \ll F$.

iii) $F \ll G$ and $G \ll H \longrightarrow F \ll H$.

Then one easily shows that :

i) the predecessors (w.r.t. $\ll$) of F are linearily ordered by $\ll$.

ii) given any $A < F(On)$, there is a unique $F' \ll F$ such that $F'(On) = A$.

This shows that we have succeeded, through the two decomposition theorems, in breaking dilators completely. The relation is well-founded (but a given point has a proper class of predecessors) and this can be used to prove arguments, or to construct functions (or functors) by induction on dilators.

II.4.5. The first use of induction on dilators has been to relate the two familiar hierarchies of number-theoretic functions :

$$g_0(n) = 0 \qquad h_0(n) = n$$
$$g_{\alpha+1}(n) = g_\alpha(n)+1 \qquad h_{\alpha+1}(n) = h_\alpha(n+1)$$
$$g_\lambda(n) = g_{[\lambda]n}(n) \qquad h_\lambda(n) = h_{[\lambda]n}(n) \qquad (\lambda \text{ limit})$$

In the limit case, one uses so-called "fundamental sequences" $[\lambda]$ n. The g-hierarchy is extremely slow, compared with h ; but g is theoretically more interesting, since in some sense, it counts the number of steps of the computation of the function.

The result proved by the author in 1976 is that

$$g_{\epsilon_0} = h_{\eta_0}$$

ϵ_0 being the familiar ordinal of arithmetic, the smallest solution of $\omega^\alpha = \alpha$; η_0 is the "Howard ordinal", traditionally attached to

II.4.6. more recently, this approach has been used in relation to the Paris-Goodstein-Kirby theorem. We assume that the reader is familiar with this nice combinatoric result. One of the key notions here is to pass from "pure base n" to pure base n+1 ; this change is easily seen as the action of a dilator, namely the function $D(E_{n-1n})$, where D is an ω-iteration of the exponential

$$D = (1+Id)^{(1+Id)^{(1+Id)^{(1+Id)^{\cdots}}}}$$

Using the decompositions of dilators, it is possible to give exact computations for the number of steps involved in the Goodstein process. This yields a new proof of the result, but also suggests (since we are free to choose arbitrary dilators) other results on the smae line of thought.

III APPLICATIONS TO COMPUTER SCIENCE

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III.1. The first attempts have been to translate more or less directly the theory of dilators into the theory of programming languages. Several simplifications were found afterwards, and lead to developments with no direct connection with the starting point, except a similarity of spirit and techniques. It is however expected that dilators will be really needed at some ulterior stage.

III.2. Coherent spaces

III.2.1. A coherent space is a set X enjoying the following properties :

i) $a \in X \wedge b \subset a \longrightarrow b \in X$

ii) if $A \subset X$ is such that : $\forall a,b \in A \quad a \cup b \in X$, then $\bigcup A \in X$.

III.2.2. The web W(X) of a coherent space consists in the following data :

$|W(X)| = \{x ; \{x\} \in X\}$

The binary relation $\frown\!\!\!\!\smile$ (mod W(X)) defined by

$$x \frown\!\!\!\!\smile y \ (\text{mod } W(X)) \text{ iff } \{x,y\} \in X.$$

Notation : $x \frown y$ for $x \frown\!\!\!\!\smile y$ and $x \neq y$, $x \asymp y$ for $\neg(x \frown y)$, $x \smile y$ for $\neg(x \frown\!\!\!\!\smile y)$.

III.2.3. The map $X \rightsquigarrow W(X)$ is a bijection between coherent spaces and reflexive-symmetric relations (unoriented graphs). The converse map is obtained from the formula :

$$a \in X \quad \text{iff} \quad \forall x,y \in a \quad x \frown\!\!\!\!\smile y \ (\text{mod } W(X)).$$

$\frown\!\!\!\!\smile$ is called coherence (modulo W(X)).

We distinguish as little as possible between X and W(X), and for instance we shall use |X| for |W(X)| and (mod X) for (mod W(X)).

III.3. the function space

III.3.1. A function F from a coherent space X to a coherent space Y is said to be stable when it enjoys the following properties :

(S 1) $a \subset b \in X \longrightarrow F(a) \subset F(b)$

(S 2) If $(a_i)_{i \in I}$ is a directed subset of X (w.r.t. inclusion), then

$$F(\bigcup a_i) = \bigcup F(a_i)$$

(S 3) if a and b are coherent, i.e. if $a \cup b \in X$, then

$$F(a \cap b) = F(a) \cap F(b)$$

III.3.2. normal form theorem

If F is a stable map from X to Y, if $a \in X$ and $z \in F(a)$, then one can find $a' \subset a$, a' finite, such that $z \in F(a')$; furthermore, if a' is chosen minimal with this property, then it is unique.

III.3.3. The proof of this property is very easy : the existence of the finite a' comes from (S 2) (because a is the directed union of its finite subsets), whereas the unicity comes from (S 3). There is more than a superficial analogy with the normal form theorem for dilators :

i) a coherent space can be seen as a (trivial) category, whose objects are the elements of X, the morphisms between a and $b \in X$ existing only when $a \subset b$; in that case there is a unique morphism e_{ab} from a to b.

ii) if one translates the words "direct limit" in this trivial category, then one gets "directed union" ; if one translates "pull-back", then one gets "intersection of coherent elements". In particular, if one translates "functor preserving direct limits and pull-backs", one gets (S 1) (functor) (S 2) (direct limits) (S 3) (pull-backs) !

III.3.4. If F is a stable map from X to Y, then the trace of F is defined by :

$$Tr(F) = \{(a,z) ; z \in F(a) \quad \forall a' \subset a (z \in F(a') \rightarrow a' = a)\}$$

Tr(F) determines F via the simple formula :

$$F(b) = \{z ; \exists a \subset b \quad (a,z) \in Tr(F)\}$$

III.3.5. The coherent space $X \Rightarrow Y$ is defined by means of its web :

$|X \Rightarrow Y| = X_{fin} \times |Y|$ (X_{fin} = finite objects of X)

$(a,y) \supset\!\subset (b,z)$ (mod $X \Rightarrow Y$) iff i) and ii) :

i) $a \cup b \in X \rightarrow y \supset\!\subset z$ (mod Y)

ii) $a \cup b \in X$ and $a \neq b \rightarrow y \frown z$ (mod Y)

III.3.6. It is easy to check that the sets of the form Tr(F) for some stable F from X to Y can be caracterized as the elements of the coherent space $X \Rightarrow Y$, i.e. as the coherent subsets of $|X \Rightarrow Y|$. In particular, the function space is interpreted as a coherent space.

It may be of some interest to see to what relation between functions correspond the inclusion of traces ; define $F \subset G$ by $Tr(F) \subset Tr(G)$; then $F \subset G$ iff $\forall a,b \in X$ ($a \subset b \longrightarrow F(a) = F(b) \cap G(a)$) ; this very nice and natural order is called the Berry order, from Gérard Berry who first found it in the context of Scott domains.

III.3.7. Compared with extant materials on the subject (so called "Scott domains"), this approach is extremely simple ; in fact Scott ignores (the analogue of) condition (S 3) ; worse, he wants, for obscure reasons, to use the ordering

$$F \subseteq_s G \text{ iff } F(a) \subset G(a) \text{ for all } a \in X.$$

This ordering (whose only advantage is the simplicity of formulation) has no nice property w.r.t. normal forms (or any kind of minimal representation), and forces to represent F by the set of all pairs (a,z) such that $z \in F(a)$ and a is finite. But the set of such representations is very complicated, and this forces to complicate very much the original pattern, leading to so called "information systems". In particular, even if it is theoretically possible to modelize programming languages within Scott's tradition, this is practically unfeasible, because of the heavy apparatus involved in the function space.

III.4. denotational semantics

III.4.1. if we take a functional language (e.g. a variant of λ-calculus) we mean by a denotational semantics an interpretation of the result of computations, not at all an interpretation of the computations themselves ; this is a static conception of semantics, that should be opposed with the dynamic one, namely an operational semantics that would modelize also the execution of programs. Up to now there is no convincing operational semantics for programming languages, even if there is now some reason to hope for the creation of such things in a near future. The traditional denotational semantics is the one due to Scott, which is unfortunately too complicated, essentially due to a superficial treatment of the function space.

III.4.2. For typed λ-calculi, using the arrow as basic type, there is a very simple coherent semantics :

i) types are interpreted as coherent spaces ; in particular, if S and T have been interpreted by X and Y respectively, then $S \Rightarrow T$ will be interpreted by $X \Rightarrow Y$.

ii) if T is interpreted by X, then an object of type T will be interpreted by an element of X ; in particular, an object depending on a variable will depend on the interpretation of the variable, and will lead to a stable function. The interpretation of the λ-abstraction will be done by means of the trace of the function ; conversely

the interpretation of application will be done in the spirit of

III.3.4. : $Aa = \{z;\ \exists b \subset a\ \ (b,z) \in A\}$

This interpretation will be compatible with η- and β- conversions, i.e. will not vary through the normalization process.

III.4.3. This basic idea can be used to replace everywhere Scott's semantics by the simpler coherent spaces, in particular for pure λ-calculus.

III.4.4. In the case of the author's system F (also called second order λ-calculus), the use of Scott domains never answered the question of defining functions which could be applied to arbitrary types, like for instance the universal identity

$$\lambda\alpha.\lambda x^{\alpha}.x^{\alpha}$$

which, when applied to a type T, yields the identity of T, $\lambda x^T.x^T$. The difficulty is that the term can be applied to any type, including its own type, and this yields some (at least apparent) circularity.

The idea will be to define objects of variable type on a small class of finite types (here finite coherent spaces), then to extend the definition to arbitrary arguments by taking direct limits. The idea is reminiscent of the definition of a dilator from its restriction to $ON < \omega$. The details, which are quite difficult, will not be explained here. Let us only mention that we need to be able to approximate an arbitrary coherent space (w.r.t. direct limits) by means of finite ones. This technical need was the reason for the abandon of Scott's semantics, which has a bad behaviour w.r.t. finitism.

III.4.5. One of the interesting outputs of the interpretation of F is the intrinsic model for λ-calculus : when we use coherent spaces, we are faced with the same problem as with Scott's interpretation, namely we have to solve an equation

$$X \underset{K}{\overset{H}{\rightleftarrows}} X \Rightarrow X \qquad (1)$$

where H and K are reciprocal isomorphisms. Now, all solutions to this question are rather arbitrary, even if some of them are quite simple. Now, given X,H,K, we can define the interpretation $t^*_{X,H,K}$ of any λ-term t, as an element of the coherent space X. The idea is to see $t^*_{X,H,K}$ as a functor of the imputs X,H,K and, by techniques of normal form (similar to those used for universal types), to obtain an interpretation t*, independant of any choice of a solution in (1). Moreover, from t*, which should be the trace of a functor, one can recover all $t^*_{X,H,K}$ by a simple formula, i.e. t* should encode all possible interpretations of t by means of coherent spaces. There is just a little problem : we cannot approximate a solution of (1) by means of finite solutions of (1). Hence we have to give up the condition "H and K reciprocal isomorphisms" in favor of the more liberal requirement $K \circ H \subset Id^X$ and $H \circ K \subset Id^{X \Rightarrow X}$. Once this modification is done, the modelization proceeds without any problem, and yields a very compact modelization. There is an unexpected feature : when $t \neq u$, then $t^* \not\subset u^*$, i.e. the intrinsic model is also operational to some extent.

III.5. linear logic

III.5.1. The interpretation of the function space by means of coherent spaces is so simple that something completely unexpected happens : <u>the connective of implication is broken into more primitive ones</u>.

Define, when X and Y are coherent spaces, a coherent space X -o Y by :

$|X \multimap Y| = |X| \times |Y|$

$(x,y) \coh (x',y')$ (mod X -o Y) iff i) and ii)

i) $x \coh x'$ (mod X) $\rightarrow$ $y \coh y'$ (mod Y)

ii) $x \frown x'$ (mod X) $\rightarrow$ $y \frown y'$ (mod Y)

The operation -o is called "<u>linear implication</u>".

Define, when X is a coherent space, the coherent space !X by

$|!X| = X_{fin}$

$a \coh b$ (mod !X) iff $a \cup b \in X$

The modality ! is read "of course".

Now, if we compare these definitions with the definition of $X \Rightarrow Y$, we immediately remark that $X \Rightarrow Y$ is equal to (!X) -o Y.

III.5.2. In fact, there is more than that : the usual implication is decomposed, not into arbitrary algebraic operations, but into <u>logical</u> operations, i.e. -o and ! should be seen as connectives of a new logic, linear logic. The subject is too big to be explained here, even quickly ; let us only say that it is a drastic reformulation of constructivism, pushing the work begun with intuitionism to its implicit consequences. Many prejudices concerning constructivism are completely destroyed by this new approach ; just think that linear logic is built on a constructive and involutive negation! The properties of this logic make us hope that it will be possible to give a semantics of proofs even for classical logic, by an appropriate translation inside linear logic !

III.5.3. Linear logic is developed in a classical framework, as a multiple conclusion logic. This involves a new kind of parallel syntax. Logically speaking, linear logic could be the tool that is needed in order to handle parallelism at a logical level.