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ON THE LOGIC OF SENSES
AN ANOMALOUS USE OF BELIEF SENTENCES
ITS RIGOROUS AND FORMAL TREATMENT

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N1. Introduction.

This lecture was concerned with the theory of sense logic developed in the work [2] on synonymy for extensional languages, in its extension [1] to the modal calculus MC^{ν} presented in [4], in the memoir [3] on a generalized synonymy notion and quasi senses (substantially) for MC^{ν}, and especially in [5] where a general interpreted language $\mathcal{SL}^{\varphi\nu}_{\alpha}$ is introduced in order to treat e.g. iterated belief sentences, whose iteration orders may be transfinite, but smaller than the ordinal α. Furthermore $\mathcal{SL}^{\varphi\nu}_{\alpha}$ contains descriptions, modal operators, non-logical operators, and wfes - i.e. well formed expressions - having both types of all finite levels and (sense) orders represented by all ordinals $< \alpha$.(1)

In [6] an axiom system valid in $\mathcal{SL}^{\nu}_{\alpha}$, and hence a logical calculus, say $\mathcal{SC}^{\nu}_{\alpha}$, is considered.

The main aim with which $\mathcal{SL}^{\varphi\nu}_{\alpha}$ was constructed was to reach a strong expressive power. On the other hand, in both [5] and [6] it is written not to claim that any completeness theorem should hold for $\mathcal{SC}^{\nu}_{\alpha}$. It can be added that certain axioms valid in ordinary extensionl calculi and e.g. in the modal claculus MC^{ν}, hold in $\mathcal{SC}^{\nu}_{\alpha}$ only in case the sense orders of certain designators occurring in them are equal or satisfy certain simple conditions. In some cases this is natural and also compulsory; in others these restrictions can be justified on the basis of the ontology underlying the semantics of $\mathcal{SL}^{\nu}_{\alpha}$, but they seem to be avoidable by means of suitable changes in this semantics (and ontology). Therefore it is natural to consider such changes in order to improve the general theory presented in [5] and [6].

Since the afore-mentioned problem concerns (sense) orders, in order to concentrate on it better, it is convenient to consider the extensional part of $\mathcal{SL}^{\nu}_{\alpha}$, deprived of non-logical operators. Let us call $\mathcal{L}^{\nu}_{\alpha}$ the sense language thus obtained, or more briefly, $\mathcal{L}_{\alpha}$.

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The first aim of the lecture was to present two successive changes in $\mathcal{L}_{\alpha}$'s semantics, which turn $\mathcal{L}_{\alpha}$ into other two interpreted languages: $\mathcal{L}_{\alpha}'$ and $\mathcal{L}_{\alpha}''$. Thus the hyper-intensional axiom

$$(1.1)\quad f=g \equiv (\forall x_1,\ldots,x_n).f(x_1,\ldots,x_n)=g(x_1,\ldots,x_n)$$
$$(\Vdash p=q \equiv_{\bullet} p \equiv q)$$

where f and g are functors or relators and x_1 to x_n have the larger of the orders of f and g, is successively improved.(2) In more detail, both the $\supset$-part of (1.1) and its converse can be asserted for $\mathcal{L}_{\alpha}$ only in case f and g have the same (sense) order. This continues to be true for $\mathcal{L}_{\alpha}'$ as far as that converse is concerned; but the $\supset$-part of (1.1) is valid in $\mathcal{L}_{\alpha}'$ no matter which orders ($<\alpha$) f and g have. Furthermore the whole wff (1.1) has this validity for $\mathcal{L}_{\alpha}''$.

At the lecture the last assertion above was considered only disregarding descriptions and λ-operators; its complete proof was performed only later. Therefore the whole subject mentioned above is planned to be published elsewhere.

The second aim of the lecture was to show, so to say, an anomalous use of belief sentences made in the ordinary language – see N2 – and a rigorous treatment of it based on e.g. $\mathcal{L}_{\alpha}$. The procedure used to reach this goal can easily be applied to most hyper- intensional languages.

The afore-mentioned treatment — see N3 — is based on and

practically consists in a certain extension $\mathcal{L}\to\mathcal{L}^{\mathcal{O}}$ of a widely arbitray formal language $\mathcal{L}$ capable to deal with belief sentences. One can identify $\mathcal{L}$ with e.g. $\mathcal{L}_{\alpha}$, $\mathcal{L}_{\alpha}''$, or $\mathcal{SL}^{\nu}_{\alpha}$. Since so far only $\mathcal{SL}^{\nu}_{\alpha}$ is available in publications, in N4 the extension $\mathcal{L}\to\mathcal{L}^{\mathcal{O}}$ is rendered explicit for $\mathcal{L}=\mathcal{SL}^{\nu}_{\alpha}$: the semantics for $\mathcal{L}^{\mathcal{O}}=\mathcal{SL}^{\mathcal{O}}_{\alpha}$ ($=_D$ $\mathcal{SL}^{\nu\mathcal{O}}_{\alpha}$) is given there directly in rather full detail, by presupposing only NN2-4 and some conventions in [5]. E.g., by the formation rules (φ_{1-10}) for $\mathcal{SL}^{\nu}_{\alpha}$ written there and the operations (1) and (2) in N3, the formation rules ($\varphi^{\mathcal{O}}_{1-10}$) for $\mathcal{SL}^{\mathcal{O}}_{\alpha}$ are completely clear.

The semantics for $\mathcal{SL}^{\nu}_{\alpha}$ – see N5 in [5] – is based on a uniqueness theorem – see Theor.6.1 in [5] – which has not been explicitely proved so far. Therefore its extension for $\mathcal{SL}^{\mathcal{O}}_{\alpha}$ is included in Theor.6.1 proved here. As preliminaries, in N5 some semantical theorems are considered. In particular Theor.5.2 is proved, where some equalities and strict inclusions among the sets QE^{β}_{t}, QI^{β}_{t}, and QS^{β}_{t} ($t\in\mathcal{T}_{\nu}$, $\beta<\alpha$) for $\mathcal{SL}^{\mathcal{O}}_{\alpha}$ are asserted. Its restriction to $\mathcal{SL}^{\nu}_{\alpha}$ (strictly) includes the relations of the above kind asserted in [5] without proof.

In N7 $\mathcal{SL}^{\mathcal{O}}_{\alpha}$ is shown to be richer than $\mathcal{SL}^{\nu}_{\alpha}$ in QSs of order β for $0\leq\beta<\alpha$, and in QEs and QIs of order β for $0<\beta<\alpha$.

Since the language $\mathcal{SL}^{\nu}_{\alpha}$, presented in [5], is referred to here, a brief errata corrige is written for it here, as well as in [6], in the Appendix. Of course the corrected designation rules for $\mathcal{SL}^{\nu}_{\alpha}$ are also included in the designation rules for $\mathcal{SL}^{\mathcal{O}}_{\alpha}$ written here in N4.

N2 Some ambiguous and anomalous uses of belief sentences.

The example presented below, quite possible in every-day life, shows an anomalous use made with belief assertions of any natural language; and thus the example also contributes to show the variety of ambiguous uses made with such assertions.

Assume that

(a) C. Rossi is charged with murder,

(b) Mr. T will witness, knows that C. Rossi is guiltless, and is honest,

(c) Mr. T. ignores that C. Rossi is Pete's father,

(d) Pete's neighbours know (a) to (c), and

(e) Pete asks his neighbours about his father's situation.

By (a) to (e), it is natural for Pete's neighbours to answer as follows: your father's situation will improve, because Mr. T will witness, he is honest, and

(f) he knows that your father is guiltless.

In speaking with Pete it would be unnatural to refer to his father as "C. Rossi". Therefore Pete's neighbours assert the pragmatic sentence (f), whose descriptive counterpart is

(g) Mr. T knows that Pete's father is guiltless.

Note that, under a normal (usual) reading, assertion (g) - as well as (f) - is false by (c), and Pete's neighbours are aware of this by (d).

The use of (f) made in the example above, shows that (f), as well as (g), is ambiguous. The most interesting feature of this ambiguity is that the reading of (f) made within the example - hence the one of (g) - is, so to say, anomalous and it does not comply with the semantics of usual formal theories of belief sentences (or those of $\mathcal{SL}_\alpha^\nu$ or $\mathcal{L}_\alpha$). This happens in spite of (g) being a simple (non-iterated) belief sentence, to which (in connection with its usual reading) also Carnap's theory written in [7] can be applied.

In fact (g) is substantially meant in the example in accordance with the following anomalous evaluation of the sense of the assertion

Pete's father is guiltless;

one regards "Pete's father" as synonymous with "C. Rossi", i.e. one attributes it an ostensive (technical) sense: C. Rossi's extension; then the usual rules are applied. Incidentally, when ostensive senses are identified with the corresponding extensions (or intensions in modal languages), which is technically useful, one ought to speak of <u>(technical) senses</u>.

<u>N3 Rigorous treatment of ambiguities and anomalies of the preceding kinds.</u>

Let us consider e.g. the interpreted modal sense language $\mathcal{SL}_\alpha^\nu$ - see [5] and N4 -. It has the formation rules (φ_{1-10}) and rules (h_{1-10}) [(ε_{1-10})] which assign every wfe Δ of $\mathcal{SL}_\alpha^\nu$ a quasi-intension $\tilde{\Delta}$ = $des_{\mathfrak{J}\mathfrak{V}}(\Delta)$ [a quasi-sense $\check{\Delta}$ = $sens_{\mathfrak{J}\mathfrak{V}}(\Delta)$] at every c-valuation $\mathfrak{J}$ and v-valuation $\mathfrak{V}$, i.e. at all assignments of (admissible) values to variables and constants - see [5] pp. 438, 451, and 452. In order to turn $\mathcal{SL}_\alpha^\nu$ into an interpreted language $\mathcal{SL}_\alpha^{\nu\mathcal{O}}$, briefly $\mathcal{SL}_\alpha^{\mathcal{O}}$, capable to deal rigorously with ambiguities and anomalies such as those considered in N2, the following four operations suffice:

(1) Add a new symbol, say $\mathcal{O}$,

(2) Duplicate rules (φ_{1-10}) in [5] p.438, by turning them into the formation rules ($\varphi^{\mathcal{O}}_{1-10}$) below respectively;

($\varphi^{\mathcal{O}}_1$) rule (φ_1) holds and[3] $c_t\mathcal{O}$, $v_{tn}\mathcal{O} \in E_t$ ($\beta<\alpha$, $t \in \tau_\nu$, $0<\mu<\alpha+\omega_0$, or $0<\mu<\beta+\omega_0$ if preferred, $n \in N_* =_D N-\{0\}$);[4]

($\varphi_h^{\mathcal{O}}$) the antecedent ($\varphi_h)_A$ of rule (φ_h) implies that Δ^h, $\Delta^h\mathcal{O} \in E_{t^h}$ (h=2,...10), where Δ^h is (the wfe arising from rule (φ_h), i.e.) $\Delta(\Delta_1,\dots,\Delta_n)$, $(\Omega x_1,\dots,x_n)\Delta$, $\sim p$, $p \supset q$, $\Box p$, $(\forall v^\beta_{tn})p$, $(\mathrm{1} v^\beta_{tn})p$, $\Delta_1 = \Delta_2$, or $(\lambda^p x_1,\dots,x_n)\Delta$, while t^h is t, ε , 0, 0, 0, 0, t, 0, or $\langle t_1,\dots,t_n,t_0\rangle$ for h=2 to 10 respectively.

(3) For r=1,...,8 require every wfe $A_{(r)}$ that arises by means of rule ($\varphi^{\mathcal{O}}_r$) and fails to end by $\mathcal{O}$, to satisfy the old designation rule (h_r); furthermore endow $A_{(r)}\mathcal{O}$ with the same quasi-extensional designatum as $A_{(r)}$:

(3.1) $des_{\mathfrak{J}\mathfrak{V}}(A_{(r)}\mathcal{O}) = des_{\mathfrak{J}\mathfrak{V}}(A_{(r)})$.

(4) Lastly, for r=1,...,8 require the arbitrary wfe $A_{(r)}$ above to satisfy the old rule (ε_r) for QS- designation; furthermore endow $\Delta =_D A_{(r)}\mathcal{O}$ with an ostensive quasi-sense as follows ;

(3.2) $\check{\Delta} = sens_{\mathfrak{J}\mathfrak{V}}(\Delta) = des_{\mathfrak{J}\mathfrak{V}}(A_{(r)})$ ($\Delta =_D A_{(r)}\mathcal{O}$).

Incidentally, on the basis of operation (4) $A_{(r)}\mathcal{O}$ can be read as $A_{(r)}$ <u>meant in an ostensive way (or sense)</u>.

It is obvious that the interpreted language $\mathcal{SL}_\alpha^{\mathcal{O}}$ reaches the aim by which it has been constructed.

Furthermore, given any sense language $\mathcal{L}$ (modal or extensional) it is rather obvious how to construct $\mathcal{L}^{\Theta}$ by means of the analogues for $\mathcal{L}$ of steps (1) to (4). In fact (1) simply introduces "Θ" and (2) to (4) duplicate in a simple way the rules of formation and the designation rules for quasi-intensions (or quasi extensions) and quasi-senses.

By regarding $\mathcal{L}$ to contain a suitable part of ordinary English, assertion (g) can obviously be translated into $\mathcal{L}^{\Theta}$ by

(g^{Θ}) Mr. T knows that (Pete's father)Θ is guiltless.

<u>N4 An explicit presentation of the rules of QI- and QS-designation for $\mathcal{ML}^{\Theta}$.</u>

Here NN2-4 and conventions 5.2-2 in [5] are presupposed - see fnt.4. The HQEs (hyper-quasi-extensions) and HQIs (hyper-quasi-intensions) for $\mathcal{ML}^{\Theta}_{\alpha}$ based on the proper individual domains $\mathcal{D}_1$ to $\mathcal{D}_\nu$ and the set Γ of possible cases - see (4.2) in [5] - will turn out to be more than those for $\mathcal{ML}^{\nu}_{\alpha}$ - see N7; hence the same occurs with the v- and c-valuations.

By rules (φ_{1-8}) in [4], p.438, and steps (1) and (2) in N3, E_t denotes here the class of the wfes in $\mathcal{ML}^{\Theta}_{\alpha}$ that have the type t ($\in \tau_\nu$). Similarly, by QI^{β}_t, QE^{β}_t, and QS^{β}_t we denote here the classes of the QIs, QEs, and QSs respectively for $\mathcal{ML}^{\Theta}_{\alpha}$, of orders $\leq \beta$ ($<\alpha$) and type t ($\in \tau_\nu$). The definitions $(2.4)_{1-2}$ in [5] of E^{β}_t and wfe^{β} are here still in force, but they refer to $\mathcal{ML}^{\Theta}_{\alpha}$.

Let $\mathcal{ML}^{\Theta}_{\lambda}$ be the λ-th segment of $\mathcal{ML}^{\Theta}_{\alpha}$, i.e. the language whose wfes are the $wfe^{<\lambda}$s, i.e. the wfes of $\mathcal{ML}^{\Theta}_{\alpha}$ whose orders are $<\lambda$. More in particular the semantics for $\mathcal{ML}^{\Theta}_{\lambda}$ will be determined by regarding $\mathcal{ML}^{\Theta}_{\lambda}$ as a theory belonging to $\mathcal{ML}_{\lambda}$, all of whose constants are primitive — see [5], pp.444-445 where $\mathcal{ML}^{\nu}_{\alpha}$ is referred to, and consider the analogue for $\mathcal{ML}^{\Theta}_{\alpha}$.

In order to determine the semantics for $\mathcal{ML}^{\Theta}_{\alpha}$ based on the sets $\mathcal{D}_1$ to $\mathcal{D}_\nu$ and Γ - see (4.2) in [5] - we want to define, for $0 \leq \beta < \lambda < \alpha$:

(1) QE^{β}_t, QI^{β}_t, and the class A^{β}_t of the entities that can be assigned to the variables and the (primitive) constants of type t ($\in \tau_\nu$) and orders $\leq \beta$,

(2) the class V^{λ} [I^{λ}] of v-valuations [c-valuations] for $\mathcal{ML}^{\Theta}_{\lambda}$,

(3) the QI $\tilde{\Delta} = des_{\mathfrak{J}\mathcal{V}}(\Delta)$ (or $des^{\lambda}_{\mathfrak{J}\mathcal{V}}(\Delta)$) designated by any $\Delta \in E^{<\lambda}_t$ - see (2.3) in [5] - at any $\mathfrak{J} \in I^{\lambda}$ and $\mathcal{V} \in V^{\lambda}$ ($t \in \tau_\nu$),

(4) the QS $\check{\Delta} = sens_{\mathfrak{J}\mathcal{V}}(\Delta)$ ($= sens^{\lambda}_{\mathfrak{J}\mathcal{V}}(\Delta)$) designated by any $\Delta \in E^{<\lambda}_t$ at any $\mathfrak{J} \in I^{\lambda}$ and $\mathcal{V} \in V^{\lambda}$ ($t \in \tau_\nu$),

(5) QS^{β}_t ($t \in \tau_\nu$), and

(6) the intension σ^{I} (or $I_{\lambda}(\sigma)$) of any $wfe^{<\lambda}$ Δ, whose QS is σ.

We do this by simultaneous transfinite induction on λ ($0 < \lambda \leq \alpha$) and β ($<\lambda$). More precisely we can consider separately the cases where λ is a limit ordinal, and the remaining case. Here the former is considered first, simply because it can be treated wery briefly. In fact, in it the objects (1) and (5) are already known for $\beta < \lambda$. We can introduce V^{λ} [I^{λ}] as the set of the functions $\mathcal{V}$ [$\mathfrak{J}$] defined (only) on the variables [constants] of orders $<\lambda$, whose restrictions $\mathcal{V}^{\prime\delta}$ [$\mathfrak{J}^{\prime\delta}$] to those of orders $<\delta$ are in V^{δ} [I^{δ}] for $0 < \delta < \lambda$. Then, for any wfe^{β} Δ with $\beta < \lambda$ and for any $\mathfrak{J} \in I^{\lambda}$ and $\mathcal{V} \in V^{\lambda}$, we can set, e.g. for $\delta = \beta + 1$:

$$(4.1)\quad des^{\lambda}_{\mathfrak{J}\mathcal{V}}(\Delta) = des^{\delta}_{\mathfrak{J}^{\prime\delta}\mathcal{V}^{\prime\delta}}(\Delta),\ sens^{\lambda}_{\mathfrak{J}\mathcal{V}}(\Delta) = sens^{\delta}_{\mathfrak{J}^{\prime\delta}\mathcal{V}^{\prime\delta}}(\Delta).$$

Then $(4.1)_{1-2}$ hold for any δ with $\beta < \delta < \lambda$.

Lastly the determination of the function $\sigma \mapsto \sigma^{I}$ for $\mathcal{ML}^{\Theta}_{\lambda}$, i.e. I_{λ}, can be defined as the union of its determinations for $\mathcal{ML}^{\Theta}_{\delta}$ ($\delta < \lambda$): $I_{\lambda} = \bigcup_{\beta < \lambda} I_{\beta}$.

Thus all objects (1) to (6) can be determined for $\mathcal{ML}^{\Theta}_{\lambda}$ when λ is a limit ordinal.

Now assume that λ is the successor of an ordinal. Then we can simply assume $\lambda = \beta + 1$; furthermore the objects (1) are determined recursively, for $t \in \tau_\nu$, by the initial clause

(4.2) $QE^{\beta}_t = D_t$ ($t = 0, \ldots, \nu$) - see (4.3) in [5]

and the recursive clauses - see (4.1) in [5]

(4.3) $QE^{\beta}_{\langle t_1,\dots,t_n,t_0\rangle} = (A^{\beta}_{t_1}\times\dots\times A^{\beta}_{t_n} \hookrightarrow \mathfrak{QE}^{\beta}_{t})\cup\{F\}$,

(4.4) $QI^{\beta}_t = (\Gamma \rightarrowtail QE^{\beta}_t)$,

and

(4.5) $A^{\beta}_t = QI^{\beta}_t \cup QS^{<\beta}_t$ (hence $A^{0}_t = QI^{0}_t$),

where

(4.6) $\mathfrak{QE}^{\beta}_t = QE^{\beta}_t - \{F\}$.

Thus we can introduce V^{λ} [I^{λ}] as the set of the functions $\mathcal{V}$ [$\mathfrak{J}$] defined (only) on the variables [constants] of $\mathcal{SL}^{\omega}_{\alpha}$ for which – see fnt.4 –

(4.7) $\mathcal{V}(v^{\delta}_{tn}) \in A^{\delta}_t$, $(c^{\delta}_{t\mu}) \in A^{\delta}_t$ ($\delta<\lambda$, $t\in\tau_\nu$, $n\in N_*$, and either $0<\mu<\lambda+\omega_0$, or $0<\mu<\alpha+\omega_0$).

In order to determine the object (3), let us first accept the determinations $(4.1)_{1-2}$ for any wfe$^{<\beta}$ Δ and any $\mathfrak{J}\in I^{\lambda}$ and $\mathcal{V}\in V^{\lambda}$. Then let N_{Δ} be the number of occurrences of (logical or non-logical) operators in the arbitrary wfe Δ of $\mathcal{SL}^{\omega}_{\alpha}$. Now we can define $\tilde\Delta = des_{\mathfrak{J}\mathcal{V}}(\Delta)$ for $\Delta\in E^{\beta *}_t$, $\mathfrak{J}\in I^{\lambda}$, and $\mathcal{V}\in V^{\lambda}$, by induction on N_{Δ} ($\in \mathbf{N}$) and recursion on t ($\in\tau_\nu$), by means of rules (h_{1-10}) below, regarded to hold for all entities that satisfy assumptions (i) to (v) below.

(i) Δ , Δ_0 to Δ_n, Δ', and $\mathcal{R}$ are wfes having the respective orders β, δ_0 to δ_n, δ', and $\delta_{\mathcal{R}}$, and the respective types t, t_0 to t_n, t', and t , where $t_0 = \langle t_1,\dots,t_n,t\rangle$, $t_{\mathcal{R}} = (t_1,\dots,t_n;t',t)$ – see (2.2) in [5] –, $n\in N_*$, and $0<\mu<\alpha+\omega_0$ (or $<\delta+\omega_0$ if preferred – see ftn.4).

(ii) x_1 to x_n are n varibles and $x_i\in E^{\delta_i *}_t$ (i=1,...,n) – see (2.3) in [5].

(iii) The orders β, $\delta_0,\dots,\delta_n$, δ', and $\delta_{\mathcal{R}}$ are $<\lambda$ while $\mathfrak{J}\in I^{\lambda}$ and $\mathcal{V}\in V^{\lambda}$.

(iv) One uses e.g. $des_{\mathfrak{J}\mathcal{V}\gamma}(\Delta)$ for $[des_{\mathfrak{J}\mathcal{V}}(\Delta)](\gamma)$ and the definitions

(4.8) $$\left[\begin{array}{l} f(\gamma)\\ \\ g\end{array}\right. =_D \left\{\langle\xi_1,\dots,\xi_n,\xi'\rangle \mid \xi' = \left[\begin{array}{l} des_{\mathfrak{J}\mathcal{V}'}(\Delta') \neq F\\ \\ sens_{\mathfrak{J}\mathcal{V}}(\Delta')\end{array}\right. \quad \xi_i\in A_t \quad (i=1,\dots,n)\right\}$$

are accepted for $\gamma\in\Gamma$, where

(4.9) $$\mathcal{V}' = \mathcal{V}\begin{pmatrix}x_1,\dots,x_n\\ \\ \xi_1,\dots,\xi_n\end{pmatrix}$$ – see Convention 5.1 in [5];

hence

(4.10) $$\left[\begin{array}{l} f(\gamma)\\ \\ g\end{array}\right. \in (A^{\delta_1}_{t_1}\times\dots\times A^{\delta_n}_{t_n} \hookrightarrow \left[\begin{array}{l}\mathfrak{QE}^{\varkappa}_{t'}\\ \\ QS^{\delta_g}_{t'}\end{array}\right.) \qquad (\varkappa = d_{f(\gamma)}, \gamma\in\Gamma)$$

where(5), by writing ξ for $\langle\xi_1,\dots,\xi_n\rangle$,

(4.11) $$d_{f(\gamma)} = \sup\{[f(\gamma)](\xi)^{ord} \mid \xi\in\mathcal{D}_{f(\gamma)}\},$$ $$d_f = \sup\{d_{f(\gamma)} \mid \gamma\in\Gamma\},$$

and – see $(6.1)_2$ below –

(4.12) $$\delta_g = \sup\{g(\xi)^{ord} \mid \xi\in\mathcal{D}_g\}, \text{ whence}$$ $$d_{f(\gamma)}\le d_f\le\delta_g<\lambda \quad (\gamma\in\Gamma).$$

(v) One accepts the additional definitions

(4.13) $$\tilde\Delta' = des_{\mathfrak{J}\mathcal{V}}(\Delta'),\quad \tilde{\mathcal{R}} = des_{\mathfrak{J}\mathcal{V}}(\mathcal{R}),$$ $$\tilde\Delta_j = des_{\mathfrak{J}\mathcal{V}}(\Delta_j) \quad (j=0,\dots,\nu),$$

and, for i=1,...,n,

(4.14) $$\hat\Delta_i = \left[\begin{array}{l} sens_{\mathfrak{J}\mathcal{V}}(\Delta_i) \text{ if } (\breve\Delta_i)^{ord} < (\Delta_0)^{ord}\\ \\ des_{\mathfrak{J}\mathcal{V}}(\Delta_i) \text{ otherwise.}\end{array}\right.$$

Rule	If Δ is	then $\hat{\Delta}(\gamma) = des^{\lambda}_{\mathcal{J}\mathcal{V}\gamma}(\Delta) = des^{\lambda}_{\mathcal{J}\mathcal{V}\gamma}(\Delta\mathcal{O})$ is
(h_1)	v^{β}_{tn} $[c^{\beta}_{t\mu}]$,	$\sigma^{I}(\gamma)$, where $\sigma = \mathcal{V}(v^{\beta}_{tn})$ $[\sigma = \mathcal{J}(c^{\beta}_{t\mu})]$, in case $\sigma \in QS^{<\beta}$; σ otherwise – see (4.14) below.
(h_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$[\hat{\Delta}_0(\gamma)](\hat{\Delta}_1, \ldots, \hat{\Delta}_n)^{\dagger}$ – see (5.8) in [5] and (4.14).
(h_3)	$\mathcal{R}(x_1, \ldots, x_n)\Delta'$,	$[\mathcal{R}(\gamma)](\langle \lambda^{\rho}, g \rangle)^{\dagger}$ if $\delta_g < \mathcal{R}^{ord}$; $[\tilde{\mathcal{R}}(\gamma)](f)^{\dagger}$ otherwise – see (5,8) in [5], (4.8), and (4.12). (6)
(h_4)	$\sim \Delta_1$ $(t_1 = 0)$,	F [T] if $\hat{\Delta}_1(\gamma)$ is T [F].
(h_5)	$\Delta_1 \supset \Delta_2$ $(t_1 = t_2 = 0)$,	T if $\hat{\Delta}_1(\gamma) = F$ or $\hat{\Delta}_2(\gamma)$ is T; otherwise F.
(h_6)	$\Box \Delta_1$	T if $des_{\mathcal{J}\mathcal{V}\gamma'}(\Delta_1) = T$ for all $\gamma' \in \Gamma$; F otherwise.
(h_7)	$(\forall x_1)\Delta'$ $(t' = 0)$,	T if $des_{\mathcal{J}\mathcal{V}'\gamma}(\Delta') = T$, where $\mathcal{V}' = \mathcal{V}\binom{x_1}{\xi}$, for all $\xi \in A^{\delta_1}_{t_1}$; otherwise F.
(h_8)	$(\imath x_1)\Delta'$ $(t' = 0)$,	η, if η is the unique element of $Q\mathcal{E}^{\delta_1}_{t_1}$ such that, for some $\xi \in A^{\delta_1}_{t_1}$, $\eta = \xi(\gamma)$ $[\eta = \xi^{I}(\gamma)$, i.e. $\eta = [I_{\delta_1}(\xi)](\gamma)]$ in case $\xi^{ord} = \delta_1$ $[\xi^{ord} < \delta_1]$ – see (4.14) below – and $des_{\mathcal{J}\mathcal{V}'\gamma}(\Delta') = T$ for $\mathcal{V}' = \mathcal{V}\binom{x_1}{\xi}$; F if no such unique η exists.
(h_9)	$\Delta_1 = \Delta_2$ $(t_1 = t_2)$,	T if $\hat{\Delta}_1(\gamma) = \hat{\Delta}_2(\gamma)$; F otherwise.
(h_{10})	$(\lambda^{\rho} x_1, \ldots, x_n)\Delta'$,	$f(\gamma)$ – see $(4.8)_1$.

At this point we can define $\breve{\Delta} = des_{\mathcal{J}\mathcal{V}}(\Delta)$ for every wfe$^{\beta}$ Δ of $\mathcal{ML}^{\nu}_{\alpha}$ by means of rules (ε_{1-10}) below, which are regarded to hold for all entities that satisfy assumptions (i) to (iv) above; they are also based on the definitions

$$(4.15) \quad \breve{\Delta}' = sens_{\mathcal{J}\mathcal{V}}(\Delta'), \quad \breve{\mathcal{R}} = sens_{\mathcal{J}\mathcal{V}}(\mathcal{R}), \quad \breve{\Delta}_j = sens_{\mathcal{J}\mathcal{V}}(\Delta_j) \ (j = 0, \ldots, n).$$

Rule	If Δ is	then $sense_{\mathcal{J}\mathcal{V}}(\Delta\mathcal{O})$ is $des_{\mathcal{J}\mathcal{V}}(\Delta)$, while $\breve{\Delta} = sens_{\mathcal{J}\mathcal{V}}(\Delta)$ is
(ε_1)	v^{β}_{tn} or $c^{\beta}_{t\mu}$,	$\mathcal{V}(v^{\beta}_{tn})$ or $\mathcal{J}(c^{\beta}_{t\mu})$ respectively.
(ε_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$,	$\langle \delta_0, \breve{\Delta}_0, \breve{\Delta}_1, \ldots, \breve{\Delta}_n \rangle$.
(ε_3)	$(\mathcal{R} x_1, \ldots, x_n)\Delta'$,	$\langle \delta_{\mathcal{R}}, \breve{\mathcal{R}}, g \rangle$ – see $(4.8)_2$.
(ε_{4-8})	$\sim \Delta_2$, $\Delta_2 \supset \Delta_3$, $\Box \Delta_2$, $(x_1)\Delta'$, or $(\imath x_1)\Delta'$, for $t_2 = t_3 = 0 = t'$,	$\langle \sim, \breve{\Delta}_2 \rangle$, $\langle \supset, \breve{\Delta}_2, \breve{\Delta}_3 \rangle$, $\langle \Box, \breve{\Delta}_2 \rangle$, $\langle \forall, g \rangle$ or $\langle \imath, g \rangle$ respectively, where $(4.8)_2$ and (4.9) hold with $n = 1$.
(ε_9)	$\Delta_1 = \Delta_2$ $(t_1 = t_2)$,	$\langle =, \breve{\Delta}_1, \breve{\Delta}_2 \rangle$.
(ε_{10})	$(\lambda^{\rho} x_1, \ldots, x_n)\Delta'$,	$\langle \lambda^{\rho}, g \rangle$ – see $(4.8)_2$.

Now the class QS_t can be defined, for $t \in \mathcal{T}_\nu$, by

(4.16) $QS_t^\beta =_D \{ sens_{\mathcal{J}\mathcal{V}}(\Delta) \mid \mathcal{J} \in I^\lambda,\ \mathcal{V} \in V^\lambda,\ \Delta \in E_t^{\beta\nless} \}$ $(\lambda = \beta+1)$.

On the basis of Theor.6.1 below, we can extend the determination of σ^I, already known for $\sigma \in QS^{<\beta}$ $(= \bigcup_{t\in\mathcal{T}_\nu} QS_t^{<\beta})$, to the case $\sigma \in QS^{\beta\nless}$:

(4.17) $\sigma^I =_D des_{\mathcal{J}\mathcal{V}}(\Delta)$, where $\sigma = sens_{\mathcal{J}\mathcal{V}}(\Delta)$

for some (constant-free) wfe$^\beta$ Δ and some $\mathcal{V} \in V^\lambda$ and $\mathcal{J} \in I^\lambda$. In other words we define $I_\lambda(\sigma)$ to be $I_\beta(\sigma)$, for $\sigma \in QS^{<\beta}$, and to be the object σ^I determined by (4.17), for $\sigma \in QS^{\beta\nless}$.

N5. Some semantical theorems, mainly on strict inclusions, among HQEs, HQIs, and QSs.

THEOR.5.1. Assume that $0 < \delta < \lambda \leq \alpha$. Then theses (a) and (b) below hold.

(a) The restrictions $\mathcal{V}'^\delta$ $[\mathcal{J}'^\delta]$ of the valuations $\mathcal{V} \in V^\lambda$ $[\mathcal{J} \in I^\lambda]$ to the variables [constants] of orders $< \delta$ are the valuations in V^δ $[I^\delta]$.

(b) If Δ is a wfe$^{<\delta}$, $\mathcal{V} \in V^\lambda$, $\mathcal{J} \in I^\lambda$, $\mathcal{V}' \in V^\lambda$, $\mathcal{J}' \in I^\lambda$, and $\mathcal{V}$ $[\mathcal{J}]$ agrees with $\mathcal{V}'$ $[\mathcal{J}']$ on the variables [constants] that occur in Δ, then

(5.1) $des_{\mathcal{J}'\mathcal{V}'}(\Delta) = des_{\mathcal{J}\mathcal{V}}(\Delta)$, $sens_{\mathcal{J}'\mathcal{V}'}(\Delta) = sens_{\mathcal{J}\mathcal{V}}(\Delta)$.

Indeed thesis (a) follows from (4.7) and thesis (b) has a proof admittely cumbersome, but as obvious as the one of its analogue for extensional languages. **q.e.d.**

By (4.16) the knowledge of the QS^βs requires the applications of rules (h_{1-10}) and (ε_{1-10}) to wfe$^\beta$s; only the first application involves the function $\sigma \mapsto \sigma^I$, and for $\sigma \in QS^{<\beta}$; i.e. it involves

the function I_β. The existence of $I_{\beta+1}$ will be proved by Theor.6.1. Incidentally Theors.5.2 and 6.1 below can be stated separately by the considerations above.

From (4.2-6), rules (ε_{1-10}), and (4.16) one sees at glance that

(α) σ is an ostensive QS, i.e. $\sigma \in$ **HQI**, only if σ has the form $sens_{\mathcal{J}\mathcal{V}}(\Delta)$ where either Δ ends by $\emptyset$ or Δ's length equals 1 and $(\mathcal{J} \cup \mathcal{V})(\Delta)$ is ostensive,[7] which certainly occurs for $\Delta^{ord}=0$; hence QS^μ**-HQI** $\neq \emptyset$ for $\mu < \alpha$, by rules (ε_{2-10})

Furthermore, if also $(4.8)_2$ is taken into account, one sees that

(β) the QS_t^βs, i.e. the elements of QS_t^β, are sets constructed (within pure set theory) starting out from some QI^βs and some symbols of $\mathcal{L}_\alpha^\nu$ (in the form of nested structures made with finite sequences - see (ε_{1-10}) - and functions - see $(4.8)_2$ -); and

(5.2) $QS_t^\beta \cap QI_t^{<\alpha} = QS_t^\beta \cap QI_t^\beta$ $(\beta < \alpha,\ t \in \mathcal{T}_\nu)$.

In fact (5.2) can be checked by inspection on the forms of the HQIs - see (4.2-6) - and those of the QS_t^βs that are not QI_t^βs - see rules (ε_{1-10}) and (4.16).

Theor.5.2. Let $0 \leq \delta < \beta < \alpha$ and $\mu < \alpha$. As a consequence

(5.3) $QE_t^\beta = QE_t^0$, $QI_t^\beta = QI_t^0$ $(t=0,\dots,\nu)$,

(5.4) $QI_t^\mu \subset QS_t^\mu$, $QI_t^\beta \subset A_t^\beta$, $A_t^\delta \subset A_t^\beta$ $(t \in \mathcal{T}_\nu)$,

(5.5) $QE_t^\delta \subset QE_t^\beta$, $QI_t^\delta \subset QI_t^\beta$; $QE_t^{<\beta} \subset QE_t^\beta$, $QI_t^{<\beta} \subset QI_t^\beta$ $(t \in \mathcal{T}_\nu - \{0,\dots,\nu\})$,

(5.6) $QS_t^{<\beta} \subset QS_t^\beta$, $QS_t^{<\beta} - QI_t^{<\alpha} \subset QS_t^\beta - QI_t^{<\alpha}$ $(t \in \mathcal{T}_\nu)$;

and, if β is a limit ordinal, then

(5.7) $A_t^{<\beta} = A_t^\beta$ $(t=0,\dots\nu)$; $A_t^{<\beta} \subset A_t^\beta$ $(t \in \mathcal{T}_\nu - \{0,\dots\nu\})$.

Indeed relations (5.3) hold for $\beta<\alpha$ by (4.2) and (4.4).

Let $(5.r)_s^{\subseteq}$ denote the non-strict counterpart of the strict inclusion relation $(5.r)_s$ $(r=4,\dots,7)$. Then (4.7), (4.5), rule (ε_1), and (4.16) easily yield $(5.4)^{\subseteq}$. In addition also $(5.5\text{-}7)^{\subseteq}$ are implied by (4.2-7), (ε_{1-10}), and (4.16) at glance.

Furthermore, by assertion (α) above some σ is in $QS_t^{\sigma} - QI_t^{<\alpha}$ $(\subseteq QS_t^{\beta} - QI_t^{<\alpha})$. Therefore $(5.4)^{\subseteq}$ and (4.5) imply that the strict implications $(5.4)_{1-2}$ hold for $0<\beta<\alpha$ and $\varkappa<\alpha$.

Now choose $\bar{\beta}<\alpha$ arbitrarily and assume that $(5.4)_3$ and (5.5-6) hold for all δ and β with $\delta<\beta<\bar{\beta}$. Furthermore suppose $\delta<\beta\leq\bar{\beta}$.

Note that by (5.2) we have the disjoint decompositions

$$(5.8)\quad A_t^{\delta} = QI_t^{\delta} \cup (QS_t^{<\delta} - QI_t^{<\alpha}), \quad A_t^{\beta} = QI_t^{\beta} \cup (QS_t^{<\beta} - QI_t^{<\alpha}).$$

Since now $\delta<\beta\leq\bar{\beta}$, by $(5.4)_1$, $(5.6)_1^{\subseteq}$, and (4.5),

$$(5.9)\quad QI_t^{\delta} \subset QS_t^{\delta} \subseteq QS_t^{<\beta} \subset A_t^{\beta};$$

hence, by (5.2), $(5.5)_2^{\subseteq}$, and (4.5),

$$(5.10)\quad QI_t^{\delta} \subset A_t^{\beta} \cap QI_t^{<\alpha}.$$

By the inductive hypothesis, $(5.6)_2$ and $(5.6)_1^{\subseteq}$ yield $QS_t^{<\delta} - QI_t^{<\alpha} \subset QS_t^{\delta} - QI_t^{<\alpha} \subseteq QS_t^{\beta} - QI_t^{<\alpha}$.
Hence, by $(5.5)_2^{\subseteq}$ and the disjoint decompositions (5.8), we have $(5.4)_3$ (for $0\leq\delta<\beta\leq\bar{\beta}$).

Now choose any $t = \langle t_1,\dots,t_n,t_0\rangle \in \mathcal{T}_\nu$. By $(5.5)_1^{\subseteq}$ for $t=t_0$ and (4.6), we have $\mathfrak{QE}_{t_0}^{\delta} \subseteq \mathfrak{QE}_t^{\beta}$. Furthermore, by the inductive hypothesis, $(5.4)_3$ implies $A_{t_i}^{\delta} \subset A_{t_i}^{\beta}$ $(i=1,\dots,n)$. Then by (4.3) we have $(5.5)_1$. Hence (4.4) implies $(5.5)_2$.

If β is a successor ordinal, then $(5.5)_{3-4}$ are practically included in $(5.5)_{1-2}$. Now let β be a limit ordinal. Then some subset S of C^{β}, where $C^{\varepsilon} =_D A_{t_1}^{\varepsilon}\times\dots\times A_{t_n}^{\varepsilon}$ for $\varepsilon<\alpha$, contains an element of each C^{ε} $(\varepsilon<\beta)$. Furthermore we know that $\mathfrak{QE}_{t_0}^{\varepsilon} \subseteq \mathfrak{QE}_{t_0}^{\beta}$ for all $\varepsilon<\beta$. Then, by (4.3),

$$QE_t^{<\beta} = \{F\} \cup \bigcup_{\varepsilon<\beta} (C^{\varepsilon} \mapsto \mathfrak{QE}_{t_0}^{\varepsilon}) \subset (C^{\beta} \mapsto \mathfrak{QE}_{t_0}^{\beta}) \cup \{F\} = QE_t^{\beta}.$$

Thus $(5.5)_3$ holds. Hence, by (4.4), $(5.5)_4$ also does.

In order to prove $(5.6)_1$, consider the wfe $\Delta =_D v_{t_0 1}^{\beta}(v_{t_1 2}^{\beta})$, where $t_1 \in \mathcal{T}_\nu - \{0,\dots,\nu\}$ and $t_0 = \langle t_1, t\rangle$. By $(5.5)_4$, for some $\mathcal{V} \in V^{\beta+1}$ we have $\mathcal{V}(v_{t_1 1}^{\beta}) \in QI_{t_1}^{\beta+\varepsilon}$. Then, for any $\mathfrak{I} \in I^{\beta+1}$, by rule (ε_2), $\breve{\Delta} = \mathrm{sens}_{\mathfrak{I}\mathcal{V}}(\Delta)$ is a non-ostensive QS_t^{β} consisting of a sequence formed with a QI outside $QI^{<\beta}$ (and with other objects). Hence – see assertion (β) – $\breve{\Delta} \notin QS_t^{<\beta}$. On the other hand $\Delta^{ord} = \beta$, so that $\breve{\Delta} \in QS_t^{\beta}$. Thus $(5.6)_1$ has been proved. By rule (ε_2) the above QS $\breve{\Delta}$ is obviously outside $QI^{<\alpha}$. Hence $(5.6)_1$ implies $(5.6)_2$.

Let now β be a limit ordinal. Then $\bigcup_{\delta<\beta} QS_t^{<\delta} = QS_t^{<\beta}$. Hence (4.5) and $(5.3)_2$ yield $(5.7)_1$. Furthermore, noting that (5.2) implies $QS_t^{<\beta} \cap QI_t^{<\beta} = QS_t^{<\beta} \cap QI_t^{\beta}$, (4.5) and $(5.5)_4$ yield $(5.7)_2$.
q.e.d.

N6. On $\mathrm{des}_{\mathfrak{I}\mathcal{V}}(\Delta)$, $\mathrm{sens}_{\mathfrak{I}\mathcal{V}}(\Delta)$, and the function $\sigma \mapsto \sigma^I$

Among the properties stated by the following theorem, the implication $(6.1)_3$ is essential for defining $\sigma \mapsto \sigma^I$, or the function I_λ $(0<\lambda\leq\alpha)$.

Theor.6.1. Assume that $0<\lambda\leq\alpha$, $\mathfrak{I}_r \in I^{\lambda}$, $\mathcal{V}_r \in V^{\lambda}$, Δ is a wfe$^{<\lambda}$, $\tilde{\Delta}_r = \mathrm{des}_{\mathfrak{I}_r\mathcal{V}_r}(\Delta_r)$, and $\breve{\Delta}_r = \mathrm{sens}_{\mathfrak{I}_r\mathcal{V}_r}(\Delta_r)$ $(r=1,2)$. Then[8]

$$(6.1)\quad \breve{\Delta}_1 \in QI^{<\lambda} \Rightarrow \breve{\Delta}_1 = \tilde{\Delta}_1;\ (\tilde{\Delta}_1)^{ord} \leq (\breve{\Delta}_1)^{ord};\ \breve{\Delta}_1 = \breve{\Delta}_2 \Rightarrow \tilde{\Delta}_1 = \tilde{\Delta}_2,$$

so that, for $\sigma \in QS^{<\lambda}$, σ^I – i.e. $I_\lambda(\sigma)$ – is determined, and we have – see rules (ε_{1-10}) – $\sigma \in QI^{<\lambda}$ if and only if $\sigma^I = \sigma$.

Remark that, if $\xi \in QI_t^{\mu}$ so that $\xi \in QS_t^{\mu}$ by $(5.4)_1$, then ξ has an order, a, as a QI and an order, b, as a QS – see ftn.8. Hence $\xi \in QI_t^{a}$, so that $\xi \in QS_t^{a}$. Then a=b by $(6.1)_{1-2}$. This justifies to denote a and b with the same notation: ξ^{ord}.

Proof of Theor.6.1. Fix π with $0<\pi\leq\alpha$, and assume the theorem to hold for $0<\lambda<\pi$. If π is a limit ordinal, then Theor.6.1 obviously holds for $0<\lambda\leq\pi$. Hence it suffices to assume that (a) $0<\pi<\alpha$ and (b) Theor.6.1 holds for $0<\lambda\leq\pi$, and to deduce that Theor.6.1 holds for $\lambda=\pi+1$. Therefore we assume (a), (b), and (c) $\lambda=\pi+1$. As preliminaries set (for r = 1,2) – see fnt.8

(6.2) $\omega_r=\Delta_r^{ord}$, $l_r=$ length of Δ_r, $l=\max(l_1,l_2)$, $\lambda=\pi+1$;

hence

(6.3) $(\tilde{\Delta}_r)^{ord}\leq\omega_r$, $(\breve{\Delta}_r)^{ord}\leq\omega_r\leq\pi$ (r=1,2).

If $\omega_1<\pi$ [and $\omega_2<\pi$ too], then (4.1) yields $(6.1)_{1-2}$ [$(6.1)_3$] immediately. Hence, by symmetry, it suffices to prove $(6.1)_{1-2}$ [$(6.1)_3$] under the assumption $\omega_1=\pi$ [and $\omega_2\leq\pi$].

In order to prove $(6.1)_1$ for $\omega_1=\pi$, let us first show that, for $l_1=1$,

(6.4) $\Delta_1=(\Delta_1)^I[=\breve{\Delta}_1]$ for $(\breve{\Delta}_1)^{ord}<\pi$ $[=\pi]$ $(\omega_1=\pi,\ l_1=1)$.

Indeed, for $l_1=1$ Δ_1 has the form $c^{\pi}_{t\mu}$ or v^{π}_{tn}; and we can momentarily set $\sigma=(\mathcal{J}\cup\mathcal{V})(\Delta_1)$ – see ftn.7 –, so that $\sigma=\breve{\Delta}_1$ by rule (ε_1), while rule (h_1) implies that $\tilde{\Delta}_1=\sigma^I$ $[\tilde{\Delta}_1=\sigma]$ for $\sigma^{ord}<\pi$ $[\sigma^{ord}=\pi]$. Thus (6.4) holds.

Now assume $\breve{\Delta}_1\in QI^{<\lambda}$. Then — see (5.2) — inspection on rules (ε_{1-10}) shows that either $l_1=1$ or ($l_1>1$ and) Δ_1 ends by $\mathcal{O}$. In the former case, for $\sigma^{ord}<\pi$ where $\sigma=\breve{\Delta}_1$, we have $\sigma^I=\sigma$ by the inductive hypothesis, so that $(6.4)_1$ yields $\tilde{\Delta}_1=\breve{\Delta}_1$; this equality also holds for $\sigma=\pi$ by $(6.4)_2$. Thus $(6.1)_1$ holds for $\omega_1=\pi$ and $l_1=1$.

In the remaining case ($l_1>1$), by rules (ε_{2-10}) Δ_1 has the form $\Delta\mathcal{O}$ and $\breve{\Delta}_1=\mathrm{des}_{\mathcal{J}_1\mathcal{V}_1}(\Delta)$, while by rules (h_{2-10}) $\tilde{\Delta}_1=\mathrm{des}_{\mathcal{J}_1\mathcal{V}_1}(\Delta)$, whence $(6.1)_1$. Thus $(6.1)_1$ holds for $\omega_1=\pi$.

Now let us prove $(6.1)_{2-3}$ for $\omega_1=\pi$ and $\omega_2\leq\pi$ by using induction on l and by noting as a preliminary, that $(6.1)_2$ holds trivially for $\sigma^{ord}=\pi$ $(\sigma=\tilde{\Delta}_1)$ – see ftn.8.

For (l=) $l_1=1$ and $\sigma^{ord}<\pi$ we have $\tilde{\Delta}_1=\sigma^I$ by $(6.4)_1$; and by the inductive hypothesis, $(6.1)_1$ and (4.17) yield $\sigma^I=\sigma$. Then $\tilde{\Delta}_1=\breve{\Delta}_1$ and $(6.1)_2$ is easily checked for $l_1=1$.

In order to prove $(6.1)_3$ for l=1, set

(6.5) $\sigma=\breve{\Delta}_1=\breve{\Delta}_2$ $(\omega_1=\pi\geq\omega_2)$.

Then $\breve{\Delta}_r\in QI^{<\lambda}$ by (ε_1), so that $(6.1)_1$ yields $\tilde{\Delta}_r=\breve{\Delta}_r$ (r=1,2). Hence $(6.1)_3$ holds.

Now fix $\bar{l}>1$, Assume $(6.1)_{2-3}$ to hold for all $l<\bar{l}$, and put $l=\bar{l}$. First consider the case when, e.g., Δ_1 ends by $\mathcal{O}$ so that, as was shown in the 2nd paragraph below (6.4), $\tilde{\Delta}_1=\breve{\Delta}_1$. Then $\sigma\in QI^{<\lambda}$; hence (6.5) and $(6.1)_1$ yield $\tilde{\Delta}_2=\breve{\Delta}_2$. Thus $(6.1)_3$ holds.

It remains to consider the case when <u>both</u> Δ_1 <u>and</u> Δ_2 <u>fail to end by</u> $\mathcal{O}$. Let $\breve{\Delta}_1$ arise by rule (ε_i), so that $\tilde{\Delta}_1$ arises by rule (h_i).

Case i=1, whence $l_1=1$ and $l_2=l=\bar{l}>1$. If $\sigma^{ord}=\pi$, by $(6.4)_2$ $\tilde{\Delta}_1=\breve{\Delta}_1$ $(=\breve{\Delta}_2)$. Hence $\breve{\Delta}_2\in QI^{<\alpha}$, so that by inspection of rules (ε_{1-10}) we see that $l_2=1$, while $l_2>1$. Therefore we consider the case $\sigma^{ord}<\pi$. If $\omega_2<\pi$, by (6.5) and the inductive hypothesis (on π), (4.17) yields $\tilde{\Delta}_2=\sigma^I$. On the other hand, rule (h_1) yields $\tilde{\Delta}_1=\sigma^I$, hence $\tilde{\Delta}_1=\tilde{\Delta}_2$ and $(6.1)_3$ holds. It remains to consider the subcase

(6.6) $\sigma^{ord}<\pi=\omega_2$ $(=\omega_1)$; hence, by (h_1) and (ε_1), $\tilde{\Delta}_1=(\breve{\Delta}_1)^I$ $(=\sigma^I)$.

Assume further that $\breve{\Delta}_2$ arises by rule (ε_j).

For j=1, we have $l_2=1$ by rule (ε_1), which is absurd.

For j=2, Δ_2 has the form $A_0(A_1,\dots,A_n)$ and $\sigma=\langle\delta_0,\breve{A}_0,\dots,\breve{A}_n\rangle$, where $\delta_0=A_0^{ord}$ and $\breve{A}_l=\mathrm{sens}_{\mathcal{J}_2\mathcal{V}_2}(A_l)$ (l=0,...,n). Since $\sigma^{ord}<\pi$, both δ_0 and $(\breve{A}_0)^{ord}$ to $(\breve{A}_n)^{ord}$ are $<\pi$ – see rule (h_2). Then we can choose n+1 (distinct) wfes B_0 to B_n of the respective orders δ_0 and $(\breve{A}_1)^{ord}$ to $(\breve{A}_n)^{ord}$, we can perform suitable changes on the values of $\mathcal{J}_2$ and $\mathcal{V}_2$ outside the constants and the variables that occur in Δ_2, by which (i) $\breve{A}_l=\breve{B}_l=_D\mathrm{sens}_{\mathcal{J}_2\mathcal{V}_2}(B_l)$ (l=0,...n). Hence, by (ε_2),

(6.7) $\sigma=\mathrm{sens}_{\mathcal{J}_2\mathcal{V}_2}(D)$, where $D=_D B_0(B_1,\dots,B_n)$.

Since $D^{ord}<\pi$, by the inductive hypothesis – see (4.17) –

(6.8) $\tilde{D}=_D\mathrm{des}_{\mathcal{J}_2\mathcal{V}_2}(D)=\sigma^I$, hence $\tilde{\Delta}_1=\tilde{D}$ by $(6.6)_{4-5}$.

On the other hand, by the inductive hypothesis on l ($\geq l_2$) $(6.1)_3$ implies, by (i), $\tilde{A}_h =_D des_{\mathfrak{J}_2\mathcal{V}_2}(A_h) = \tilde{B}_h =_D des_{\mathfrak{J}_2\mathcal{V}_2}(B_h)$ ($h=0,\dots n$) (even if $A_h^{ord} = \pi$ for some $h\in\{1,\dots,n\}$). Since $A_0^{ord} = \delta_0 = B_0^{ord}$, $\check{A}_h=\check{B}_h$, and $\tilde{A}_h=\tilde{B}_h$ ($h=0,\dots,n$), by (4.14) $\hat{A}_h=\hat{B}_h$ ($h=1,\dots n$). Then, by rule (h_2),

(6.9) $\tilde{D} = \hat{\Delta}_2$, hence $\tilde{\Delta}_1 = \tilde{\Delta}_2$ by $(6.8)_3$.

Thus $(6.1)_3$ holds for j=2. The remaining values of j will be treated in similar ways.

For j=3, Δ_2 has the form $(\mathcal{R}x_1,\dots,x_n)\Delta'$ and $\sigma = \langle \delta_{\mathcal{R}}, \check{\mathcal{R}}, g\rangle$, where conditions $(4.8)_2$ and $(4.15)_2$ hold for $\mathfrak{J}=\mathfrak{J}_2$ and $\mathcal{V}=\mathcal{V}_2$. Since $\sigma^{ord} < \pi$,

(6.10) $\delta_{\mathcal{R}} < \pi$, $(\check{\mathcal{R}})^{ord} < \pi$, $\delta_g < \pi$ – see $(4.12)_1$.

Then there are wfes $\mathcal{O}$ and D' with $\mathcal{O}^{ord} = \delta_{\mathcal{R}}$ and $D'^{ord} < \pi$, such that, setting $D =_D (\mathcal{O}x_1,\dots,x_n)D'$, we have $\check{D} =_D sens_{\mathfrak{J}_2\mathcal{V}_2}(D) = \sigma$ after a change of $\mathfrak{J}_2$ and $\mathcal{V}_2$ of the above type. Hence, with obvious notations, $\check{\mathcal{O}} = \check{\mathcal{R}}$; furthermore $\mathfrak{J}_2$, $\mathcal{V}_2$, and D' [Δ'] satisfy condition $(4.8)_2$ (and (4.9)) in $\mathfrak{J}$, $\mathcal{V}$, and Δ'. Then

(6.11) $g(\xi_1,\dots,\xi_n) = sens_{\mathfrak{J}_2\mathcal{V}'}(D') = sens_{\mathfrak{J}_2\mathcal{V}'}(\Delta')$ for all $\xi_i \in A_{t_i}^{\delta_i}$ (i=1,...n),

where (4.9) holds for $\mathcal{V}=\mathcal{V}_2$.

By $(6.10)_3$ and (4.11-12), $(6.11)_3$ implies that the QS $g(\xi_1,\dots,\xi_n)$ has an order $<\pi$. Then, by the inductive hypothesis on l, $(6.11)_2$ yields $des_{\mathfrak{J}_2\mathcal{V}'}(D') = des_{\mathfrak{J}_2\mathcal{V}'}(\Delta')$ whenever $(6.11)_3$ holds. Furthermore let $f(\gamma)$ [$\mathbf{f}(\gamma)$], Δ' [D'], $\mathfrak{J}_1$, and $\mathcal{V}_1$ satisfy conditions $(4.8)_2$ in $f(\gamma)$, Δ', $\mathfrak{J}$, and $\mathcal{V}$. Then $f(\gamma)=\mathbf{f}(\gamma)$; and, by the arbitrariness of $\gamma\in\Gamma$, f=**f**.

By applying rule (h_3) to Δ_2 and D, and by using an obvious symbolism, we obtain that, for all $\gamma\in\Gamma$,

(6.12) $\hat{\Delta}_2(\gamma) = [\hat{\mathcal{R}}(\gamma)](\langle\lambda^p,g\rangle)^\dagger$ and $\tilde{D}(\gamma) = [\tilde{\mathcal{O}}(\gamma)](\langle\lambda^p,g\rangle)^\dagger$ if $\delta_g < \mathcal{R}^{ord}$ ($= \mathcal{O}^{ord}$), and

(6.13) $\hat{\Delta}_2(\gamma) = [\hat{\mathcal{R}}(\gamma)](f)^\dagger$ and $\hat{D}(\gamma) = [\hat{\mathcal{O}}(\gamma)](f)^\dagger$ otherwise.

Since $\check{\mathcal{O}} = \check{\mathcal{R}}$, by the inductive hypothesis on l, $(6.1)_3$ yields

$\tilde{\mathcal{O}} = \tilde{\mathcal{R}}$. Hence the equality f=**f** and (6.12-13) yield $(6.9)_1$.

Since $D^{ord} < \pi$, we deduce (6.8) in the above way. Hence $(6.8)_3$ and $(6.9)_1$ yield $(6.9)_2$. Thus $(6.1)_3$ holds for j=3.

For j=4,5,6, Δ_2 has the form $\sim p_1$, $p_1 \supset p_2$, or $\Box p_1$ respectively. Furthermore by (6.5-6) there are wffs q_1 and q_2, of orders $<\pi$ such that by a change of $\mathfrak{J}_2$ and $\mathcal{V}_2$ of the above type we have that $\check{\Delta}_2 = \check{D} =_D des_{\mathfrak{J}_2\mathcal{V}_2}(D)$, where D is $\sim q_1$, $q_1\supset q_2$, or $\Box q_1$ respectively. By (ε_j) this is equivalent to $\check{p}_r = \check{q}_r =_D sens_{\mathfrak{J}_2\mathcal{V}_2}(q_r)$ for r=1, and for r=2 too if j=5. Then, by the inductive hypothesis on l, $(6.1)_3$ yields $\hat{p}_r=\hat{q}_r$ for r=1, and for r=1,2 if j=5; and rules (h_{4-6}) yield $(6.9)_1$. Since $D^{ord}<\pi$, (6.8) and $(6.9)_2$ can be deduced as in the case j=3. Then $(6.1)_3$holds for j=4,5,6.

For j=7,8, Δ_2 has the form $(\forall x_1)\Delta'$ or $(\imath x_1)\Delta'$ respectively; and $(4.8)_2$ (with (4.9)) holds for n=1, t'=0, $\mathfrak{J}=\mathfrak{J}_2$, and $\mathcal{V}=\mathcal{V}_2$. Furthermore, by (6.5) and $(6.6)_1$, $\delta_1 = x_1^{ord} < \pi$ – see rules (ε_{7-8}) – and, for some wff$^{<\pi}$ p, after changes on $\mathfrak{J}_2$ and $\mathcal{V}_2$ of the above type, we have $(6.9)_1$, where

$$D =_D (\forall x_1)p \text{ or } D =_D (\imath x_1)p,$$
(6.14)
$$\check{D} = sens_{\mathfrak{J}_2\mathcal{V}_2}(D)\ (\tilde{D} = des_{\mathfrak{J}_2\mathcal{V}_2}(D)).$$

Then, by rules (ε_{7-8}), Δ' [p], $\mathfrak{J}_2$, and $\mathcal{V}_2$ satysfy condition $(4.8)_2$ in Δ', $\mathfrak{J}$, and $\mathcal{V}$, for n=1 and t'=0. Hence – see (4.9) for $\mathcal{V}=\mathcal{V}_2$ –

(6.15) $g(\xi_1) = sens_{\mathfrak{J}_2\mathcal{V}'}(\Delta') = sens_{\mathfrak{J}_2\mathcal{V}'}(p)$, $g(\xi_1)^{ord}<\pi$ for $\xi_1 \in A_{t_1}^{\delta_1}$.

Therefore, by the inductive hypothesis on l, $(6.1)_3$ yields

(6.16) $des_{\mathfrak{J}_2\mathcal{V}'}(p) = des_{\mathfrak{J}\mathcal{V}'}(\Delta')$ for $\xi_1\in A_{t_1}^{\delta_1}$,

Furthermore let $f(\gamma)$ [$\mathbf{f}(\gamma)$], Δ' [p], $\mathfrak{J}_2$, and $\mathcal{V}_2$ satisfy condition $(4.8)_1$ (with (4.9)), for $\gamma\in\Gamma$. Then we easily see that

$f=\mathbf{f}$. Hence, on the one hand, $(6.14)_1$ implies by (h_7) that, for all $\gamma \in \Gamma$, $\tilde{\Delta}_2(\gamma) = T \iff f(\gamma)(\xi_1) = T$ $(\forall \xi_1 \in A_{t_1}^{\delta_1}) \iff \mathbf{f}(\gamma)(\xi_1) = T$ $(\forall \xi_1 \in A_{t_1}^{\delta_1}) \iff \tilde{D}(\gamma) = T$. Thus $(6.9)_1$ holds.

On the other hand, by applying (h_8) to Δ_2 [D] in the case $(6.14)_2$, we have that, for all $\gamma \in \Gamma$,

(a) [(b)] $\tilde{\Delta}_2(\gamma)$ [$\tilde{D}(\gamma)$] is the unique $\eta \in \mathcal{Q}\mathcal{E}_{t_1}^{\delta_1}$ such that for some $\xi \in A_{t_1}^{\delta_1}$, ($\alpha$) $\eta = \xi(\gamma)$ if $\xi^{ord} = \delta_1$, while $\eta = \xi^I(\gamma)$ if $\xi^{ord} < \delta_1$, and (β) $[f(\gamma)](\xi) = T$ $[[\mathbf{f}(\gamma)](\xi) = T]$, provided such unique η exists; and $\tilde{\Delta}_2(\gamma) = F$ $[\tilde{D}(\gamma) = F]$ otherwise.

Since $f(\gamma) = \mathbf{f}(\gamma)$, $\tilde{\Delta}_2(\gamma) = \tilde{D}(\gamma)$; and by the arbitrariness of $\gamma \in \Gamma$ $(6.9)_1$ holds again. At this point, since $D^{ord} < \aleph$, (6.8) and $(6.9)_2$ can be deduced in the above way. Thus $(6.1)_3$ holds for j=7,8.

For j=9, Δ_2 has the form $A_1 = A_2$; by (6.5) and $(6.6)_1$, for some wfe$^{<\aleph}$ D, $\check{D} = \check{\Delta}_2$ - see $(6.9)_2$ - after changes on $\mathcal{J}_2$ and $\mathcal{V}_2$ of the above type. Hence, by (ε_{1-10}), D has the form $B_1 = B_2$; furthermore, by using (here and in the sequel) an obvious symbolism of the above kind, we have that $\check{B}_r = \check{A}_r$ and $B_r^{ord} < \aleph$ (r=1,2). Then, by the inductive hypothesis on l, $(6.1)_3$ yields $\tilde{B}_r = \tilde{A}_r$ (r=1,2). Hence, by applying (h_9) to Δ_2 and D, we obtain that, for all $\gamma \in \Gamma$, $\tilde{\Delta}_2(\gamma) = T$ $[F] \iff \tilde{A}_1(\gamma) = [\neq]\ \tilde{A}_2(\gamma) \iff \tilde{B}_1(\gamma) = [\neq]\ \tilde{B}_2(\gamma) \iff \tilde{D}(\gamma) = T$ [F].

Hence $(6.9)_1$ holds. Furthermore (6.8) and $(6.9)_2$ can be deduced as before. Hence $(6.1)_3$ holds for j=9.

For j=10, Δ_2 has the form $(\lambda^p x_1,\dots,x_n)\Delta'$. By (6.5) and $(6.6)_1$, for some wfe$^{<\aleph}$ D, $\check{D} = \check{\Delta}_2$ - see $(6.13)_3$ - after changes on $\mathcal{J}_2$ and $\mathcal{V}_2$ of the above type. Hence, by (ε_{1-10}), D has the form $(\lambda^p y_1,\dots,y_n)D'$. More, one easily sees that it can be chosen of the form $(\lambda^p x_1,\dots,x_n)D'$. At this point (ε_{10}) implies that Δ' [D'], $\mathcal{J}_2$, and $\mathcal{V}_2$ satisfy condition $(4.8)_2$ in Δ', $\mathcal{J}$, and $\mathcal{V}$. Hence (6.11) holds.

By (6.5) and $(6.6)_1$, $\delta_g < \aleph$ - see (4.11-12) and (4.9) for $\mathcal{V} = \mathcal{V}_2$. Hence $(6.11)_3$ yields $g(\xi_1,\dots,\xi_n)^{ord} < \aleph$, so that, by the inductive hypothesis on l, $(6.1)_3$ and $(6.11)_2$ yield

(6.17) $des_{\mathcal{J}_2\mathcal{V}'}(D') = des_{\mathcal{J}_2\mathcal{V}'}(\Delta')$ - see (4.9) for $\mathcal{V} = \mathcal{V}_2$ - $\forall \xi_i \in A_{t_i}^{\delta_i}$ (i=1,...,n).

Thence we deduce $f=\mathbf{f}$ in the usual way, where f [$\mathbf{f}$] is the function

determined by the requirement of satisfying condition $(4.8)_1$ in f, Δ', $\mathcal{J}$, and $\mathcal{V}$, together with Δ' [D'], $\mathcal{J}_2$, and $\mathcal{V}_2$. As a consequence, by (h_{10}) we have, for all $\gamma \in \Gamma$, that $\tilde{\Delta}_2(\gamma) = f(\gamma) = \mathbf{f}(\gamma) = \tilde{D}(\gamma)$, so that $(6.9)_1$ holds. We can deduce (6.8) and $(6.9)_2$ as before. Thus $(6.1)_3$ holds also for j=10, i.e. it holds for i=1.

x x x

Now let us prove $(6.1)_2$ for (l=) $l_1 = \bar{l} > 1$. We can consider only the case $\sigma^{ord} < \aleph$ ($\sigma = \check{\Delta}_1$) - see two paragraphs above (6.5). In it $\sigma \in A_t^{\aleph}$ for some $t \in T_\nu$. Hence we can choose $\mathcal{J}_2$. $\mathcal{V}_2$, and Δ_2 in such a way that (6.5) holds for $\omega_2 = \aleph$ and $l_2 = 1$. Then, by the analogue of $(6.4)_1$ for Δ_2, $\tilde{\Delta}_2 = (\check{\Delta}_2)^I = \sigma^I$. Furthermore $(6.5)_2$ and $(6.1)_3$ imply $\tilde{\Delta}_1 = \tilde{\Delta}_2$; hence $\tilde{\Delta}_1 = \sigma^I$. By the inductive hypothesis on $\aleph$, $(\sigma^I)^{ord} \leq \sigma^{ord}$. Hence $(6.1)_2$ holds.

At this point we can assert the validity of $(6.1)_{1-2}$ for any wfe Δ_1 of order $\leq \aleph$; and the one of $(6.1)_3$ for $\Delta_r^{ord} \leq \aleph$, $l_r \leq \bar{l}$ (r=1,2), and i=1.

x x x

In order to complete the proof of Theor.5.2, it is sufficient to prove $(6.1)_3$ in the case $\omega_1 = \aleph \geq \omega_2$ and $i \in \{2,\dots,10\}$, so that $l_1 > 0$. To this end assume (6.5) again. By symmetry it is sufficient to consider the subcase $j > 1$, hence $l_2 > 1$. By (ε_{1-10}) this implies j=i. The treatments of the afore-mentioned cases will have some features in common; e.g. (6.5) is assumed.

Case i=2. By $(6.6)_2$ (and (ε_2)) Δ_r has the form $\Delta_{r0}(\Delta_{r1},\dots,\Delta_{rn})$ (r=1,2); furthermore, by $(6.5)_2$,

$$(6.18)\quad \Delta_{10}^{ord} = \Delta_{20}^{ord},\quad \check{\Delta}_{1h} = \check{\Delta}_{2h},\ \text{where}\quad \check{\Delta}_{rh} = sens_{\mathcal{J}_r\mathcal{V}_r}(\Delta_{rh})\ (r=1,2;\ h=0,\dots,n).$$

Then, by the inductive hypothesis on l,

$$(6.19)\quad \tilde{\Delta}_{1h} = \tilde{\Delta}_{2h}\ \text{where}\ \tilde{\Delta}_{rh} = des_{\mathcal{J}_r\mathcal{V}_r}(\Delta_{rh})\quad (r=1,2;\ h=0,\dots,n).$$

By (6.17-18) we have $\hat{\Delta}_{1h} = \hat{\Delta}_{2h}$ (h=1,...,n) - see (4.14). Then rule (h_2) yields $\hat{\Delta}_{10}(\gamma) = [\hat{\Delta}_{10}(\gamma)](\hat{\Delta}_{11},\ldots,\hat{\Delta}_{1n})^{\dagger} = [\tilde{\Delta}_{20}(\gamma)](\hat{\Delta}_{21},\ldots,\hat{\Delta}_{2n})^{\dagger} = \tilde{\Delta}_2(\gamma)$ for all $\gamma \in \Gamma$. Hence $\hat{\Delta}_1 = \hat{\Delta}_2$. Then $(6.1)_3$ holds.

For i=3, Δ_r can be regarded to have the form $(\Omega_r x_1,\ldots,x_n)\Delta'_r$ with x_1 to x_n independent of r,(9) so that by (ε_3) $\check{\Delta}_r = \langle \Omega_r^{ord}, \check{\Omega}_r, g_r \rangle$, where g_r, Δ'_r, $\mathfrak{J}_r$, $\mathcal{V}_r$, and $\mathcal{V}'_r$ satisfy condition $(4.8)_2$ in g, Δ', $\mathfrak{J}$, $\mathcal{V}$, and $\mathcal{V}'$ (r=1,2); and, by (6.5), for some δ_i and $t_i \in T_\nu$ (i=1,...,n)

$$(6.20)\quad \Omega_1^{ord} = \Omega_2^{ord},\ \check{\Omega}_1 = \check{\Omega}_2,\ g_1 = g_2,\ x_i \in E_t^{\delta_i *}\ (i=1,\ldots,n;\ r=1,2);$$

in addition — see (4.9) for $\mathcal{V} = \mathcal{V}_r$ and $\mathcal{V}' = \mathcal{V}'_r$ —

$$(6.21)\quad g_r(\xi_1,\ldots,\xi_n) = \text{sens}_{\mathfrak{J}_r \mathcal{V}_r}(\Delta'_r), \quad \text{for all } \xi_i \in A_{t_i}^{\delta_i}\ (i=1,\ldots n;\ r=1,2).$$

For r=1,2, define f_r by requiring that f_r, Δ'_r, $\mathfrak{J}_r$, and $\mathcal{V}_r$ should satisfy condition $(4.8)_1$ in f, Δ', $\mathfrak{J}$, and $\mathcal{V}$. Hence

(α_r) <u>for all</u> $\gamma \in \Gamma$ <u>and</u> $\xi_i \in A_{t_i}^{\delta_i}$ (i=1,...,n) <u>the</u> HQE $[f_r(\gamma)](\xi_1,\ldots,\xi_n)$ <u>equals the</u> HQE $[\text{des}_{\mathfrak{J}_r \mathcal{V}'_r}(\Delta'_r)](\gamma)$, <u>where</u> (4.9) <u>holds for</u> $\mathcal{V} = \mathcal{V}_r$ <u>and</u> $\mathcal{V}' = \mathcal{V}'_r$, <u>whenever the latter</u> HQE <u>is</u> $\neq$ F; <u>and the former</u> HQE <u>is inexistent otherwise.</u>

By the inductive hypothesis on l, $(6.1)_3$ and equalities $(6.20)_3$ and $(6.21)_1$ for r=1,2, yield that $\text{des}_{\mathfrak{J}_1 \mathcal{V}'_1}(\Delta'_1) = \text{des}_{\mathfrak{J}_2 \mathcal{V}'_2}(\Delta'_2)$ for all $\xi_i \in A_{t_i}^{\delta_i}$ (i=1,...,n). Hence (α_{1-2}) yield $f_1 = f_2$. This equality and (6.20) imply, by rule (h_3), that $\hat{\Delta}_1(\gamma) = \hat{\Delta}_2(\gamma)$ for all $\gamma \in \Gamma$, i.e. $\hat{\Delta}_1 = \hat{\Delta}_2$. Thus $(6.1)_2$ holds.

For i=4,5,6, Δ_r has the form $\sim p_r$, $p_r \supset q_r$, or $\Box p_r$ respectively (r=1,2). By (6.5) (and (ε_{4-6})) $\check{p}_1 = \check{p}_2$ and, for i=5, also $\check{q}_1 = \check{q}_2$. Then, by the inductive hypothesis, $(6.1)_3$ implies $\tilde{p}_1 = \tilde{p}_2$ and, for i=5, also $\tilde{q}_1 = \tilde{q}_2$. Hence, by rules (h_{4-6}), $\hat{\Delta}_1 = \hat{\Delta}_2$. Thus $(6.1)_3$ holds.

For j=7,8, Δ_r can be regarded to have the form $(\forall x_1)\Delta'_r$ or $(\imath x_1)\Delta'_r$ respectively, with x_1 independent of r — in a sense analogous to that explained in ftn.9 —; hence, by (ε_{7-8}), $\check{\Delta}_r$ is $\langle \forall, g_r \rangle$ or $\langle \imath, g_r \rangle$ respectively, where $(4.8)_2$ and (4.9) are regarded to hold for n=1, t'=0, $\mathfrak{J} = \mathfrak{J}_r$, $\mathcal{V} = \mathcal{V}_r$, $\mathcal{V}' = \mathcal{V}'_r$, and $g = g_r$ (r=1,2). Then, by (6.5), $g_1 = g_2$. Hence the afore-mentioned validity of (4.9) yields $\text{sens}_{\mathfrak{J}_1 \mathcal{V}_1}(\Delta'_1) = \text{sens}_{\mathfrak{J}_2 \mathcal{V}_2}(\Delta'_2)$ for all $\xi_1 \in A_{t_1}^{\delta_1}$. Hence, by the inductive hypothesis on l, $(6.1)_3$ yields $\text{des}_{\mathfrak{J}_1 \mathcal{V}_1}(\Delta'_1) = \text{des}_{\mathfrak{J}_2 \mathcal{V}_2}(\Delta'_2)$ for all $\xi_1 \in A_{t_1}^{\delta_1}$. Furthermore, for r=1,2 define f_r by the requirement that

(β_r) <u>for all</u> $\gamma \in \Gamma$ <u>and</u> $\xi_1 \in A_{t_1}^{\delta_1}$ <u>the</u> HQE $[f_r(\gamma)](\xi_1)$ <u>equals the</u> HQE $[\text{des}_{\mathfrak{J}_2 \mathcal{V}'_2}(\Delta'_r)](\gamma)$ <u>whenever the latter</u> HQE <u>is</u> $\neq$ F, <u>while it fails to exist otherwise</u>.

Then $f_1 = f_2$. Now it is easy to check by rules (h_{7-8}), that $\hat{\Delta}_1 = \hat{\Delta}_2$. Hence $(6.1)_3$ holds.

For j=9, Δ_r has the form $\Delta_{r1} = \Delta_{r2}$; and by (6.5) and (ε_9), using an obvious symbolism, we have that, for s=1,2, $\check{\Delta}_{1s} = \check{\Delta}_{2s}$; hence, by the inductive hypothesis on l, $(6.1)_3$ implies $\tilde{\Delta}_{1s} = \tilde{\Delta}_{2s}$ (s=1,2). Then $\hat{\Delta}_1 = \hat{\Delta}_2$ by rule (h_9). Thus $(6.1)_3$ holds.

For j=10, Δ_r can be regarded to have the form $(\lambda^p x_1,\ldots,x_n)\Delta'_r$ with x_1 to x_n independent of r — see ftn.9 —; hence, by (ε_{10}), $\check{\Delta}_r = \langle \lambda^p, g_r \rangle$ where g_r, Δ'_r, $\mathcal{V}_r$, and $\mathcal{V}'_r$ satisfy conditions $(4.8)_2$ and (4.9) in g, Δ', $\mathcal{V}$, and $\mathcal{V}'$ (r=1,2). Then, by (6.5), $g_1 = g_2$; hence — by the afore-mentioned validity of (4.9) — $\text{sens}_{\mathfrak{J}_1 \mathcal{V}'_1}(\Delta'_1) = \text{sens}_{\mathfrak{J}_2 \mathcal{V}'_2}(\Delta'_2)$ for all $\xi_i \in A_{t_i}^{\delta_i}$ (i=1,...,n). Then by the inductive hypothesis on l, $(6.1)_3$ yields $\text{des}_{\mathfrak{J}_1 \mathcal{V}'_1}(\Delta'_1) = \text{des}_{\mathfrak{J}_2 \mathcal{V}'_2}(\Delta'_2)$.

Furthermore, for r=1,2 define f_r by condition (α_r) above. Hence $f_1 = f_2$. Now it is easy to check by rule (h_9) that $\hat{\Delta}_1 = \hat{\Delta}_2$. Hence $(6.1)_3$ holds. **q.e.d.**

N7. Comparison of the HQIs and QSs for $\mathcal{SL}_\alpha^{\omega}$ with those for $\mathcal{SL}_\alpha^{\nu}$

We want to show that <u>the</u> HQIs <u>and</u> QSs <u>for</u> $\mathcal{SL}_\alpha^{\omega}$ <u>are more than those for</u> $\mathcal{SL}_\alpha^{\nu}$, and that this occurs through the QSs of e.g. universal operators. To this end set, within the ordinary language,

$$(7.1)\quad p_n \equiv_D m+1 > n, \text{ where } m = n^2 \quad (n \in \mathbf{N});$$

or, from a perhaps less intuitive or more rigorous point of view, denote the standard name of n in the ordinary language by $\bar{n}$, and use 'p_n' as the expression: $\overline{n}^2 \,\mathrm{J}\, "+1>" \,\mathrm{J}\, \bar{n}$, for all $n \in \mathbf{N}$ - see ftn.3. Furthermore consider the intuitive assertion

(α) <u>for every</u> $n \in \mathbf{N}$, p_n (<u>holds</u>).

It has the following translation into $\mathcal{ML}^{\omega}_{\alpha}$, to be regarded as of order zero.

(7.2) $A \equiv_D (\forall n)\, n^{2\omega}+1 > n \qquad (A \in E^{\omega}_0)$,

where n is a variable running over $\mathbf{N}$, e.g. in that n is v^{ω}_{11} and $\mathcal{D}_1 = \mathbf{N}$. By using an obvious symbolism, rules (ε_{1-3}) yield

(7.3) $\breve{A} = \langle \forall, g \rangle$, where $g(n) = (\breve{p}_n =) \langle >, \langle +, n^2, 1 \rangle, n \rangle$ for $n \in \mathbf{N}$.

Incidentally, if "ω" in (7.2) would be crossed out, then "n^2" in (7.3) ought to be replaced by e.g. "$\langle exp, n, 2 \rangle$".

As an hypothesis for reduction ad absurdum let $\breve{A}$ be a QS for $\mathcal{ML}^{\nu}_{\alpha}$. Then $\breve{A}=\breve{B}$, where B has the form $(\forall x) c+1 > x$ and the term c has an ostensive QS. Hence c is a constant or a variable. The first case must be discarded because n^2 is not constant. The second case is also unacceptable, because n^2 fails to run over the whole set $\mathbf{N}$. Hence $\breve{A}$ fails to be a QS for $\mathcal{ML}^{\nu}_{\alpha}$, which proves the underlined assertion at the outset.

By (4.2-6) it is now obvious that the QE^{β}_ts [QI^{β}_ts] for $\mathcal{ML}^{\omega}_{\alpha}$ are more than those for $\mathcal{ML}^{\nu}_{\alpha}$ in case $t \in \mathcal{T}_\nu - \{0, \dots, \nu\}$ and $0 < \beta < \alpha$. This and the example above imply the analogue for the QS^{β}_ts in case $t \in \mathcal{T}_\nu$ and $0 \leq \beta < \alpha$.

Of course, by (5.3) the QE^{β}_t s [QI^{β}_t s] for $\mathcal{ML}^{\omega}_{\alpha}$ are those for $\mathcal{ML}^{\nu}_{\alpha}$ in case $t=0, \dots, \nu$ and $0 \leq \beta < \alpha$. Furthermore, by the semantical rules in N4, the same can obviously be said of the QE^{ω}_ts and QI^{ω}_ts ($t \in \mathcal{T}_\nu$).

If for some purposes, presently unknown, one likes to consider a finer (weaker) basic semantics for $\mathcal{ML}^{\omega}_{\alpha}$, in which e.g. "$3^{2\omega}$" fails to be synonymous with "9", this obviously causes QSs to increase strictly. Instead the strict inclusions proved above are nontrivial.

<u>APPENDIX ON PAPER 5</u>

Since my the present work is tightly based on [5], the following errata-corrige for it - already written in [6] - is included here.

p.433, 1.18	$\lg_1 8 \longrightarrow \lg_2 8$	
p.438. 1.14	$\mathfrak{p} \in \mathcal{T}_\nu \longrightarrow \vartheta \in \mathcal{T}_\nu$	
p.443. 1.7	- see (A) - $\longrightarrow$ according to (A)	
p.447, 1.2b	(2) $\longrightarrow$ (2), then (3) and (4) for $\Delta^{ord} < \beta$,	
p.448, 1.7	(α) the QE^{β}_t's $\longrightarrow$ (α) the QI^{β}_t's	
p.449, 1.7b	$\delta < \beta \longrightarrow \delta \leq \beta$	
p.450, 1.5	(5.1) and $(4.6)_3 \longrightarrow$ (5.1)	
p.450, (5.13)	$= \begin{cases} \mathrm{sens}_{\mathcal{I}\mathcal{V}'}(\Delta') \neq F \\ \mathrm{des}_{\mathcal{I}\mathcal{V}'\gamma}(\Delta')\ (\gamma \in \Gamma) \end{cases} \longrightarrow = \begin{cases} \mathrm{sens}_{\mathcal{I}\mathcal{V}'}(\Delta') \\ \mathrm{des}_{\mathcal{I}\mathcal{V}'\gamma}(\Delta') \neq F\ (\gamma \in \Gamma) \end{cases}$	
p.450, (5.13')	$g^I \longrightarrow g^I(\gamma)$	$QI^{\delta_g}_{t'} \longrightarrow QI^{d_g(\gamma)}_{t'}$

p.450, Replace formula (5.14) with

$$(5.14)\ \left[\begin{matrix} \delta_g \\ d_g(\gamma) \end{matrix}\right. =_D \sup\left\{ \left[\begin{matrix} g(\xi)^{ord} \\ [g^I(\gamma)](\xi)^{ord} \end{matrix}\right. \middle| \ \xi \in \left[\begin{matrix} \mathcal{D}_g \\ \mathcal{D}_{g^I(\gamma)} \end{matrix}\right. \right\}$$

for $\xi = \langle \xi_1, \dots, \xi_n \rangle$.

p.451, (h_4) $\qquad \dot{\Delta}_1$ is T $\longrightarrow \tilde{\dot{\Delta}}_1(\gamma)$ is T p.451, (h_5) $\tilde{\Delta}_1$=F or $\tilde{\Delta}_2$=T $\longrightarrow$ $\Delta_1(\gamma)$=F or $\tilde{\Delta}_2(\gamma)$=T

p.452, (ε_1) $\langle \Delta_\circ \longrightarrow \langle \delta_\circ, \Delta_\circ$

p.452, (ε_2) $\langle \breve{\mathcal{A}} \longrightarrow \langle \delta_{\mathcal{A}}, \breve{\mathcal{A}}$

p.452, 1.7b HQSs $\longrightarrow$ HQIs

p.456, (e) from $\mathfrak{A}$; $\longrightarrow$ from $\mathcal{A}$,

p.457, (7.3) $= (\eta^* x^1)p = (\eta_* x^1)p. \longrightarrow$
$= (\eta^* x^1)p^1 = (\eta_* x_1)p^1$, where $p^1 =_D p[x/x^1]$ and $x^1 \in E_1 \cup \dots \cup E_\nu$.

p.459, 1.8 for $p \equiv_D B^1(A, z=\mathfrak{R}_1) \longrightarrow$
for $p \equiv_D B^1(A, z=\mathfrak{R}_1)$ and $z \in E_1 \cup \dots \cup E_\nu$

p.459, 1.9-12 $p \longrightarrow p(z^1)$

FOOTNOTES

(*) The present work has been performed within the activity sphere of the CNR (Consiglio Nazionale delle Ricerche) in group n.3, during the academic year 1984-85.

(1) In the papers mentioned here and in the present one only formal theories are considered, where any wfe Δ has one (hyper-) extension and one sense. Several wfes, e.g. "3", "5-2", and "$\lg_2 8$" have the same extension, 3, and different senses. The simplest among these can be called <u>ostensive</u> - see [3], p.484, for more detail -. It is often identified with its extension, 3.

(2) In $\mathcal{L}_\alpha$ and $\mathcal{SL}^\nu_\alpha$ the equality of two propositions is equivalent to the equivalence of these. Suppose that f and g are relators (of the same sense order), and that one of them is the relation in M and p: the man M believes that p. Then it is clear that the variables in (1.1) have to take quasi-senses as values (or something equivalent). This justifies the name given to (1.1).

(3) By e.g. "$\Delta\Theta$" it is meant $\Delta J \Theta$ (to be read as Δ juxtaposed to Θ) where $aJb =_D \langle a_1,\dots,a_r,b_1,\dots,b_s\rangle$ for $a = \langle a_1,\dots,a_r\rangle$ and $b = \langle b_1,\dots,b_s\rangle$.

(4) However, in [5], p.438 - see rule (φ_1) -, the index μ in $c^{\beta}_{t\mu}$ is allowed to run from 1 to $\alpha+\omega_\circ$. If preferred, we can regard it to run from 1 to $\beta+\omega_\circ$. Thus the λ-th segment of $\mathcal{SL}^{\Theta}_{\alpha}$, i.e. the language formed with the wfes of $\mathcal{SL}^{\Theta}_{\alpha}$ that have orders $\leq \lambda$, turns out to be of the same kind as $\mathcal{SL}^{\Theta}_{\alpha}$ $(0<\lambda\leq\alpha)$.

(5) By (4.10-11) we have $f(\gamma) \in QE^{\varkappa}_{\langle t_1,\dots,t_n,t'\rangle}$. Furthermore (4.12) and rule (ε_{10}) below imply that $\langle \lambda^p, g\rangle \in QS^{\delta_g}_{\langle t_1,\dots,t_n,t'\rangle}$.

(6) By rule (ε_{10}) below, $\breve{\tilde{\Delta}} =_D des_{\mathcal{JV}}[(\lambda^p x_1,\dots,x_n)\Delta'] = \langle \lambda^p, g\rangle$ so that, by ftn.(5), $(\breve{\Delta}')^{ord} =_D \delta_g$. Furthermore, by rule (h_{10}), $\tilde{\breve{\Delta}} =_D$ $=_D des_{\mathcal{JV}}[(\lambda^p x_1,\dots,x_n)\Delta'] = f$. Then the consequent of rule (h_3) can also be framed as follows: $\tilde{\Delta}(\gamma)$ = ... <u>is</u> $[\tilde{\mathcal{A}}(\gamma)](\breve{\Delta}')$ <u>if</u>

$(\check{\tilde{\mathcal{X}}})^{ord} < \mathcal{A}^{ord}$; $[\hat{\mathcal{A}}(\gamma)](\tilde{\hat{\mathcal{X}}})$ otherwise.

(7) Since Δ's length is 1, Δ has the form $c^{\varkappa}_{t\mu}$ or $v^{\varkappa}_{tn}$. Furthermore $\mathfrak{I}\cup\mathcal{V}$ is a function and $(\mathfrak{I}\cup\mathcal{V})(\Delta)$ is $\mathfrak{I}(c^{\varkappa}_{t\mu})$ or $\mathcal{V}(v^{\varkappa}_{tn})$ respectively.

(8) The order σ^{ord} of the HQI σ [QS σ] is the least among the orders of the wfes Δ that can designate it, i.e. such that $\sigma = des_{\mathfrak{I}\mathcal{V}}(\Delta)$ [$\sigma = sens_{\mathfrak{I}\mathcal{V}}(\Delta)$] at some $\mathfrak{I}\in I^{<\alpha}$ and $\mathcal{V}\in V^{<\alpha}$ - see the remark below Theor.6.1.

(9) More precisely x_1 to x_n, Δ'_1, and Δ'_2 can be so chosen that, for r=1,2, Δ_r has the same QS-designatum ($sens_{\mathfrak{I}_r\mathcal{V}_r}$) and the same QI-designatum ($des_{\mathfrak{I}_r\mathcal{V}_r}$) as $(\mathcal{A}_r x_1,\dots,x_n)\Delta'_r$.

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