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ON THE ORDER-STRUCTURE OF THE REAL NUMBERS IN NON-STANDARD MODELS OF THE ANALYSIS

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§1. The order-theoretic and topological properties of the real axis in non-standard models of the analysis have been extensively studied by E.ZAKON in [7]. Further results, including partial answers to [7, Questions 1-5], can be found in [3], [4], [5], [6].

This note summarizes the achievements of a joint research, carried on in this area by F.HONSELL and the author: more details and complete proofs will appear in [1] (see also [2]).

Given any non-standard model of the analysis, we consider the order-types:

$\omega, \lambda, \nu, \lambda'$ of N, R, *N, *R (the standard and non-standard naturals and reals);

λ_0 of $G_0 = \{ x \in {}^*R \mid \exists n \in N \ |x| < n \}$ (the *galaxy* of 0);

μ of $M_0 = \{ x \in {}^*R \mid \forall n \in N \ |x| < 1/n \}$ (the *monad* of 0);

θ of ${}^*R^+/G_0$ (the positive *infinites* modulo equivalence);

τ_x of $T_x = \{ y \in \theta \mid y < x \}$ (the initial *segment* determined by x in θ).

We are mainly concerned here with the symmetry of θ, the equality of θ and τ_x (resp. of λ' and μ), and the possible cofinalities of these order-types. We shall also consider some connected questions about pseudometrizability and completeness of the natural group topologies on *R (all these problems have been posed in [7]).

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§2. We recall that the order-types μ, λ', λ_0, τ_x are *symmetric* and satisfy the following identities, where ξ^* denotes the *converse order-type* of ξ (see [7]):

$\nu = \omega+(\omega^*+\omega)\theta$, $\lambda' = \lambda_0(\theta^*+1+\theta)$, $\mu = \lambda_0\theta+1+\lambda_0\theta^*$, $\theta = \tau_x+1+\theta$, $\lambda_0 = \mu\lambda$,

hence $\operatorname{cof}\nu = \operatorname{cof}\lambda' = \operatorname{cof}\theta$ and $\operatorname{cof}\mu = \operatorname{cof}\tau_x = \operatorname{cof}\theta^*$.

We begin by considering the following eight equalities:

(1) $\theta = \theta^*$, (2) $\theta = \theta+1+\theta$, (3) $\theta = \theta^*+1+\theta$, (4) $\lambda' = \mu$, (5) $\operatorname{cof}\theta = \operatorname{cof}\theta^*$,
(6) $\theta = \tau_x$ for some x, (7) $\theta = \tau_x$ for any x, (8) $\tau_x = \tau_y$ for any x, y.

The following theorems give an almost complete insight of the connections existing among the above conditions.

THEOREM 1. *The implications of the following diagram hold in any non-standard model of the analysis:*

$$(\star)\qquad \begin{array}{ccccccc} (1) & \longleftrightarrow & (2) & \longleftrightarrow & (3) & \longrightarrow & (4) \\ \updownarrow & & & & & & \downarrow \\ (6) & \longleftrightarrow & (7) & \longrightarrow & (8) & & (5) \end{array}$$

Moreover, if $\operatorname{cof}\theta = \omega$, *then* (5) *implies in turn* (4), *and any one of* (1)-(3), (6)-(7) *is equivalent to the conjunction* (5)&(8).

On the other hand we obtain, via the models of section 4,

THEOREM 2. *Let* κ_1 *and* κ_2 *be regular cardinals. There are non-standard models verifying both*

(2.1) $\operatorname{cof}\theta = \kappa_1$, $\operatorname{cof}\theta^* = \kappa_2$, *and*

(2.2) $|\tau_x| \neq |\tau_y|$ *for suitable* x, y (*hence* $\theta \neq \theta^*$).

If $\kappa_1 = \kappa_2$, *then there are also models verifying both* (2.1) *and*

(2.3) $\theta = \mu$ (*hence* $\theta = \theta^* = \lambda'$).

Moreover $|\theta|$ *can be any cardinal not less than* $2^{\omega}\kappa_1$ *in models of* (2.1) & (2.3), *and any cardinal not less than* $2^{\kappa^{\omega}}\kappa_1$ *in models of* (2.1) & (2.2).

THEOREM 3. *Apart from compositions, no arrow can be added to Diagram* ($\star$) *except perhaps* (5) $\longrightarrow$ (4).

Notice that, although the equivalence between (4) and (5) is claimed in [3, Lemma 3.2], the proof given there works only if θ has countable cofinality; thus also the implication (1) $\longrightarrow$ (4) lacks a proof in [3].

Moreover, the consistency of (1) is proved in [3] and in [6] only for saturated models, hence, in fact, under the very strong assumption $|\theta| = \operatorname{cof}\theta = \kappa^+ = 2^{\kappa}$.

The remaining non-trivial assertions of Theorems 1-3 seem to be new, except the implication (3) $\longrightarrow$ (4), which follows from the proof of [6, Theorem 2.12], and the independence of (5), already stated in [5, Theorem 4.4] and proved in a similar way also in [3, Theorem 2.7].

Finally, the questions are open, whether any of the implications (5) $\longrightarrow$ (4) and (5)&(8) $\longrightarrow$ (1) holds for arbitrary cofinalities, and whether (8) can hold with arbitrary values of $\operatorname{cof}\theta \neq \operatorname{cof}\theta^*$ and $|\theta|$. A connected problem is the consistency of $\theta = \theta^* \neq \mu$.

§3. Any *isolated* subgroup Δ of ${}^*\mathbb{R}$ gives a *uniformity* $U(\Delta)$ with basis

$$\{\, E(x) \mid x \in {}^*\mathbb{R}^+\setminus\Delta \,\}, \quad \text{where} \quad E(x) = \{\, (u,v) \in {}^*\mathbb{R}^2 \mid |u-v| < x \,\}.$$

In particular the natural Hausdorff topology on ${}^*\mathbb{R}$ as a dense ordered group is given by the uniformity $U(0)$ (the *Q*-uniformity of [7]).

If we define the *rank* of x as $R_x = \{\, y \in {}^*\mathbb{R} \mid \exists n\in\mathbb{N}\ 1/n < x/y < n \,\}$, the isolated subgroups of ${}^*\mathbb{R}$ are in 1-1 correspondence with the Dedekind *cuts* in the (multiplicative) ordered group of the ranks $K = {}^*\mathbb{R}^+/R_1$, whose order-type is $\theta^*+1+\theta$.

For any $x \in {}^*\mathbb{R}^+$, let Δ_x^- (resp. Δ_x^+) be the *largest* (resp. *least*) isolated subgroup of ${}^*\mathbb{R}$ to which x *does not belong* (resp. *belongs*): then $U_x^- = U(\Delta_x^-)$ and $U_x^+ = U(\Delta_x^+)$ are two "adjacent" uniformities, associated to the cuts determined in K by the rank R_x. Other uniformities correspond to the "*monadic*" gaps of ${}^*\mathbb{R}$, i.e. those gaps which break no rank (see [7, §§5-6]).

Pseudometrizability and completeness of the uniformities $U(\Delta)$ depend on simple properties of the order-types θ and λ', namely: $U(\Delta)$ is complete iff ${}^*\mathbb{R}/\Delta$ has no regular gaps and $U(\Delta)$ is pseudometrizable iff ${}^*R^+\setminus\Delta$ has countable coinitiality.

Hence one has in any non-standard model:

LEMMA 1. *(i) All uniformities* U_x^- *are pseudometrizable;*

(ii) any uniformity U_x^+ *is pseudometrizable iff* $\mathrm{cof}\ \theta^* = \omega$;

(iii) the Q-uniformity $U(0)$ *is metrizable iff* $\mathrm{cof}\ \theta = \omega$;

(iv) any increasing or decreasing sequence in θ *is associated to a pseudometrizable uniformity* $U(\Delta)$.

It follows from Theorem 2 that, without the comprehensiveness hypothesis, $U(0)$ as well as all U_x^+'s can be or not be pseudometrizable; moreover, there are in any model pseudometrizable uniformities which correspond to monadic gaps of ${}^*R$: this answers completely Question 1 of [7] (partial answers can be found in [3] and [5]).

The question on completeness is more intriguing: we only derive here two sufficient conditions from the followig general lemma on dense ordered groups:

LEMMA 2. *Let* A *be any dense ordered group. Then:*

(i) A *has no regular gaps if it has a least positive gap;*

(ii) $\mathrm{cof}\ A^- \leq \min\{\mathrm{cof}\ X, \mathrm{cof}\ Y^*\}$ *for any regular gap* (X, Y) *in* A.

THEOREM 4. *(i) All uniformities* U_x^- *are complete in any non-standard model.*
(ii) Assume that in λ' *there are no cuts* (X,Y) *with* $\mathrm{cof}\ X = \mathrm{cof}\ Y^* = \kappa$: *then* $U(0)$ *(resp. any* U_x^+*) is complete whenever* $\mathrm{cof}\ \theta \geq \kappa$ *(resp.* $\mathrm{cof}\ \theta^* \geq \kappa$*). More generally,* $U(\Delta)$ *is complete unless* $\mathrm{cof}\ ({}^*R^-\setminus\Delta) < \kappa$.

Note that (i) above provides a new proof of a conjecture of Zakon's, which was already settled in [3, §4].

We conclude this section by remarking that the models $\mathcal{S}_\alpha$, given by Theorem 5 below, have regular gaps both in θ and in λ', whenever α is an ordinal with countable cofinality; hence $U(0)$ and all U_x^+'s are incomplete there. Unfortunately, no models are known, where any of these uniformities can be proved complete: therefore Questions 2-3 of [7] are still waiting for a satisfactory answer.

§4. We need only quite general model-theoretic techniques in order to obtain structures fulfilling the desired order-theoretic properties; hence all the results of this section hold, no matter what kind of structures we consider as models of the analysis. We assume an arbitrary ground model $\mathfrak{M}_0$ to be fixed throughout the section and we consider only elementary extensions of $\mathfrak{M}_0$, which we call simply models: thus our models can be elementary extensions of the "superstructure" $\Pi_\omega(R)$ or simply of R, but further conditions can be also required, e.g. that of being an enlargement (see [7, §1]).

We use the subscript 0 when referring to the order-types of the ground model, and we reserve the usual notation for entities of the various models we introduce in the theorems below.

The first theorem, which gives "symmetric" models, can be proved by taking the direct limits of elementary chains obtained by adding at each non-limit step a suitable set of constants and "filling" some cut.

THEOREM 5. *For any limit ordinal* α *there is a model* $\mathcal{S}_\alpha$ *verifying*

(4.1) $\mathrm{cof}\ \theta = \mathrm{cof}\ \theta^* = \mathrm{cof}\ \alpha$;

(4.2) $\theta = \mu$;

(4.3) $|\theta| = |\alpha||\mu_0|$.

Moreover, if (4.3) *is weakened to*

(4.4) $|\theta| \leq \beth_\alpha(|\theta_0|)$,

then $\mathcal{S}_\alpha$ *can be taken such that any cut* (X,Y), *either in* θ *or in* λ', *verifies at least one of the equalities*

(4.5) $\mathrm{cof}\ X = \mathrm{cof}\ \alpha$, $\mathrm{cof}\ Y^* = \mathrm{cof}\ \alpha$.

Note that, if α is inaccessible, then both (4.3) and (4.5) hold: actually, in this case, θ, μ and λ' are η_α-sets of Hausdorff.

In order to obtain "asymmetric" models, we state the following lemma, which improves [3, Lemma 2.3] and [5, Satz 2.1]:

LEMMA 3. *Let* D *be an* ω*-incomplete ultrafilter over* I *and let* F *be any filter of partitions of* I. *Let* $\mathcal{N}$ *and* $\mathcal{N}_1$ *be the corresponding ultralimits of the ground model* $\mathcal{M}_0$ *and of the standard model, respectively. Then* $\mathcal{N}_1 \prec \mathcal{N}$, *hence in particular* $\mathrm{R}^I/D|_F \prec {}^*\mathrm{R}^I/D|_F$, *and*

(*i*) *if* T *is an initial segment of* ${}^*\mathrm{N}_0$, *then* $T^I/D|_F$ *is an initial segment of* ${}^*\mathrm{N}$;

(*ii*) ν_1 *is an initial segment of* ν, *hence* $\theta = \theta_1 + (\theta_1{}^*{+}1{+}\theta_1)\xi$, *where* ξ *is a dense order-type without endpoints;*

(*iii*) θ_0 *is cofinal in* θ (*resp. coinitial in* $\theta\backslash\theta_1$) *iff* F *is coarser than the filter of all partitions of* I *in less than* $\mathrm{cof}\,\theta$ (*resp.* $\mathrm{cof}\,\theta^*$) *pieces.*

Taking the limits of elementary chains obtained by suitable applications of the above lemma at each non-limit step, we get

THEOREM 6. *Let* α *be a limit ordinal and* κ *any cardinal less than* $\mathrm{cof}\,\theta_0$. *There is a model* $\mathcal{U}_{\kappa,\alpha}$ *such that:*

(*i*) θ_0 *is cofinal in* θ (*hence* $\mathrm{cof}\,\theta = \mathrm{cof}\,\theta_0$),

(*ii*) $\mathrm{cof}\,\theta^* = \mathrm{cof}\,\alpha$,

(*iii*) $|\theta| \le (|\alpha||\theta_0|)^\kappa$,

(*iv*) *for any ordinal decomposition* $\alpha = \beta + \gamma$ *there is some* x *such that*
$$|\tau_x| = \sum_{\delta<\gamma} |\delta|^\kappa.$$

THEOREM 7. *For any limit ordinal* α *there is a model* $\mathcal{B}_\alpha$ *such that:*

(*i*) ${}^*\mathrm{N}_0$ *is an initial segment of* ${}^*\mathrm{N}$, *hence* $\theta = \theta_0 + (\theta_0{}^*{+}1{+}\theta_0)\xi$, *for a suitable dense order-type* ξ *without endpoints* (*in particular* $\mathrm{cof}\,\theta^* = \mathrm{cof}\,\theta_0{}^*$);

(*ii*) $\mathrm{cof}\,\theta = \mathrm{cof}\,\alpha$;

(*iii*) $|\theta| \le 2^{|\alpha||\mu_0|}$.

As remarked in §1, we still lack a general procedure for constructing models where all intervals of θ are isomorphic and $\mathrm{cof}\,\theta \ne \mathrm{cof}\,\theta^*$ are arbitrarily fixed. However, in order to completely state Theorem 3, any model of (8) & non-(5) suffices now, and a suitable direct limit of a sequence of κ-saturated models gives

COROLLARY 7.1. *Let* κ *be a regular cardinal verifying* $2^{<\kappa} = \kappa \ge |\mu_0|$. *Then there is a model* $\mathcal{C}_\kappa$ *such that:*

(*i*) $|\theta| = \mathrm{cof}\,\theta = \kappa$;

(*ii*) $\mathrm{cof}\,\theta^* = \omega$;

(*iii*) $\tau_x = \tau_y$ *for any* x,y.

REFERENCES

[1] M.FORTI & F.HONSELL - *On the order structure of the non-standard real axis* (in preparation).

[2] ——— - *Sull'ordinamento dei numeri reali non-standard.* Atti Accad. Naz. Lincei, Rend. Sci. Mat. Fis. Nat. (1986, to appear).

[3] S.KAMO - *Nonstandard Natural Number Systems and Nonstandard Models.* J. Symb. Logic 46 (1981), 365-376.

[4] ——— - *Nonstandard Real Number Systems with Regular Gaps.* Tsukuba J. Math. 5 (1981), 21-24.

[5] K.POTTHOF - *Ordnungseigenschaften von Nichtstandard Modellen*, in "*Theory of sets and Topology (in Honour of Felix Hausdorff)*" (G.Asser, J.Flaschmeyer & W.Rinow, eds.). VEB, Berlin 1972, 403-426.

[6] S.YANG, D.WANG & J.WANG - *On the order structure of the* ω*-regular extension hyperreal numbers* (chinese). Acta Math. Sin. 23 (1980), 658-667.

[7] E.ZAKON - *Remarks on the Nonstandard Real Axis*, in "*Applications of Model Theory to Algebra, Analysis and Probability*" (W.A.J.Luxemburg, ed.). New York 1969, 195-227.