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ON THE ABSTRACT MODEL THEORETIC NEIGHBOURHOOD OF THE LOGICS OF COMPUTER LANGUAGES I. COMPACTNESS

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Aims. The logics of computers languages split into two classes: the *programming logics*, which are theoretically implementable, and the *logics of programs*, which express some properties of the computers languages. The aim of this study is to isolate some relations between the compactness properties of such logics.

Basics. Referring to [BF] for the properties of the classic logics, we note that many logics of computers languages don't fall into this framework (see [Vi] for the particulaires). Whence, throughout this paper, a *logic* $\mathscr{L} = \langle Voc_{\mathscr{L}}, Str_{\mathscr{L}}, Stc_{\mathscr{L}}, \models_{\mathscr{L}} \rangle$ can be not regular and can have *(i)* a semantic domain strictly contained in the first-order one, *(ii)* a syntax which does not allow all the first-order sentences, and *(iii)* a satisfaction relation which does not satisfy the reduct, atom, negation, substitution, and relativization properties. Let us look at some new properties for such logics.

(RT) A logic $\mathscr{L}$ is *recursion-theoretic* if *(i)* for every recursive first-order L-structure $\|\mathfrak{A}\|$ there is an $\mathscr{L}$–vocabulary $H \supseteq L$ and an $H_{\mathscr{L}}$–structure $\mathfrak{A}^+$ such that $\|\mathfrak{A}^+\| \upharpoonright L = \|\mathfrak{A}\|$, and *(ii)* for every recursive $L_{\mathscr{L}}$–structure $\mathfrak{A}$, $Th_{\omega\omega}(\|\mathfrak{A}\|) = Th_{\mathscr{L}}(\mathfrak{A}) \cap Stc_{\omega\omega}(L)$. (See [BF, *XIX.2*] for the notations.)

(CS) A logic $\mathscr{L}$ (with dependence number $< \kappa$) has the *code structure property* if for every $\mathscr{L}$–theory $\mathbf{T} = \{\varphi_\alpha | \alpha < \kappa\}$ relative to a vocabulary $L = \{c_h, f_m, R_n \mid h,m,n<\omega\}$ and for every collection $\{\mathfrak{A}_\beta \mid \beta<\kappa\}$ of $\mathscr{L}$–models $\mathfrak{A}_\beta = \langle A_\beta, a_h, f_m, R_n$ $(h,m,n<\omega)\rangle$ of $T_\beta = \{\varphi_\alpha \mid \alpha<\beta\}$, $\beta<\kappa$ (of cardinality $\mu \geq \kappa$), there is an $\mathscr{L}$*–code structure*, i.e. an $\mathscr{L}$–structure $\mathfrak{H}$ which satisfies the following conditions:

(1) $\mathfrak{H} = \langle H, <, c_\beta, g, a_h, \bar{f}_m, \bar{R}_n (\beta<\kappa,\ h,m,n<\omega)\rangle$,

(2) there is an isomorphism $i: \langle field(<), <\rangle \cong \langle \kappa, <\rangle$ such that $i(c_\beta) = \beta$,

(3) if $x \in A \setminus field(<)$ then $g(x) = \beta < \kappa$,

(4) $\langle \{x \mid g(x) = \beta\}, a_h, \bar{f}_m(c_\beta,-,\ldots,-), \bar{R}_n(c_\beta,-,\ldots,-)$ $(h,m,n<\omega)\rangle \cong \mathfrak{A}_\beta$ for every $\beta<\kappa$.

(TOT) A logic $\mathscr{L}$ *expresses the totality* if given a function symbol f, there is an $\mathscr{L}$–vocabulary $L \supseteq \{f\}$ and a set $\Phi(f)$ of $L_{\mathscr{L}}$–sentences such that *(i)* $\Phi(f)$ has a countable $\mathscr{L}$–model and *(ii)* $\mathfrak{M}$ is a countable $\mathscr{L}$–model of $\Phi(f)$ iff $f^{\mathfrak{M}}$ is a total recursive function.

(WTOT) A logic $\mathscr{L}$ *expresses weakly the totality* if given a function symbol f, there is a constant symbol c, an $\mathscr{L}$–vocabulary $L \supseteq \{f,c\}$, and a set $\Phi(f)$ of $L_{\mathscr{L}}$–sentences such that *(i)* $\Phi(f)$ has a countable $\mathscr{L}$–model and *(ii)* $\mathfrak{M}$ is a countable $\mathscr{L}$–model of $\Phi(f)$ iff $f^{\mathfrak{M}} \restriction \{t^{\mathfrak{M}} \mid t$ is a closed L–term$\}$ is a total recursive function (on $\{t^{\mathfrak{M}} \mid t$ is a closed L–term$\}$).

Examples. For $\mathscr{L}_{\omega\omega}$, $\mathscr{L}^{ck}_{\omega_1\omega}$, $\mathscr{L}^{2w}$, $\mathscr{L}_{\omega\omega}(Q_0)$, and $\mathscr{L}_{\omega\omega}(\omega,<)$ see [BF, *II*]. $\mathscr{L}^{fin}_{\omega\omega}$ is the first-order logic restricted to the finite structures (see [Gu]) and $\mathscr{L}^{rc}_{\omega\omega}$ is the first-order logic restricted to the reachable ones. $\mathscr{L}^{2re}$ is the logic with relational variables where these variables range over r.e. relations. $\mathscr{L}_{\omega\omega}(Q_{re})$ and $\mathscr{L}_{\omega\omega}(Q^*_{re})$ are the logics in which the $\langle 1,1\rangle$-quantifiers Q_{re} and Q^*_{re} are defined respectively by

$\|\mathfrak{A}\| \models Q_{re}\, x,y\varphi(x,y)$ iff $\{\langle a,b\rangle \in A^2 \mid \|\mathfrak{A}\| \models \varphi[a,b]\}$ is r.e.

$\|\mathfrak{A}\| \models Q^*_{re}\, x,y\varphi(x,y)$ iff $\{\langle t^{\mathfrak{A}}, r^{\mathfrak{A}}\rangle \mid t,r$ are closed terms such that $\|\mathfrak{A}\| \models \varphi(t,r)\}$ is r.e.

$\mathscr{L}_e$ is the *effective logic* (see [Vi]). $\mathcal{DL}_{td}$ is the *non-standard dynamic logic* introduced in [ANS]. For this logic we have

	(RT)	(CS)	(TOT)	(WTOT)
$\mathscr{L}_{\omega\omega}$	yes	yes	no	no
$\mathscr{L}^{ck}_{\omega_1\omega}$	yes	yes	no	no
$\mathscr{L}_{\omega\omega}(Q_0)$	yes	yes	no	no
$\mathscr{L}_{\omega\omega}(\omega,<)$	yes	yes	no	no
$\mathscr{L}^{2w}$	yes	yes	no	no
$\mathscr{L}^{2re}$	yes	yes	yes	yes
$\mathscr{L}_{\omega\omega}(Q_{re})$	yes	yes	yes	yes
$\mathscr{L}_{\omega\omega}(Q^*_{re})$	yes	yes	no	yes
$\mathscr{L}_e$	yes	no	yes	no
$\mathscr{L}^{rc}_{\omega\omega}$	yes	no	no	no
$\mathscr{L}^{fin}_{\omega\omega}$	no	no	yes	yes
$\mathcal{DL}_{td}$	no	yes	no	no

Some abstract model-theoretic results. Recalling that $w_\kappa(\mathscr{L})$ is the *well-ordering number* of $\mathscr{L}$, that ω_1^{ck} is the first non-recursive ordinal, and that $\mathscr{L}$ is $\aleph_0$*–compact* if every countable $\mathscr{L}$–inconsistent set of $\mathscr{L}$–sentences has a finite $\mathscr{L}$–inconsistent subset, we have:

Result 1. *If $\mathscr{L}$ obeys to relativization,* (RT) *and* (CS), *then the following are equivalent:*

(i) $\omega_{\aleph_0}(\mathcal{L}) = \omega$.

(ii) $\mathcal{L}$ *is* $\aleph_0$*–compact.*

Result 2. *If* $\mathcal{L}$ *is a recursion-theoretic logic and* κ *is an infinite cardinal, then the following are equivalent:*

(i) $\mathcal{L}$ *expresses the totality with a set of at most* κ ***sentences.***

(ii) $\omega_\kappa(\mathcal{L}) \geqslant \omega_1^{ck}$.

Result 3. $\mathcal{L}_{\omega\omega}(Q^*_{re})$ *expresses weakly the totality but does not characterize* $(\omega,<)$.

Result 4. *If a recursion-theoretic logic* $\mathcal{L}$ *expresses weakly the totality with a countable set of* $\mathcal{L}$*–sentences, then* $\mathcal{L}$ *is not* $\aleph_0$*–compact.*

Note. The above results fall in a join research of the author with J.A. Makowsky of the *TECHNION – Israel Institure of Technology* of Haifa.

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