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UNIVERSAL ALGEBRAIC SEMANTICS

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The purpose of the present work is two-fold: to offer a new algebraic approach for the development of a universal syntax and semantics of formal languages (both machine and logic oriented) and to define a new general notion of logic, called ***universal logic***. The main motivation for introducing such logics is clearly expressed in Barwise's paper [Barw 85]: "...there are many topics [...] which are not usually considered part of extended model theory since they do not fit so well under the general framework that has been developed in abstract model theory. [...] the most glaring omission of this sort is work on the semantics and logic of computer languages. [...] in the long run, it seems that a unified view of logic and semantics will require us to come up with a framework that encompasses both fields, but we are far from such a conception at present."

Let U be a set of *sorts*. We say that A is a U-*set* if A is just a set with a *sort assignment*, which is a function s_A:A→ U. The set $A_u = \{a \in A : s_A(a) = u\}$ is called set of elements of sort u. If A and B are U-sets, then a U-map is a function f: A → B such that $s_B \circ f = s_A$. If A and B are U-sets, it is convenient to denote by ${}^{A}B$ or (A → B) the set of all the U-maps from A into B.

1. Definition. An *α-dimensional domain* F *of formulas* is a quintuple <U,α,L,τ,σ> such that

(i) U is a set of sorts;

(ii) α is a U-ordinal, called set of *variables*;

(iii) L is a set, called set of *formulas*. Given a formula F, the U-ordinal ρF is called *rank* of F;

(iv) Given a formula F, $\tau(F) = \tau_F$: ρF → α is a U-map called *assignment*;

(v) Given a U-map $\delta \in {}^{\alpha}\alpha$, $\sigma(\delta) = \sigma_\delta$: L •→ L is a partial map called *substitution*.

The following axioms are satisfied for every F, G ∈ L and $\varepsilon, \gamma, \delta \in {}^{\alpha}\alpha$:

(a) If σ_δ is defined on F (hereto denoted by $\sigma_\delta \downarrow F$), then $\tau_{\sigma_\delta F} = \delta \circ \tau_F$;

(b) If $\gamma \circ \tau_F = \delta \circ \tau_F$, then $\sigma_\gamma F =_d \sigma_\delta F$; (Refer to the list of notation for the definition of $=_d$)

(c) $\sigma_{id}F = F$ (id: α → α is the identity function);

(d) $\sigma_{\gamma \circ \delta} = \sigma_\gamma \circ \sigma_\delta$;

(e) If $\sigma_\gamma \downarrow F$, $\sigma_\varepsilon \downarrow F$ and $\tau_{\sigma_\gamma F} = \delta \circ \tau_{\sigma_\varepsilon F}$, then $\sigma_{\delta \circ \varepsilon} F = \sigma_\gamma F$.

(f) If $\sigma_{\delta 1}F = \sigma_{\delta 2}G$ for two substitutions $\sigma_{\delta 1}$ and $\sigma_{\delta 2}$, then there exist H ∈ L and $\sigma_{\gamma 1}$, $\sigma_{\gamma 2}$ such that $\sigma_{\gamma 1}H = F$ and $\sigma_{\gamma 2}H = G$. ♦

Each element F ∈ L is called *formula* of rank ρF. If $\tau_F(\lambda) = \kappa$, the variable v_κ is said to *occur free* in the λ^{th} place of F. The interpretation of the axioms is straightforward: $\sigma_\delta F$ is the formula obtained from F by substituting, for every $\kappa \in \alpha$, the variable v_κ with the variable $v_{\delta(\kappa)}$.

The following definition introduces a binary relation $\Diamond$ on L, which expresses the property: "to have the same predicate symbol".

2. Definition. Let F,G $\in$ L. We say that F and G *have the same predicate symbol* and we write F$\Diamond$G iff there exist a formula H and two substitutions σ_δ, σ_γ such that σ_δH =F and σ_γH=G.♦

Each equivalence class $[F]_\Diamond$ is said to be a *predicate symbol*. If G $\in$ $[F]_\Diamond$, G is said to be a formula of predicate symbol $[F]_\Diamond$ and, if $\lambda = \tau_F$, we write F = $[F]_\Diamond(v \bullet \lambda)$.

3. Definition. A *logic signature* Δ of dimension α is a quintuple $<U,\alpha,\Sigma,\eta,\Xi>$ where

(a) U is a set of sorts;

(b) α is a U-ordinal, called set of *variables*;

(c) Σ is an ω-set, called set of *non-logical operators*;

(d) Ξ is an ω-set, called set of *logical operators*;

(e) For every operator f, $\eta(f)$ is the set of variables quantified by f. If $<i,\lambda> \in \eta(f)$ ($\lambda \in \alpha$), we say that the logical operator f *quantifies* the variable v_λ in its i^{th} argument;

(f) $\eta(f) = \emptyset$ for every non-logical operator $f \in \Sigma$.

The index n $\in$ ω of an operator $f \in \Sigma \cup \Xi$ is its *arity* and, if $\eta(f)=\{<i_1,\lambda_1>,....\}$, we call f a *quantifier on* $<i_1,\lambda_1>,....$ ♦

Some variables are possibly quantified by an operator f because of Property (e). For example, the binary operation $\mathbf{h}_{\kappa\lambda}$ defined by $\mathbf{h}_{\kappa\lambda}(F_1,F_2) = \exists v_\kappa(F_1) \wedge \forall v_\lambda(F_2)$ where F_1 and F_2 are first-order formulas, is a simple example of quantifier on the variable v_κ in the first argument and on the variable v_λ in the second one.

4. Example

1. (*first-order signature*) Let U = {0}. The first-order signature $\Delta_{\omega\omega} = <U,\alpha,\eta,\Xi_{\omega\omega}>$ of dimension α consists of the following logical operators: three sentential connectives: the *disjunction symbol* $\vee$, the *conjunction symbol* $\wedge$ and the *negation symbol* $\neg$; the *existential quantifiers* c_κ ($\kappa \in \alpha$), the *truth symbol* T_o, the *falsehood symbol* F_o and the *equality symbols* $d_{\kappa\mu}$ ($\mu,\kappa \in \alpha$). The logical operators have the following arity: $\Xi_0 = \{T_o,F_o,d_{\kappa\mu},...\}$; $\Xi_1 = \{\neg,c_\kappa,...\}$; $\Xi_2 = \{\vee,\wedge\}$; $\eta(\neg) = \eta(\vee) = \eta(\wedge) = \eta(F_o) = \eta(T_o) = \eta(d_{\kappa\mu}) = \emptyset$; $\eta(c_\kappa) = \{<1,\kappa>\}$ for every κ.
2. (λ-*calculus signature*) Let U = {0}. The λ-calculus signature $\Delta_\lambda = <U,\alpha,\eta,\Xi_\lambda>$ of dimension α consists of the following logical operators: the *projection symbols* π_κ ($\kappa \in \alpha$), the λ-*abstraction quantifiers* λ_κ ($\kappa \in \alpha$), and the *application symbol* **App**. The operators have the following arity: $\Xi_0 = \{\pi_\kappa,...\}$; $\Xi_1 = \{\lambda_\kappa,...\}$; $\Xi_2 = \{\mathbf{App}\}$; $\eta(\mathbf{App}) = \eta(\pi_\kappa) = \emptyset$; $\eta(\lambda_\kappa) = \{<1,\kappa>\}$ for every $\kappa \in \alpha$.
3. (*computer signature*) Let U = {0}. The computer signature $\Delta_c = <U,\alpha,\eta,\Xi_c>$ of dimension α consists of the following logical operators: the *assignment symbols* $\mathbf{assign}_{\kappa\mu}$ ($\mu,\kappa \in \alpha$), the *sequencing symbol* ; , the *union symbol* $\cup$, the *star symbol* *, and the *test symbols* $?_{\kappa\mu}$ ($\kappa,\mu \in \alpha$). The operators have the following arity $\Xi_0 = \{\mathbf{assign}_{\kappa\mu},...,?_{\kappa\mu},...\}$; $\Xi_1 = \{*\}$; $\Xi_2 = \{;,\cup\}$; $\eta(\mathbf{assign}_{\kappa\mu}) = \eta(*) = \eta(;) = \eta(\cup) = \eta(?_{\kappa\mu}) = \emptyset$ for every $\kappa,\mu \in \alpha$.♦

Let β be a subset of U-ordinal α. We call a U-function $\delta \in {}^\alpha\alpha$ *suitable for* β if $\delta(\lambda) = \lambda$ for every $\lambda \in \beta$, and $\delta(\lambda) \notin \beta$ if $\lambda \notin \beta$. In the sequel, we let "δ is suitable for the operator f" stand for "δ is suitable for $\{\lambda : <r,\lambda> \in \eta(f)$ for some r}".

5. Definition. Let Δ be a logic signature. By a Δ-*language* **L** *of dimension* α we understand a pair <F,**L**> where F is an α-dimensional domain of formulas and **L** is a $(\Sigma \cup \Xi)$-algebra of carrier L (set of formulas). The following axioms are satisfied for every $f \in \Sigma \cup \Xi$, formulas $F_1,...,F_n \in$ L and function $\delta \in {}^\alpha\alpha$ suitable for f:

(a) *quantification property* : If $F = f(F_1,...,F_n)$, then

$\lambda \in Rg\ \tau_F$ iff there exists an i such that $\lambda \in Rg\ \tau_{F_i}$ and $<i,\lambda> \notin \eta(f)$;

(b) *endomorphism property* : Whenever $\sigma_\delta(f(F_1,...,F_n))$ is defined, then $f(\sigma_\delta F_1,...,\sigma_\delta F_n)$ is also defined and $\sigma_\delta(f(F_1,...,F_n)) = f(\sigma_\delta F_1,...,\sigma_\delta F_n)$. ♦

We generalize the standard notion of semantic domain or possible world so as to include those logics which deal with computer languages.

6. Definition. The set Tp of *derived types* is defined as follows:

1. $\mathbf{0} \in$ Tp;
2. If p $\in$ Tp, then **Pow**(p) $\in$ Tp;
3. If ($p_i \in$ Tp: i $\in$ I) is an I-indexed family of types, then $(\Pi_{i\in I}p_i) \in$ Tp. ♦

7. Definition. Let N be a U-set. Then the α-*ary generalized Cartesian space* p *with base* N associated with (or induced by) the type p is defined as follows:

(i) **0** induces the Cartesian power ${}^\alpha N = \{x : x: \alpha \to N$ is a U-map, i.e. $x_\lambda \in N_u$ iff $s_\alpha(\lambda) = u\}$;

(ii) if p = **Pow**(p_o) and p_o induces p_o, then p induces $Pow_\emptyset(p_o)$ (i.e. $Pow(p_o)\setminus\{\emptyset\}$);

(iii) if p = $(\Pi_{i\in I}p_i)$ and p_i induces p_i (i $\in$ I), then p induces $\Pi_{i\in I}p_i$ (Cartesian product). ♦

If p is a type, $p_N{}^\alpha$ will denote the α-ary generalized Cartesian space with base N induced by **p**. When there is no ambiguity we write p^α for $p_N{}^\alpha$. It is possible to give now the definition of possible world or semantic domain.

8. Definition. An α-ary *possible world* W with base N is the set $(p^\alpha \to A)$ of the functions, the domain of which is an α-ary generalized Cartesian space p^α with base N, and the range is a set A.♦

For example, if U = ω and A is a set, then the ω-set N defined by N_0 = A; $N_n = Pow({}^nA)$ for every n > 0, is the base related to second-order logic. The possible world $(\mathbf{0}^\alpha \to Bool) = Pow({}^\alpha N)$ is the semantic domain of second-order logic.

Let p^α be an α-ary generalized Cartesian space and $\beta \subseteq \alpha$ be a U-ordinal. The composition function $\bullet : p^\alpha \times {}^\beta\alpha \to p^\beta$ and the restriction function $\rceil$ are defined as follows, for every element x $\in p^\alpha$, subset $\Gamma \subseteq \alpha$ and map $\lambda \in {}^\beta\alpha$:

(1) p = **0**	$x \bullet \lambda = x \circ \lambda$;	$\Gamma\rceil x = \Gamma \upharpoonright x$;
(2) p = **Pow**(p_1)	$x \bullet \lambda = \{y \bullet \lambda : y \in x\}$;	$\Gamma\rceil x = \{\Gamma\rceil y : y \in x\}$;

(3) $p = \Pi_{i \in I} p_i$ $\qquad (x \bullet \lambda)_i = x_i \bullet \lambda \ (i \in I);$ $\qquad (\Gamma \rceil x)_i = \Gamma \rceil x_i \ (i \in I).$

In what follows it is necessary to extend the *substitutions* also to possible worlds: $(\sigma_\delta Z)x = Z(x \bullet \delta)$ for every $Z \in (p^\alpha \to A)$, $x \in p^\alpha$ and $\delta \in {}^\alpha\alpha$.

The generalized cylindric operations Q_Γ ($\Gamma \subseteq \alpha$) on possible worlds can be defined (see [Sal 86]). These operations generalize the usual cylindric operations of polyadic set algebras to each possible world $(p^\alpha \to A)$: $Q_\Gamma(Z)$ is the generalized "cylinder" generated by translating Z parallel to the Γ^{th} axis of the space p^α. Furthermore, by using the following logical interpretation of generalized cylindrifications,

$Q_\Gamma(Z) = Z$ (i.e. Z is a Γ-cylinder) iff Z does not depend from the variables v_λ with $\lambda \in \Gamma$,

the notions of semantic quantifier and interpretation morphism can be stated.

9. Definition. Let Δ be a logic signature and $W = (p^\alpha \to A)$ be a possible world with base N. Then we say that

(i) $\underline{D}$ is a *weak semantic set Δ-algebra of dimension α with base* N (in symbols $\underline{D} \in WSS_{N,\Delta}$) provided that $\underline{D}$ is a $(\Sigma \cup \Xi)$-algebra, the universe of which is a possible world W with base N. $\underline{D}$ has to verify the following conditions for every operator $f \in \Sigma \cup \Xi$, $Z_1,\dots,Z_n \in W$, and function $\delta \in {}^\alpha\alpha$ suitable for f:

(a) *endomorphism set property*:

$$\sigma_\delta[f(Z_1,\dots,Z_n)] = f(\sigma_\delta Z_1,\dots,\sigma_\delta Z_n);$$

(b) *quantification set property*: let Γ be a subset of α, let $\Gamma_i = \{\lambda \in \Gamma : \langle i,\lambda\rangle \notin . \eta(f)\}$, and let $Q_{\Gamma i} Z_i = Z_i$ for every i=1,...,n. Then $Q_\Gamma f(Z_1,\dots,Z_n) = f(Z_1,\dots,Z_n)$

(ii) $\underline{D}$ is a *semantic set Δ-algebra of dimension α with base* N (in symbols $\underline{D} \in SS_{N,\Delta}$) provided that $\underline{D} = \langle D, f,\dots,Q_\Gamma,\dots,\sigma_\delta,\dots\rangle_{\Gamma \subseteq \alpha, \delta \in {}^\alpha\alpha}$ and the reduct $\langle D,f,\dots\rangle$ is a weak semantic set Δ-algebra with base N. ♦

Remark. Quantification set property clarifies the notion of semantic quantifier. If

- the sets $Z_1,\dots,Z_{i-1},Z_{i+1},\dots,Z_n$ does not depend from the variable v_λ (that is, $Q_{\{\lambda\}} Z_k = Z_k$ for every $k \neq i$),
- the set Z_i depends from the variable v_λ (that is, $Q_{\{\lambda\}} Z_i \neq Z_i$), and
- the operator f quantifies the variable v_λ in the i^{th} argument, then

$f(Z_1,\dots,Z_n)$ does not depend from v_λ (that is, $Q_{\{\lambda\}} f(Z_1,\dots,Z_n) = f(Z_1,\dots,Z_n)$). ♦

10. Example.

(a) (*Existential quantifier*) The cylindric operation C_λ: Pow$({}^\alpha A) \to$ Pow$({}^\alpha A)$, defined by $C_\lambda(Z) = \{x \in {}^\alpha A : x[\lambda/d] \in Z$ for some $d \in A\}$ is a semantic quantifier which quantifies the variable v_λ (see E.4.(1)).

(b) (*λ-abstraction*) A triple $\langle A,\times,\psi\rangle$ is called a *λ-calculus algebra* if

- $(A,\times)$ is an applicative structure (that is, A is a set and $\times$ is a binary operation on A).

- $\psi: (A \to A) \to A$ is a function verifying the condition $\psi(f) \times a = f(a)$ for every $a \in A$ and every function $f \in (A \to A)$ representable in $(A,\times)$. A function $f: A \to A$ is representable in $(A,\times)$ if there exists $m \in A$ such that $f(a) = m \times a$ for every $a \in A$. Then, the λ-abstraction $\Lambda_\kappa: ({}^\alpha A \to A) \to ({}^\alpha A \to A)$ associated with the λ-calculus algebra $\langle A,\times,\psi\rangle$, is defined as follows:

$$\Lambda_\kappa(Z)y = \psi(f) \text{ for every } Z \in ({}^\alpha A \to A) \text{ and every } y \in {}^\alpha A,$$

where the function $f: A \to A$ is defined by $f(a) = Z(y[\kappa/a])$ for every $a \in A$. Function Λ_κ is a semantic quantifier which quantifies the variable v_κ (see E.4.(2)). ♦

The general notion of a semantic algebra is obtained by abstraction from the notion of semantic set algebra.

11. Definition. Let Δ be a logic signature. By a *semantic Δ-algebra of dimension* α we mean an algebraic structure $\underline{D} = \langle D, f,\dots, q_\Gamma,\dots, \sigma_\gamma \dots\rangle_{\Gamma \subseteq \alpha, \gamma \in {}^\alpha\alpha, f \in \Sigma \cup \Xi}$ (in simbols $\underline{D} \in SA_{\alpha,\Delta}$) such that $\langle D,f,\dots\rangle$ is a $(\Sigma \cup \Xi)$-algebra, and q_Γ and σ_γ are unary operations on D. The following axioms are satisfied for every $f \in (\Sigma \cup \Xi)_n$, $x,x_1,\dots,x_n \in D$, $\Gamma,\Omega \subseteq \alpha$, and $\gamma,\delta \in {}^\alpha\alpha$:

(P1) The reduct $\langle D,f,\dots\rangle$ is a $(\Sigma \cup \Xi)$-algebra; (P2) $q_\emptyset x = x$;

(P3) $\sigma_{Id} x = x$; (P4) $\sigma_{\gamma \circ \delta} = \sigma_\gamma \circ \sigma_\delta$;

(P5) If $(\alpha \backslash \Gamma) \upharpoonright \gamma = (\alpha \backslash \Gamma) \upharpoonright \delta$, then $\sigma_\gamma q_\Gamma = \sigma_\delta q_\Gamma$; (P6) If $\Gamma \subseteq \Omega$ and $q_\Omega x = x$, then $q_\Gamma x = x$;

(P7) If $q_\Gamma x = x$ and $q_\Omega x = x$, then $q_{(\Gamma \cup \Omega)} x = x$;

(P8) If $\delta \in {}^\alpha\alpha$ is suitable for f, then $\sigma_\delta[f(x_1,\dots,x_n)] = f(\sigma_\delta x_1,\dots,\sigma_\delta x_n)$;

(P9) Let Γ be a subset of α, let $\Gamma_i = \{\lambda \in \Gamma : \langle i,\lambda\rangle \notin \eta(f)\}$, and let $q_{\Gamma i} x_i = x_i$ for every i=1,...,n. Then $q_\Gamma f(x_1,\dots,x_n) = f(x_1,\dots,x_n)$. ♦

12. Example. (*λ-calculus logic*)

Let Δ_λ be the λ-calculus signature defined in E.4.(2) and $\langle A,\times,\psi\rangle$ be a λ-calculus algebra (see E.10.(b)). By the *λ-calculus set Δ_λ-algebra* associated with $\langle A,\times,\psi\rangle$ we mean a weak semantic set Δ_λ-algebra:

$\underline{M}_\lambda = \langle ({}^\alpha A \to A), \mathbf{APP}, \Pi_\kappa, \Lambda_\kappa, \dots\rangle_{\kappa \in \alpha}$ such that

$(Z\ \mathbf{APP}\ T)y = (Zy) \times (Ty)$ ($\times$ is the binary operation on A); $\qquad \Pi_\kappa y = y_\kappa$

Λ_κ is the unary quantifier defined in E.10.(b)

for every $Z,T \in ({}^\alpha A \to A)$ and every $y \in {}^\alpha A$.

By a *λ-calculus semantic Δ_λ-algebra of dimension* α we mean an algebraic structure

$$\underline{B} = \langle B, \mathrm{App}, \pi_\kappa, \lambda_\kappa, \dots, q_\Gamma, \sigma_\gamma\rangle_{\kappa \in \alpha, \Gamma \subseteq \alpha, \gamma \in {}^\alpha\alpha}$$

such that the following axioms are satisfied:

1. $\underline{B}$ is a semantic Δ_λ-algebra of dimension α; 2. $q_\Gamma \pi_\kappa = \pi_\kappa$ for every $\Gamma \cap \{\kappa\} = \emptyset$;

3. $\sigma_\gamma \pi_\kappa = \pi_{\gamma\kappa}$; 4. $kuy = u$ with $k = \lambda_\kappa \lambda_\mu \pi_\kappa$;

5. $suyz = (uz)(yz)$ with $s = \lambda_\kappa \lambda_\mu \lambda_\nu [\pi_\kappa \pi_\nu (\pi_\mu \pi_\nu)]$; 6. $\varepsilon xy = xy$ with $\varepsilon = (\lambda_\kappa \lambda_\mu (\pi_\kappa \pi_\mu))$;

7. if $\forall z\ (xz = yz)$, then $\varepsilon x = \varepsilon y$;

where xy stands for x **App** y, and $(x_1 x_2 .... x_n)$ stands for $(...(x_1 x_2)x_3)...)x_n)$. ♦

13. Example. (*first-order logic*)

Let $\Delta_{\omega\omega}$ be the first-order signature defined in E.4.(1). By a *first-order set* $\Delta_{\omega\omega}$*-algebra of dimension* α we mean a cylindric set algebra of dimension α. By a *first-order semantic* $\Delta_{\omega\omega}$*-algebra of dimension* α we mean an algebraic structure

$$\underline{B} = \langle B,\vee,\wedge,\neg,T_o,F_o,c_\kappa,d_{\kappa\lambda},...,q_\Gamma,\sigma_\gamma\rangle_{\kappa,\lambda\in\alpha,\Gamma\subseteq\alpha,\gamma\in{}^\alpha\alpha}$$

such that the following axioms are satisfied: 1. $\underline{B}$ is a semantic $\Delta_{\omega\omega}$-algebra of dimension α;

2. the reduct $\langle B,\vee,\wedge,\neg,T_o,F_o,c_\kappa,d_{\kappa\lambda}\rangle_{\kappa,\lambda\in\alpha}$ is a cylindric algebra of dimension α;

3. $x \wedge d_{\kappa\lambda} \le \sigma_{[\kappa/\lambda]}x$ where $x \le y$ iff $x = x \wedge y$, and $[\kappa/\lambda] \in {}^\alpha\alpha$ is so defined: $[\kappa/\lambda]\mu = \mu$ if $\mu\neq\kappa$, λ if $\mu=\kappa$; 4. $\sigma_\gamma d_{\kappa\lambda} = d_{\gamma\kappa,\gamma\lambda}$; 5. $c_\kappa q_\Gamma = q_\Gamma$ if $\{\kappa\} \subseteq \Gamma$. ♦

14. Example. (*computer logic*)

Let Δ_c be the computer signature defined in E.4.(3). By the *computer semantic set* Δ_c*-algebra* we mean a weak semantic set Δ_c-algebra: $\underline{B}_c = \langle Pow({}^\alpha A\times{}^\alpha A), assign_{\kappa\mu}, ?_{\kappa\mu}, ;, \cup, *,...\rangle_{\kappa,\mu\in\alpha}$ such that

$\langle x,y\rangle \in assign_{\kappa\mu}$ iff $y_\lambda = x_\lambda$ for every $\lambda\neq\kappa$, and $y_\kappa = x_\mu$ (assignment);

$\langle x,y\rangle \in\ ?_{\kappa\mu}$ iff $x = y$ and $x_\kappa = x_\mu$ (testing for =);

$\langle x,y\rangle \in Z_1 ; Z_2$ iff there exists z such that $\langle x,z\rangle \in Z_1$ and $\langle z,y\rangle \in Z_2$ (sequencing);

$\langle x,y\rangle \in Z_1 \cup Z_2$ iff $\langle x,y\rangle \in Z_1$ or $\langle x,y\rangle\in Z_2$ (non-deterministic union);

Z^* = the reflexive and transitive closure of the binary relation Z (iteration).

By a *computer semantic* Δ_c*-algebra of dimension* α we mean an algebraic structure

$$\underline{B} = \langle B, assign_{\kappa\mu},?_{\kappa\mu}, ;,\cup,*,...,q_\Gamma,\sigma_\gamma\rangle_{\kappa,\mu\in\alpha,\Gamma\subseteq\alpha,\gamma\in{}^\alpha\alpha}$$

such that the following axioms are satisfied:

1. $\underline{B}$ is a semantic Δ_c-algebra of dimension α;
2. $(x ; y) ; z = x ; (y ; z)$
3. $(?_{\kappa\mu} ; assign_{\kappa\mu}) = ?_{\kappa\mu}$
4. $(x \cup y) \cup z = x \cup (y \cup z)$
5. $(x \cup y) = (y \cup x)$
6. $(x^* ; x) = x^*$ and $(x ; x^*) = x^*$
7. $\sigma_\gamma(assign_{\kappa\mu}) = assign_{\gamma(\kappa)\gamma(\mu)}$
8. $\sigma_\gamma(?_{\kappa\mu}) = ?_{\gamma(\kappa)\gamma(\mu)}$. ♦

15. Definition. . An *interpretation morphism* from $\underline{L}$ to $\underline{D}$ is a morphism I: $\underline{L} \to \underline{D}$ verifying, for every $F\in L$ and $\sigma_\delta\!\downarrow\! F$, the following conditions: (i) $Q_{(\alpha\backslash Rg\ \tau_F)}I(F) = I(F)$;

(ii) $\sigma_\delta(I(F)) = I(\sigma_\delta F)$. ♦

References

[Barw 85] Barwise J., **Model-Theoretic Logics: Background and Aims.** In Barwise, J. Feferman, S. (Eds.): *Model-Theoretic Logics*, 1985.

[Sal 86] Salibra A., **A Universal Algebraic Approach to Languages, Logics and Semantics,** Universita' di Pisa, 1986.