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VALIDITY AND COMPLETENESS FOR INTUITIONISTIC PREDICATE LOGIC

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The text below is a shortened version of the two lectures presented at the conference. Full details may be found in the book "Constructivism in Mathematics" by A.S. Troelstra and D. van Dalen, chapter 13, sections 1 and 2. We shall in particular discuss the connection between "intuitive intuitionistic validity" and validity in Beth models, and describe an intuitionistic completeness proof due to H.M. Friedman.

Preliminaries.

$(K,\preccurlyeq)$ is said to be a spread if K is a tree of finite sequences of natural numbers, partially ordered by the initial-segment relation $\prec$ such that

$$k \in K \vee k \notin K,$$
$$k \in K \wedge k' \preccurlyeq k \Rightarrow k' \in K, \quad \langle\rangle \in K,$$
$$k \in K \Rightarrow \exists n \in \mathbb{N}(k * \langle n \rangle \in K)$$

($*$ is concatenation). A fan is a finitely branching spread. We let k, k', k" range over the nodes of K, and α, β, γ over infinite branches of the spread. So, if

$$\bar{\alpha}n = \langle \alpha 0, \ldots, \alpha(n-1)\rangle, \bar{\alpha}0 = \langle\,\rangle,$$

then

$$\alpha \in K \Leftrightarrow \forall n(\bar{\alpha}n \in K).$$

DEFINITION. A Beth model is a quadruple $\mathcal{B} \equiv (K,\preccurlyeq,D,\Vdash)$ such that
(i) $(K,\preccurlyeq)$ is a spread;
(ii) D is a domain, i.e. an inhabited set;
(iii) the forcing relation $\Vdash$ between elements of K and prime sentences P with constants from D satisfies

B1 $\quad k \Vdash P \Leftrightarrow \forall \alpha \in k \exists m(\bar{\alpha}m \Vdash P), \quad k \nVdash \bot$ for all $k \in K$. □

Note that B1 implies monotonicity of forcing, i.e. if $k \Vdash P$ and $k' \succcurlyeq k$ then $k' \Vdash P$.

DEFINITION. For *compound* sentences with constants from D we extend $\Vdash$ by the clauses

B2	$k \Vdash A \wedge B$	$:= k \Vdash A$ and $k \Vdash B$;
B3	$k \Vdash A \vee B$	$:= \forall\alpha\in k\exists n(\bar{\alpha}n \Vdash A$ or $\bar{\alpha}n \Vdash B)$
B4	$k \Vdash A \to B$	$:= \forall k' \geqslant k(k' \Vdash A \Rightarrow k' \Vdash B)$
B5	$k \Vdash \exists x\, A(x)$	$:= \forall\alpha\in k\exists n\exists d\in D\,(\bar{\alpha}n \Vdash A(d))$
B6	$k \Vdash \forall x\, A(x)$	$:= \forall d\in D\,(k \Vdash A(d))$. □

We have the following

1.2. LEMMA. For all sentences A

(i) $\forall\alpha\in k\exists n(\bar{\alpha}n \Vdash A) \Rightarrow k \Vdash A$ (*covering property*),
which is classically equivalent to $k \nVdash A \Rightarrow \exists\alpha\in k\forall n(\bar{\alpha}n \nVdash A)$;

(ii) $k' \geqslant k$ and $k \Vdash A \Rightarrow k' \Vdash A$ (*monotonicity*). □

REMARK. Beth models are topological models: giving trees their usual topology with basis the collection of sets $V_n := \{\alpha : \alpha \in n\}$, the forcing definition assigns to each sentence A the open set $[\![A]\!] = \bigcup\{V_n : n \Vdash A\}$.

The preceding clauses B1-6 constitute the so-called *weak definition of forcing*. Let now $(K,\leqslant)$ be a fan. We can then give a strong definition (classically equivalent to the weak definition) for B1, B3, B5; with the abbreviation

$$\forall k' \geqslant_z k(\ldots) := \forall k' \geqslant k(\mathrm{lth}(k') = \mathrm{lth}(k) + z \to \ldots)$$

the clauses become:

B1'	$k \Vdash P \Leftrightarrow \exists z\, \forall k' \geqslant_z k\, \exists k'' \leqslant k' (k'' \Vdash P)$
	or $k \Vdash P \Leftrightarrow \exists m \forall\alpha\in k\exists n \leq m(\bar{\alpha}n \Vdash P)$;
B3'	$k \Vdash A \vee B := \exists z \forall k' \geqslant_z k(k' \Vdash A$ or $k' \Vdash B)$
	or $k \Vdash A \vee B := \exists m \forall\alpha\in k(\bar{\alpha}m \Vdash A$ or $\bar{\alpha}m \Vdash B)$;
B5'	$k \Vdash \exists x\, A(x) := \exists z\, \forall k' \geqslant_z k\, \exists d\in D(k' \Vdash A(d))$
	or $k \Vdash \exists x\, A(x) := \exists m \forall\alpha\in k\exists d\in D(\bar{\alpha}m \Vdash A(d))$.

If we assume α to range over branches *dense* in the tree $(K,\leqslant)$, i.e. $\forall k\exists\alpha(\alpha \in k)$, then the second variant listed after "or" under each of these three cases is equivalent to the first one.

DEFINITION. We call the forcing definition based on B1', B2, B3', B4, B5' *the strong definition of forcing*. □

If we assume *compactness* of the space of infinite branches (intuitionistically: the fan theorem), the strong definition is equivalent to the weak definition.

Intuitionistic validity.

For simplicity we restrict from now on attention to the language of pure predicate logic without equality and function symbols.

Let $F(R_1, \ldots, R_n)$ be any formula of intuitionistic predicate logic **IQC**, containing the predicate symbols $R_1, \ldots, R_n$ (and no others) and suppose R_i to have n_i argument places.

F is *intuitionistically valid* ("valid" for short, or $\vDash_i F$) by

$$\vDash_i F(R_1, \ldots, R_n) := \forall D \forall \underline{R}_1 \ldots \underline{R}_n F^D(\underline{R}_1, \ldots, \underline{R}_n),$$

where D ranges over all intuitionistically meaningful domains, and $\underline{R}_i$ ranges over all n_i-ary relations over D; $F^D(\underline{R}_1, \ldots, \underline{R}_n)$ results from F by relativizing all quantifiers to D, and substituting $\underline{R}_i$ for R_i.

Of course, the logical operations in the right hand formula should be read intuitionistically. This definition is completely similar to Tarski's definition of truth.

Lawless sequences.

The lawless sequences are an extreme version of Brouwer's notion of choice sequence. A lawless sequence is a process α of producing values $\alpha 0$, $\alpha 1$, $\alpha 2$, ... such that at any moment only a *finite* initial segment is known.

Let $(K,\leqslant)$ be a fan as before, and let LS_K be the *lawless* infinite branches of the spread, i.e. sequences subject only to the restriction that their initial segments have to belong to K. We shall assume the following principles:

Density $\forall n\exists\alpha(\alpha \in n)$,

Open data $A(\alpha) \to \exists n(\alpha \in n \wedge \forall\beta\in n\, A(\beta))$

(a schema for A containing at most one lawless parameter, motivated by the fact that $A(\alpha)$ must be asserted on the basis of knowledge of an initial segment only).

Fan axiom (FAN); A does not contain choice parameters:

$$\forall\alpha\exists n\, A(\bar{\alpha}n) \to \exists m \forall\alpha\exists n \leq m\, A(\bar{\alpha}m).$$

The justification of FAN involves the same problems as arise in the case of Brouwer's bar theorem, and we shall not discuss that topic here. We only note that the special case of the decidable fan theorem

FAN_D $\quad \forall n(An \vee \neg An) \wedge \forall\alpha\exists n\, A(\bar{\alpha}n) \rightarrow \exists m\forall\alpha\exists n{\le}m\, A(\bar{\alpha}m)$

together with the neighbourhood function principle

NFP $\quad \forall\alpha\exists n\, A(\bar{\alpha}n) \rightarrow \exists f{\in}K_0\forall n(fn{\ne}0 \rightarrow \exists m\forall\alpha{\in}n\, A(\bar{\alpha}n))$,

where K_0 is the set of neighbourhood functions, yields FAN.

Connection between intuitionistic validity and Beth models.

Let $\mathfrak{B} \equiv (K,\preccurlyeq,\Vdash,D)$ be a Beth model, $(K,\preccurlyeq)$ a fan and let $\langle R_n\rangle_n$ enumerate the proposition- and relation symbols, R_n with $r(n)$ argument places. Let $\langle \underline{R}_n\rangle_n$ be a sequence of relations, $\underline{R}_n \subset LS_{\underline{K}} \times D^{r(n)}$. We write $\underline{R}_n{}^{\alpha}(\vec{t})$ for $\underline{R}_n(\alpha,\vec{t})$.

PROPOSITION. Each of the formulas (1) and (2) below defines a relationship between $\mathfrak{B}$ and the model specified by $\langle \underline{R}_n\rangle_n$:

(1) $\quad \exists x(\bar{\alpha}x \Vdash R_n(d_1,\dots,d_{r(n)}))$ iff $\underline{R}_n{}^{\alpha}(d_1,\dots,d_{r(n)})$,

(2) $\quad k \Vdash R_n(d_1,\dots,d_{r(n)})$ iff $\forall\alpha{\in}k\, \underline{R}_n{}^{\alpha}(d_1,\dots,d_{r(n)})$.

On the weak definition of forcing, and assuming density and open data (1) is equivalent to (2).

Each $\mathfrak{B}$ gets $\langle \underline{R}_n\rangle_n$ assigned by (1), and each $\langle \underline{R}_n\rangle_n$ gets a $\mathfrak{B}$ assigned to it by (2), and these two constructions are inverse to each other. □

For arbitrary formulas we have the following theorem describing the relation between intuitive validity and Beth forcing.

PROPOSITION. Let $D = \mathbb{N}$, and let $(\underline{K},\preccurlyeq)$ be a fan. Assume the Beth model $\mathfrak{B} \equiv (K,\preccurlyeq,\Vdash,D)$ and the sequence $\langle \underline{R}_n\rangle_n$ to be related by (1) or (2) as in the previous proposition.

(i) With the strong definition of forcing, and density, open data and the fan axiom for all A

$$(3) \qquad \begin{cases} \exists n(\bar{\alpha}n \Vdash A) \Leftrightarrow A^{\alpha} \\ k \Vdash A \Leftrightarrow \forall\alpha{\in}k(A^{\alpha}) \end{cases}$$

where A^{α} is obtained from A by substitution of $\underline{R}_n{}^{\alpha}$ for R_n in A everywhere.

(ii) (3) also holds if $\mathfrak{B}$ is based on the weak definition of forcing, and assuming density and open data. □

We are now ready for

Friedman's completeness proof.

Let $\mathcal{L}$ be a language without function symbols or = and with at least one predicate symbol of arity one. Let $\underline{C} \equiv \{c_n : n \in \mathbb{N}\}$ be a collection of constants not in $\mathcal{L}$. Let $\mathcal{L}' := \mathcal{L}(\underline{C}) = \mathcal{L} \cup \underline{C}$. For any set of formulas Γ, let $\mathcal{L}(\Gamma)$ be the language containing only constants from $\underline{C}$ occurring in formulas of Γ.

The proof covers three situations:

(i) the ⊥-free fragment of **IQC**;

(ii) minimal predicate logic **MQC**, obtained by dropping $\bot \rightarrow A$ ("ex falso sequitur quodlibet") from **IQC** and

(iii) **IQC**, for a notion of "fallible" Beth model, where ⊥ may be true at certain nodes.

As to case (ii), in a Beth model for **MQC** we permit ⊥ to be true at certain nodes, provided ⊥ is not true in the whole model; this corresponds to the idea that in minimal logic ⊥ plays the role of an arbitrary unprovable proposition. in case (iii), if ⊥ is true at node k, then all sentences are forced at k.

The proof produces in each of the cases (i)-(iii) a Beth model $\mathcal{M}$ which is universal in the sense that for sentences A in the language and logic (**IQC** or **MQC**) under consideration $\vdash A \Leftrightarrow \mathcal{M} \Vdash A$ (A is Beth-valid in the model). The first step in constructing $\mathcal{M}$ is the construction of a certain labelled binary tree.

Construction of the labelled binary tree.

Let $\langle A_n\rangle_n$ be an enumeration with infinite repetitions, i.e. $\forall k\exists m{>}n(A_m \equiv A_k)$, such that in case (i) all ⊥-free sentences of $\mathcal{L}'$ are enumerated and in cases (ii) and (iii) all sentences of $\mathcal{L}'$ are enumerated. Let $\Gamma \vdash_m A$ mean that A is deducible from Γ by a deduction with code number less than m, where deducibility is interpreted in case (i), (iii) as deducibility in **IQC**, and in case (ii) as deducibility in **MQC**. (So $\Gamma \vdash_m A$ is decidable.)

We now construct a Beth model over the binary tree of finite

01-sequences; we shall let k, k' range over such sequences. To each node k we are going to assign a finite set Γ_k of sentences of $\mathcal{L}'$, the <u>label</u> of k.

The labels are defined by induction on the length of the nodes as follows. We put

$$\Gamma_{\langle\,\rangle} = \emptyset.$$

Let lth(k) = u. In the definition of $\Gamma_{k*\langle i\rangle}$ we consider four cases:

(i) $A_u \notin \mathcal{L}(\Gamma_k)$.
Take $\Gamma_{k*\langle 0\rangle} \equiv \Gamma_{k*\langle 1\rangle} := \Gamma_k$.

(ii) $A_u \in \mathcal{L}(\Gamma_k)$, $A_u \equiv B \vee C$, $\Gamma_k \vdash_u B \vee C$.
Take $\Gamma_{k*\langle 0\rangle} := \Gamma_k \cup \{B\}$, $\Gamma_{k*\langle 1\rangle} := \Gamma_k \cup \{C\}$.

(iii) $A_u \in \mathcal{L}(\Gamma_k)$, $A_u \equiv \exists x\, B(x)$, $\Gamma_k \vdash_u \exists x\, B(x)$.
Let c_i be the first constant of C not in $\Gamma_k \cup \{B(x)\}$ and take
$\Gamma_{k*\langle 0\rangle} \equiv \Gamma_{k*\langle 1\rangle} := \Gamma_k \cup \{B(c_i)\}$.

(iv) If (i)-(iv) do not apply, take
$\Gamma_{k*\langle 0\rangle} := \Gamma_k$, $\Gamma_{k*\langle 1\rangle} := \Gamma_k \cup \{A_u\}$.

Observe that for some formula Bx with FV(B) = {x} $B' \equiv \exists x(Bx \to Bx)$ is provable, and this B' is repeated infinitely often in $\langle A_n\rangle_n$, so ultimately each c_i will occur in $\mathcal{L}(\Gamma_k)$ provided k is sufficiently long; it is easily seen a bound on lth(k) can be given, depending on i, which guarantees that $c_i \in \mathcal{L}(\Gamma_k)$.

Below we shall write $\vdash \Gamma \to A$ for $\vdash \bigwedge\Gamma \to A$.
We can now define the Beth model.

DEFINITION. $\mathcal{B}^* \equiv (K, \preccurlyeq, \underline{C}, \Vdash)$, where $(K, \preccurlyeq)$ is the binary tree of finite 01-sequences, $\underline{C} \equiv \{c_n : n \in \mathbb{N}\}$ is the constant domain which may be identified with $\mathbb{N}$, and $\Vdash$ for prime sentences is given by

$$k \Vdash P := \vdash \Gamma_k \to P. \ \square$$

The completeness result now follows from two lemmas.

LEMMA A. For all $A \in \mathcal{L}(\Gamma_k)$ and all $x \in \mathbb{N}$

$$(\vdash \Gamma_k \to A) \Leftrightarrow \forall k' \succcurlyeq_x k(\vdash \Gamma_{k'} \to A)).$$

PROOF by induction on x. □

LEMMA B. In the model $\mathcal{B}^*$ for all sentences $A \in \mathcal{L}(\Gamma_k)$

$$k \Vdash A \Leftrightarrow \vdash \Gamma_k \to A.$$

PROOF by induction on A. □
We summarize the result:

THEOREM.

(i) For the ⊥-free sentences of **IQC** there is a "universal" Beth model $\mathcal{B}_i$ such that for all sentences A

$$\mathcal{B}_i \Vdash A \text{ iff } \vdash A.$$

(ii) For **MQC** there is a Beth model $\mathcal{B}_m$ such that for all sentences A

$$\mathcal{B}_m \Vdash A \text{ iff } \vdash A.$$

(iii) For **IQC** there exists a fallible Beth model $\mathcal{B}_f$ such that for all sentences A

$$\mathcal{B}_f \Vdash A \Leftrightarrow \vdash A. \qquad \square$$

The result on the connection between Beth validity and intuitionistic validity now immediately leads to the following

COROLLARY. Let F be ⊥-free. There is a sequence of relations $\langle \underline{R}_n^{\alpha}\rangle_n$ depending on a parameter α ranging over lawless 01-sequences, such that

$$\forall\alpha(F^{\mathbb{N}}(\underline{R}^{\alpha}_{i(1)}, \dots, \underline{R}^{\alpha}_{i(n)})) \Leftrightarrow \vdash F(R_{i(1)}, \dots, R_{i(n)})$$

where $R_{i(k)}$ is a relation symbol with the same number of arguments as $\underline{R}^{\alpha}_{i(k)}$. The $\underline{R}^{\alpha}_n$ are enumerable in α, the other arguments range over $\mathbb{N}$; as axioms for lawless parameters we need only FAN, density, and open data. □