Estratto da

A. Zanardo (a cura di), *Atti degli incontri di logica matematica* Volume 4, Siena 27-30 maggio 1987.

Disponibile in rete su http://www.ailalogica.it

SOME CONSISTENCY RESULTS CONCERNING A THEORY OF PROPERTIES

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1.We present some consistent extensions of PI, a theory of partial properties and total operations. PI contains:
a) classical logic, extended by the operators T(rue), F(alse), which internalize strong 3-valued logic and allow to define an intensional equivalence relation $\equiv$;
b) a theory OP of total operations (essentially a type-free λ-calculus with a predicate N, which satisfies induction axioms and Peano axioms);
c) the abstraction principle AP: $\forall u(u \in \{x:A\} \equiv A(u))$ (A arbitrary) ;for a survey of related theories ,see [Fa].
In current literature we only find descriptions of the least inductive model L_o of type-free theories, similar to PI, because L_o is naturally linked to a predicative interpretation of the abstraction process. However, by lifting some ideas of Kripke's theory of truth ([K]) to PI, we can prove (details are given in [Ca]):
1.1. If M is a model of OP, there are 2^m non-isomorphic expansions of M(m=card(M)), which are models of PI;
1.2. the collection INT(M) of the expansions of M, which are intrinsic models of PI, form a complete non-distributive lattice of cardinality 2^m.

1.1-2 show that PI is essentially an abstract theory and the predicative interpretation is quite special. Below we list some candidates for enriching the structure of PI-properties and operations.

2.Consistency results concerning properties.Let P(a):= = a is a property;Cl(a):=a is a total property or,in short,a is a class.

A.1.Cl-compactness: there is an operation ξ ,which associates to each non-empty property b, a non-empty class $\xi b \subseteq b$.

A.2.Separation:there is an operation sep such that,if b,c are properties for which $\forall x(x\in b \vee x \in c)$ holds,then sep(b,c) is a class and sep(b,c)$\subseteq$ c , $\forall x(\neg x \in b \rightarrow x \in$ sep(b,c)) hold.

A.3.Exact Representation: there is an operation ER such that,if b,c are disjoint properties,then $b =_e ER(b,c)$ and $c =_e -ER(b,c)$ $(-d:=\{x: \neg x \in d\}$; $a =_e b := a \subseteq b \wedge b \subseteq a)$.

A.4."Myhill-Shepherdson": if $F:P \longrightarrow P$ is extensional, then $Fa =_e \cup\{Fc: c \subseteq a \wedge Cl(c)\}$,for every property a; (F extensional:= $a =_e b \longrightarrow Fa =_e Fb$).

2.1.Theorem. PI+A.1-A.4 is a conservative extension of PI (which is equiconsistent with Peano Arithmetic).

Proof of 2.1: one pursues a recursion theoretic analysis of suitable inductive models of PI over possibly non-N-standard models of OP. The main tool is a uniform version of the ordinal comparison theorem for such structures.Incidentally,we note that the language of PI can be used for an alternative approach to generalized recursion theory.

A.5. Let PWO(<) (QWO(<)) stand for " < is a class encoding a linear ordering,which is well-founded with respect to classes (respectively properties)";

Axiom (β): $Cl(<) \wedge PWO(<) \longrightarrow QWO(<)$;

Axiom ($\beta\rho$): there is an operation Rc,which yields-uniformly in the classes b,< and the elementary condition A- the class Rc(< ,b,A), defined by transfinite recursion along < by means of A and b .

2.2.Theorem.(i) PI+ N-induction for classes+($\beta\rho$) is equiconsistent with predicative analysis.

(ii) The same theory with ($\beta\rho$) replaced by (β) contains Σ^1_1-BI_0.

Proof of 2.2.(i):by a series of proof-theoretic reductions which combine standard cut elimination with the tecniques of [Cb.].

A.6.It is worth mentioning that in **PI** the fixed point of the operation $\lambda v.\{x:A(x,v)\}$ exactly represents a fixed point (in set-theoretic sense) of the operator defined by A(x,-),if A is positive elementary in the sense of [M]. Indeed,by OP and AP,we can find a property I_A such that

$$\forall u(u \in I_A \longleftrightarrow A(u,I_A)).$$

One may wonder whether I_A actually represents the least fixed point of the operator induced by A.

2.3.Theorem. There are models of PI,in which,for every elementary positive operator A,

(i) I_A defines the $\subseteq$-least fixed point of A;

(ii) I_A defines the $\subseteq$-greatest fixed point of A;

(iii) it can be assumed that I_A is a class.

Proof of 2.3: by constructing suitable non-minimal non intrinsic models of PI.

On the proof-theoretic side, let $Clos_A(B) := \forall x(A(x,B) \rightarrow B(x))$; WGI is the axiom schema: $Pa \wedge Clos_A(a) \rightarrow I_A \subseteq a$; GI is the schema: $Clos_A(B) \rightarrow \forall x(x \in I_A \rightarrow B(x))$, B arbitrary condition; PI^+ is PI with the schema of N-induction.

2.4. Theorem. (i) PI^+ +WGI+A.1-4 is strictly weaker than predicative analysis.

(ii) PI^+ +GI is equiconsistent with ID_1.

A.6. Finally, it is possible to prove the existence of PI-models, in which there is a class SET of classes, satisfying extensionality and (a sort of strengthened) impredicative second order comprehension (see [Cc]).

3. Consistency results concerning operations. There exist models of OP, plus each one of the following statements:

O.1. There is a surjective $f: N \rightarrow V$ (=universal class).

O.2. CT: every $f: N \rightarrow N$ is recursive.

O.3. Every $f: N^N \rightarrow N$ is continuous.

O.4. Every $f: N \rightarrow N$ is hyperarithmetical.

O.5. N-choice: $\forall x \in N \exists y A(x,y) \rightarrow \exists f \forall x \in N. A(x,fx)$ (A arbitrary).

O.1-3 hold in the recursive graph model RE (for O.3 and refinements, see [B]); O.4 is true in the "admissible" analogue of RE, made up with Π^1_1-sets. O.5 holds in $P\omega$ by applying the extension theorem of [S].

Recently Myhill and Flagg greatly improved O.5 in the context of Frege structures by using D_∞. At present, we do not know if their results extend to PI.

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