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ON DUMMETT's LC QUANTIFIED

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In § 1. we present a sequent calculus G-LC for the logic Q-LC that results from adding Dummett' s axiom A -> B v B -> A to intuitionistic predicate logic and we prove the cut elimination theorem for it. Q-LC is valid with respect to the class K^C of connected Kripke models with nested domains.

In § 2. we extend the language of Q-LC by adding the existence predicate E and we give an axiom system, QE-LC, which results to be characterized by the class E^C of connected (Kripke) E-models with nested domains.

§ 1. The calculus G-LC.

Let *L* be an elementary language containing countably many individual variables x, y, w,.... (with or without subscripts), a non empty set of predicate letters P^n, $n \geq 1$, the symbol of falsehood *f* , the connectives & (and), v (or), -> (if ... then), the quantifiers ∀ (for all), ∃ (there is) and the auxiliary symbols (,). We use A, B, C as metavariables for formulas, which are defined in the usual way. The negation is defined thus : $\neg A =_{df} A \rightarrow f$.

The letters M, N, P, Q designate sets (possibly empty) of wff.

A sequent M —» N is an ordered pair of finite sets (possibly empty) of formulas and the notation M, A —» N is an abbreviation of M U {A} —» N. M —» A, N is treated analogously.

Axioms : M, A —» A, N ; M, *f* —» N.

Inference rules :

$$\frac{M \longrightarrow N}{P, M \longrightarrow N, Q}\ \text{w (weakening)}$$

$$\frac{M, A, B \longrightarrow N}{M, A \,\&\, B \longrightarrow N}\ \&: \qquad \frac{M \longrightarrow A, N \qquad P \longrightarrow B, Q}{M, P \longrightarrow A \,\&\, B,\ N, Q}\ :\&$$

$$\frac{M, A \longrightarrow N \qquad P, B \longrightarrow Q}{M, P, A \vee B \longrightarrow N, Q}\ \vee: \qquad \frac{M \longrightarrow A, B,\ N}{M \longrightarrow A \vee B,\ N}\ :\vee$$

$$\frac{M \twoheadrightarrow A, N \qquad P, B \twoheadrightarrow Q}{M, P, A \to B \twoheadrightarrow N, Q} \to :$$

$$\frac{M, A(w) \twoheadrightarrow N}{M, \exists x A(x) \twoheadrightarrow N} \exists : \quad \text{where } w \text{ does not occur in } M \text{ or } \exists x A(x) \text{ or } N \qquad \frac{M \twoheadrightarrow A(x), N}{M \twoheadrightarrow \exists x A(x), N} : \exists$$

In $\exists :$ w is said to be the *special variable* of $\exists :$.

$$\frac{M, A(x) \twoheadrightarrow N}{M, \forall x A(x) \twoheadrightarrow N} \forall :$$

$$\frac{\begin{array}{l} M_1, A_1 \twoheadrightarrow B_1, A_2 \to B_2, \ldots, A_n \to B_n, \forall x D_1(x), \ldots, \forall x D_m(x) \\ \vdots \\ M_n, A_n \twoheadrightarrow A_1 \to B_1, \ldots, A_{n-1} \to B_{n-1}, B_n, \forall x D_1(x), \ldots, \forall x D_m(x) \\ M_{n+1} \twoheadrightarrow A_1 \to B_1, \ldots, A_n \to B_n, D_1(w_1), \forall x D_2(x), \ldots, \forall x D_m(x) \\ \vdots \\ M_{n+m} \twoheadrightarrow A_1 \to B_1, \ldots, A_n \to B_n, \forall x D_1(x), \ldots, \forall x D_{m-1}(x), D_m(w_m) \end{array}}{\bigcup_{i=1}^{n+m} M_i \twoheadrightarrow A_1 \to B_1, \ldots, A_n \to B_n, \forall x D_1(x), \ldots, \forall x D_m(x)} L_{n,m}$$

where $w_1,\ldots,w_m$ are all different variables and none of them occurs in the conclusion of $L_{n,m}$ and $n, m \geq 0$.
$w_1,\ldots,w_m$ are said to be the *special variables* of $L_{n,m}$.

Let G-LC# be the calculus G-LC plus the following cut rule :

$$\frac{M \twoheadrightarrow A, N \qquad A, P \twoheadrightarrow Q}{M, P \twoheadrightarrow N, Q} \text{cut}$$

DEFINITION 1.1. Let Π be a proof in G-LC# where the last rule is the cut rule and no other rule in Π is a cut rule. Then Π is said to be *simple*.

DEFINITION 1.2. A proof Π is said to be *quasi-regular* iff the free variables occurring in Π are all different from the bound variables occuring in Π.

A proof Π is said to be *regular* iff it is quasi-regular and any special variable occurring in a premiss S of $L_{n,m}$ or of $\exists :$ occurs only in the subproof of Π ending with S.

LEMMA 1.1 Let Π be a regular and simple proof in G-LC# of the sequent M, P $\twoheadrightarrow$ N, Q, then there is a cut-free proof Σ of M, P $\twoheadrightarrow$ N, Q.

THEOREM 1.2 Let Ω be a proof of the sequent M $\twoheadrightarrow$ N in G-LC#. Then there is a proof Ω^* of M $\twoheadrightarrow$ N in G-LC.

THEOREM 1.3 A sequent M $\twoheadrightarrow$ N is provable in G-LC iff Q-LC $\vdash$ M $\to$ $\bigvee$N for M, N $\neq \emptyset$.
A sequent $\twoheadrightarrow$ N is provable in G-LC iff Q-LC $\vdash$ $\bigvee$N , for N $\neq \emptyset$.
A sequent M $\twoheadrightarrow$ is provable in G-LC iff Q-LC $\vdash$ M $\to$ *f*, for M $\neq \emptyset$.

PROOF. By theorem 1.2 and the fact that for any $n, m \geq 0$,

$$\{ A_1 \to [B_1 \vee \ldots \vee (A_n \to B_n) \vee \forall x D_1(x) \vee \ldots \vee \forall x D_m(x)] \ \& \ \ldots \ \& \ A_n \to [(A_1 \to B_1) \vee \ldots \vee B_n \vee \forall x D_1(x) \vee \ldots \vee \forall x D_m(x)] \ \& \ [(A_1 \to B_1) \vee \ldots \vee (A_n \to B_n) \vee D_1(w_1) \vee \ldots \vee \forall x D_m(x)] \ \& \ \ldots \ \& \ [(A_1 \to B_1) \vee \ldots \vee (A_n \to B_n) \vee \forall x D_1(x) \vee \ldots \vee D_m(w_m)] \} \to [(A_1 \to B_1) \vee \ldots \vee (A_n \to B_n) \vee \forall x D_1(x) \vee \ldots \vee \forall x D_m(x)]$$

is a theorem of Q-LC, where $w_1,\ldots,w_m$ are all different variables and none of them occurs in the consequent of the implication.

§ 2. The calculus QE-LC.

The *axioms* of QE-LC are the propositional axioms of the intuitionistic logic plus the following ones:

$A \to B \vee B \to A$

$\forall x A(x) \ \& \ E(y) \to A(y)$ where y is free for x in $\forall x A(x)$

$A(y) \ \& \ E(y) \to \exists x A(x)$ where y is free for x in $\exists x A(x)$

$\exists x A(x)$

Inference rules :

$$\frac{A \qquad A \to B}{B} \qquad\qquad \frac{E(x) \ \& \ B(x) \to A}{\exists x B(x) \to A} \quad x \text{ not free in } A$$

$$\frac{C_1 \to (B_1 \vee (C_2 \to (B_2 \vee (\ldots \vee (C_n \to (B_n \vee (E(x) \to A(x)))) \ldots))))}{C_1 \to (B_1 \vee (C_2 \to (B_2 \vee (\ldots \vee (C_n \to (B_n \vee \forall x A(x))) \ldots))))} \quad n \geq 0 \text{ and x is not free in the conclusion of the rule}$$

DEFINITION 2.1 An E-model ME is a quadruple <W,R,D,V> where W is a non-emty set, R is a partial order of W, D is the domain function that associates with each $w \in W$ a non-empty set D_w, such that if wRv then $D_w \subseteq D_v$. Let $U = \bigcup \{D_w\}_{w \in W}$. V is an assignment function such that $V(P^n)w \subseteq U^n$, $V(E)w = D_w$ and if wRv then $V(P^n)w \subseteq V(P^n)v$.

DEFINITION 2.2 An interpretation μ is a function from the variables of the language into U. By $\mu^{(x/d)}$ we denote the interpretation μ' which coincides with μ except that $\mu'(x) = d$, where $d \in U$.

DEFINITION 2.3 The notion of a wff A's being true in M at w under μ, $ME\ {}^{\mu}|=_w A$, is so defined :

$ME\ {}^{\mu}|\neq_w f$

$ME\ {}^{\mu}|=_w P^n(x_1,...,x_n)$ iff $< \mu(x_1),..., \mu(x_n) > \in V(P^n)w$

$ME\ {}^{\mu}|=_w A \,\&\, B$ iff $ME\ {}^{\mu}|=_w A$ and $ME\ {}^{\mu}|=_w B$

$ME\ {}^{\mu}|=_w A \vee B$ iff $ME\ {}^{\mu}|=_w A$ or $ME\ {}^{\mu}|=_w B$

$ME\ {}^{\mu}|=_w A \rightarrow B$ iff for all v. wRv. if $ME\ {}^{\mu}|=_v A$ then $ME\ {}^{\mu}|=_v B$

$ME\ {}^{\mu}|=_w \forall xA(x)$ iff for all v.wRv. and for all $d \in D_v$, $ME\ {}^{\mu(x/d)}|=_v A(x)$

$ME\ {}^{\mu}|=_w \exists xA(x)$ iff there is a $d \in D_w$, such that $ME\ {}^{\mu(x/d)}|=_w A(x)$.

DEFINITION 2.4 A wff $A(x_1,...,x_n)$ is true in ME, $ME \models A(x_1,...,x_n)$, iff for all μ and all w, $ME\ {}^{\mu}|=_w A(x_1,...,x_n)$. A wff $A(x_1,...,x_n)$ is E- valid iff for all E-models ME, $ME \models A(x_1,...,x_n)$.

THEOREM 2.1 The logic QE-LC is characterized by the class $\mathbf{E^c}$ of connected E-models.

COROLLARY 2.2 The logic Q-LC is characterized by the class $\mathbf{K^c}$ of connected Kripke models iff QE-LC is a conservative extension of Q-LC.

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