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LOGIC AS A SUBJECT OR AS AN ATTITUDE

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Logic as a subject or as an attitude

Mathematics, in the history of mankind, arose as common sense, and it still does so in individual histories. Everyday life language is the linguistic vehicle of common sense. Spoken numerals as a vehicle of whole numbers is a marvelous feature. As the child acquires the syntactic structural means to build new sentences, so it acquires the morphological structural means to build spoken numerals, even far beyond need and necessity, and far beyond the physical and (at least initially) mental grasp of number. Going on and on in this sequence is the first striking expression of the mathematical mind – which can hardly be overestimated, although it has hardly drawn anybody's attention, because it is simply a matter of fact (or it looks so), and the knowledge about it is common sense, which is not to be questioned.

Mathematics, in the history of mankind, not only arose but also continued and developed as (ever and ever more sophisticated) common sense – common to crowds of people, of educated people, of learned people. Only a short time are children allowed to acquire mathematics in the way of common sense but soon the teacher impatiently takes over, imposing his own mathematics, as institutionalised instruction obliges him to do so.

My statement that researchers have paid little attention to children's first and spontaneous arithmetical activities seems to be contradicted by the title of Piaget's work (with Szeminska) 'La génèse du nombre chez l'enfant' (1941). The contradiction, however, resides in the title only, since the work itself is not at all concerned with the genesis of number. In fact, genesis of the mind has abundantly and marvelously been studied by Piaget in his earlier work. This genetic approach should sharply be distinguished from the epistemological one he chose in his later work. In order to make clear what this switch meant, let me briefly sketch the history of the number concept and its teaching.

As opposed to his geometry, Euclid's arithmetic is rooted in the whole number of common sense, and up to the midcentury the teaching approach to number was still common-sensical. As soon as infinities are to be dealt with *arithmetically* this common sense approach is blemished by paradox. Cantor removed the blemish by abandoning the then common sense approach in favour of a more sophisticated one, later on extended by Frege-Russell in an even more refined way from *infinities* to *finite* natural number and its foundation.

In Piaget's epistemology *episteme* means a state of knowledge as advanced as accessible to himself, which in the case of number happened to be Frege-Russell, at least as understood by him (and at a later stage, for mathematics in general, Bourbaki's system). This sophisticated number concept is Piaget's touchstone, his tool to assess childrens' understanding of number at various ages, in far from common sense situations and using an artificial language, both of them created for the sake of assessment. Obviously this is no way to trace the *genesis* of number. It proved rather to be the source of such oddities as non-conservation, never observed in the unfortunately rare exeamples of *genetic* research. It is a pity that Piaget's monumental work has not yet been appreciated as a whole. I even wonder whether his switch from genesiology to epistemology has ever been noticed.

Of course, Piaget is not the main responsible for New Math as far as it meant, with regard to teaching, rejecting the common sense approach in favour of a new one which pretended sophistication, although eventually it realised nothing else but a caricature. Indeed from olden times, teachers, in particular of mathematics, rather than respecting the children's common sense version were inclined to impose their own version. There is, however, an increasing trend to allow learners to reinvent mathematics on their own: guided reinvention as I called it. If allowed to be reinvented, newly acquired mathematics has a good chance to attain the state of common sense of higher order.

'Logic' and 'logical' belong to everyday life language. So one may guess that these words are carriers of common sense ideas, which not necessarily do match those of Aristotelians or modern logicians, unfamiliar to the great majority. It may be regretted that as a consequence of teaching habits most people consider mathematics as a meaningless formal language. Logic, however, is still associated by most people with thinking — ' logical' meaning the right way of thinking, and 'illogical' the wrong way — and even with regard to contents logic, as generally understood, comprises much more, and quite different things, than poor formal logic.

Let me tell you a little story, which reveals the wealth of common sense logic.

> *It happened years ago. Three boys in a family: John (7;7), Matthew (5;10), Tom (3;10). John and Matthew attend different schools. John tells that, while playing on street with Matthew, he was greeted by Matthew's teacher 'Hi, John!' 'How did she know my name?', John asks his father. 'What do you think yourself?', Father answers. 'Once Matthew should have told her about his brothers', John explained and immediately continued: 'How did she know that I was John (rather than Tom)?' 'What do you think yourself?' "She saw that I was older than Matthew, rather than younger as Tom should be', John himself answered his second question.*

John thought about other people's thinking — logic in action. He did so according to certain patterns, which it would be revealing to analyse. His father could have tried to have him analyse them himself. Thus he should have continued after 'What do you think yourself?' asking 'why do you think so?' but I am pretty sure that at this age John would have answered 'cause'. Thinking about one's own thoughts — reflective thinking — is a hard thing, even for adults.

After this example of logic according to common sense, a counter-example is not out of place. As such I choose the famous syllogism:

> All men are mortal.
> Socrates is a man.
> Ergo: Socrates is mortal.

Since it is a well-worn truth that Socrates had died as early as 399 B.C., this is an idle reasoning. It even looks like begging the question since Socrates' death is rather one of the supports of what pretends to be a major premise.

If you know Lewis Carroll's 'Alice in Wonderland', which is a rich treasure of marvelous logic, it is worthwhile mentioning that Carroll also authored a 'Symbolic Logic', a waste-basket of such monstruosities like

> Babies are illogical.
> Nobody is despised who can manage a crocodile.
> Illogical persons are despised.
> Ergo: Babies cannot manage crocodiles.

Let us grant babies a logic of their own! According to common sense besides statements and reasonings, also questions can be illogical. So is the question

> Do you still beat your wife?

if the questioned husband is expected to answer by 'yeah and nay'. So are Piaget's famous test questions like.

> Are there more tulips than flowers in the bouquet?

if the tested child is not allowed to think

> Other flowers — he means.

Up to now I did not tackle logic as related to mathematics or mathematical instruction. Honestly said, I am not sure whether this is expected from me. I am a bit confused about the theme proposed to this conference, and whether my own lecture as prepared meaningfully contributes to the intended subject. About a year ago I was asked to contribute to a conference on 'Logic in Mathematics Education'. I was somewhat surprised to read now the title 'La Logica Matematica nella Didattica', which means the part played by symbolic logic in general educational theory, while the various round tables are concerned with questions whether and how to teach logic at school. It is only logical to state that these three themes have little in common. But let me henceforth suppose that the proper theme is that prescribed for the round tables, that is, teaching logic at school, yet not necessarily related to mathematics. You can guess my answer at least partially. Logic in the sense of formal logic means as little in school instruction as it does in life. So it is neither worth teaching nor learning . Logic as understood by common sense is an all pervading stream — both life and school pervading — and deserves to be respected and nourished as such, not only in mathematics but in all education. 'All pervading' means that it can remain implicit on long stretches, or even all the way if it is too difficult to be verbalised. It is the learning process itself that defines the subjects to be made explicit and the moment where this can happen, and it is the teacher's task to guide this process and to look for its progress.

As all pervading a feature is language and it is so on various levels. Such levels can be distinguished in everyday life language as they can in any technical language. Logic is most often, if not always, concerned with the relation between phenomena in the physical and mental world and their linguistic counterparts, that is, with formulation and translation, which includes translation from one level to the other. Rarely is this impact of logic respected or accounted for in actual instruction, because this logic cannot be taught as a subject matter. It can, however, be instilled as an attitude by seizing any opportunity to have the learner reflect on the logic implicit to mental experiences and actions. I am afraid most teachers are not aware of these opportunities, and as far as this fear is justified, a change of teachers' attitude is badly needed. If we wish to foster logical culture we have first to teach teachers observing learning processes, to analyse them, and to avail themselves of opportunities to make explicit the implicit logical behaviour.

Let me adduce a few examples. According to Euclid's 'Definitions' at the start of his 'Elements' (most of which are never used in the sequel) equilateral triangles are *not* isosceles, squares are neither rectangles nor rhombs, and all of them are no rhomboids (= parallelograms), whereas at present we consider 'equilateral' as a special case of 'isosceles', squares as special rectangles and rhombs and all of them as special parallelograms. But even colloquial language still adheres to the old terminology as used in Euclid's 'Definitions'. This is quite natural. There is a natural distinction between *truth* and *whole truth*, which indeed matters when a judge interrogates a witness. Telling about a square only that is a rectangle (or a rhomb) is not the whole truth, which under certain circumstances a witness is obliged to disclose. There are good reasons why as geometers and mathematicians we act differently. The great majority of properties, and the most important ones, of rectangles with unequal sides (of rhombs with unequal angles) are shared by squares. So by considering squares as special rectangles and rhombs we can dispense with stating and proving these properties anew for squares. Distinguishing between truth and whole truth, and calling in at any moment as much or as little evidence as needed is an important global logical principle, which is implicit to a lot of mathematics acted out. To what degree is this principle made explicit? Do teachers tell pupils that, unlike everyday life language, geometry deals with squares as though they were rectangles and rhombs. Probably they do. But do they also touch the question why? Or even worse, have they ever asked this question themselves and are they aware of the general principle of *truth* and *whole truth*?

I cannot answer these questions but I am quite pessimistic. I am pessimistic because of extensive reports on actual instruction — observations by Rijkje Dekker and Paul Herfs in 7th grades where teachers used experimental collections of worksheets on graphs, functions and relations, produced by the mathematics section of SLO (Institute for Curriculum Development at Enschede)

I am going to show one of the sheets, a weather diary (Fig. 1.) Which graph belongs to which day? Obviously the order is 6, 5, 1, 4, 2, 3. In her group Jane worked alone while the others discussed. After a few minutes she had found 6, 3, 1, 4, 2, 5, which is almost correct, that is up to the absurd transposition 3 < - > 5, possibly due to bad writing. The others copy it and pass to the following sheet. Next day when visiting the group, the teacher laconically states 'wrong'. The discussion is resumed with Jane abstaining. The pupils try to find a graph matching one day after the other in the given order, rather than first examining the graphs with their striking visual features. Cary finds 6 for Sunday. Then Rik

first assigns 5 to Sunday , but switches to Monday, which is also wrong. The breaking through is Rik's assigning 4 to Tuesday, which is indeed convincing. After 2 for Wednesday, which is now obvious, the remainder is easy. The teacher did not contribute anything, possibly did not even know what and how to contribute.

The foregoing could be an example of what I call jigsaw puzzle logic. What do children do with jigsaw puzzles? They look for striking features of the particular pieces: a rectilinear border, a corner, blue sky, grass, a seagull in the air, and so on, and after a while for contours and colours matching each other. This is the jigsaw puzzle strategy of the striking features: positive ones, that is, clues; negative ones, that is, alibis. But no such strategy can be detected in the report, even no hindsight, no reflection, no stimulus to reflect. No 'how have you found it?' No 'how could one have found it in the shortest way?', which is a typically mathematical attitude: cutting roundabouts and dead ends, refurbishing a solution in order to increase its degree of obviousness. In the present case this would have been, looking first to the graphs: very high, very low temperatures, sudden changes, and so on. Strangely enough, the pupils first reacted on catchwords such as 'variable'. No help by the teacher, no suggestion to afterwards reflect on the solving procedure. No wonder! Whoever told her that this is mathematics, that this is logic?

Another sheet I will show you is the first of the series 'Biking'. (Fig. 2.) It is about four children living at Losser and going by bike to school at Enschede — 10 km distance. School starts at a quarter past eight but children leave home as early as half past seven. Again stories are to be assigned to graphs. A blank leaves one story to be invented by the pupils themselves.

Again a case of jigsaw puzzle logic, although easier than the weather diary. Again the texts are being read first and the graphs identified afterwards. 'Yury' causes hesitations, 'Hermine' is immediately identified. In a parallel group she is even the first. Yet again no explicitation of reasonings.

The next sheet shows Yury's graph at a larger scale. (Fig. 3.) It exemplifies what I call the 'logic of linearity and steepness. From the report I show you an extract of the discussion on question 2e:

> How can you see that the first twenty
> minutes Yury went equally fast all the time?

An informative discussion. (Fig. 4.) The first, and only valid, move was Erna's. But Erna gets confused by the numerical approach of the others and finally yields to the authority of the teacher, who has not even tried to make sure who in the group shares which opinion. Without knowing it she formalises — that is to say, quite inadequately — Erna's intuition: 'straight, not once steeper and once less steep'. This is the correct argument, indeed, at least if the relation 'steeper — faster, less steep — slower' is made explicit. But is it? It seems so. Question 2g asks:

> Between which moments is Yury's speed the greatest?

A pupil motivates it by 'climbs higher', and the teacher adds 'travels the same distance in a shorter time'. In a parallel class the teacher's explanation is 'the line is steep'. In neither class is the intuitive feeling for what matters, made conscious, which would have been easy. There is not even the slightest attempt, the slightest idea that this making conscious would pay off. Indeed, again and again pupils are tempted to associate 'steeper' with 'slower', which is not far-fetched.

I should confess that acting out the 'logic of linearity and steepness' is obstructed by a flaw in the design: the rigid lattice coupling 5 minutes with 1 km (on this sheet and others). Didactically it should rather have been a blurred background feature. This shortcoming reinforces the natural temptation to confound time and distance, which indeed happened quite often, for instance when 'halfway' was interpreted as 'halftime'. Even when corrected, the logic of distinguishing halfway and halftime has never been made explicit, with the consequence that confounding the two variables remained an obstinate obstacle.
As an example how to improve it, I show you another sheet. (Fig. 5.) The problem is:

> Fred and Frank are joggers. They go from Hengelo to Enschede.
> Fred runs half of the distance and walks the other half.
> Frank runs half the time and walks the other half. Both of them are equally fast joggers and equally fast walkers. Who is the first to arrive?

The problem was discussed by two girls, Leonie and Yvonne (7th grade). They started with drawings — the second is a clock, which however was not used in the sequel. After a few useless attempts (guessing distances and times, and numerical substitutions) Leonie took Yvonne's drawing, saying:

> While running Frank covers more than half the distance.

She could not argue it but now it dawned on Yvonne:

> When Fred is midway, he has used less than half the time.

The conclusion is clear but again it is not made conscious by verbalising.

The last sheet I am going to show is typical of a vast number of problems and an example of what I call the 'fixed charge logic': the price of a certain commodity is the sum of two components, a fixed charge and an amount of money proportional to the quantity purchased — graphically represented by a linear function. On the present sheet it means tiles, for which one dealer charges transportation fees while the other does not but asks a higher price a piece. (Fig. 6.) Yet you may as well think of electricity or phone bills, or renting a car, or buying more or less expensive and economic light bulbs as examples of fixed charge logic: under which circumstances and to what extent are higher constant charges compensated by lower variables ones?
Class room observations do not reveal the faintest awareness of the isomorphic nature

of this kind of problems. Even worse: teachers do not seem to feel that this is the proper thing that matters mathematically and didactically. Again: if somebody or something is to be blamed for this shortcoming, it is not the teachers but rather teachers' education. They learnt ready made mathematics rather than having been guided to reinvent the implicit thinking structures and to make them explicit. They swallowed the stones of formal logic rather than digesting the bread of common sense logic.

Fig. 1

Saturday 26 September
What a day! Sunshine in the morning and quite hot. With our jackets we biked to the swimming pool. Suddenly wind rose, and it got cold. A short time it was even raining. Fortunately the sun came back for a while.

Sunday 27 September
Sunshine. Not as changeable as yesterday. For tomorrow even higher temperatures are previded. It is a pity that this is a long shoolday.

Monday 28 September
Very hot and oppressive. Late in the afternoon colder. Thunderstorm!

Tuesday 29 September
Yesterday's thunderstorm has confused the weather. The whole day less than 19°.

Wednesday 30 September
Variable overcast, variable temperature. More sun late the afternoon.

Thursday 1 October
About 11 o'clock a shower but otherwise fine. In the afternoon somewhat cloudy, but not cold.

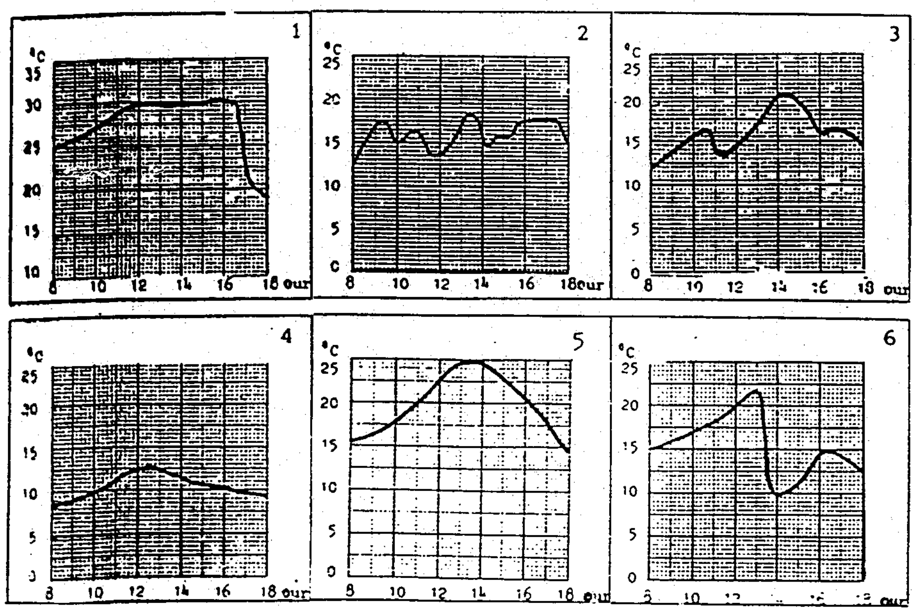
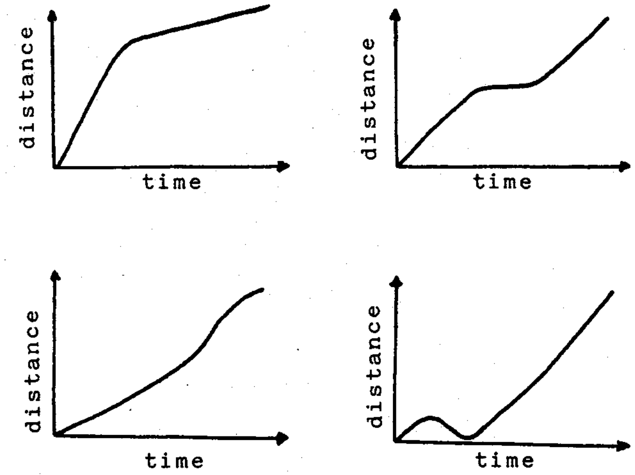


Fig. 2



Yuri
I always start quietly...
underway I go faster.....
I don't like to be late.

Hermina
After I had left, it crossed my mind that we had gymnastics today. I had forgotten my gym shoes. So I returned. Then I had to go hard to be on time.

Fritz
I went to school on moped,
but of course underway,..
bop bop bop, no gasoline.
Moped aside. Marching. Just
on time.

Mary
....

Fig. 3

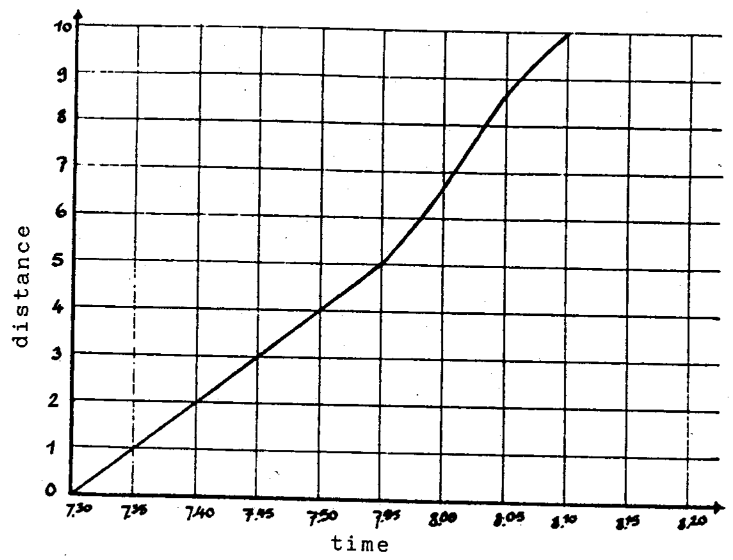


Fig. 4

Berdin reads the question: "Wait a moment".
Erna: It climbs the same. Always climbing the same.
Tasha agrees.
Erna: The line doesn't do like this  . It stays climbing the same.
Berdin is not convinced. After reading once more, he says: Because of the distance and the time.
Tasha: Each day he goes 5 minutes a kilometer.
Erna: but how fast does he go? You didn't explain why he goes as fast all the time.
Tasha: this is not the question. It is how you can see that he goes as fast. Not why he does so.
Berdin: Because of the kilometers, the distance, the time.
Colette asks Erna what she thinks about it. Erna doesn't answer.

. . .
. . .

Tasha: We are three against one, and still you (Erna) give a different answer.
Berdin: If you (Erna) are right, you get a sweet from me.
Teacher (arriving): Look, Erna. After 5 minutes he went 1km, after 10 minutes 2 km. Every five minutes 1 km more. It looks like a staircase and so you can see that he went regurarly. From 7.55 to 8.00 he went 1½ km. If he bikes regurarly the line shall be straight. Did you have it?
Teacher: ...Fine. Do you agree.
Erna: Yes.

. . .
. . .

Fig. 5

Yvonne: 

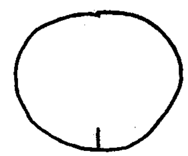
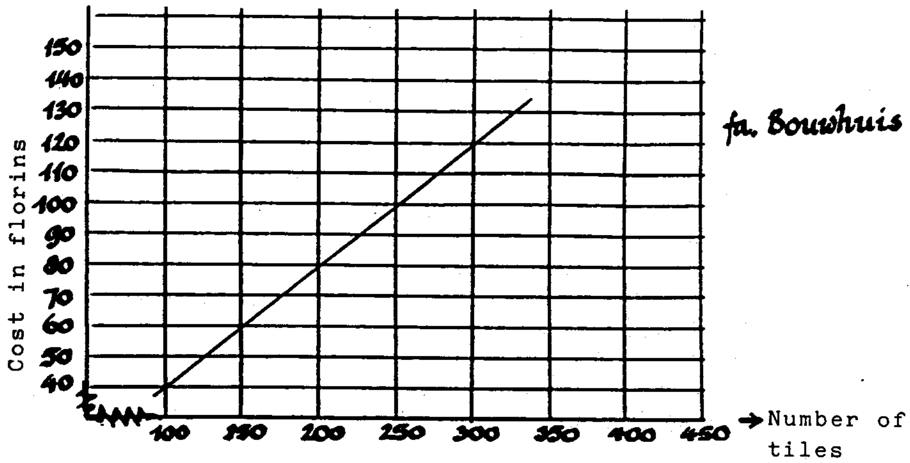
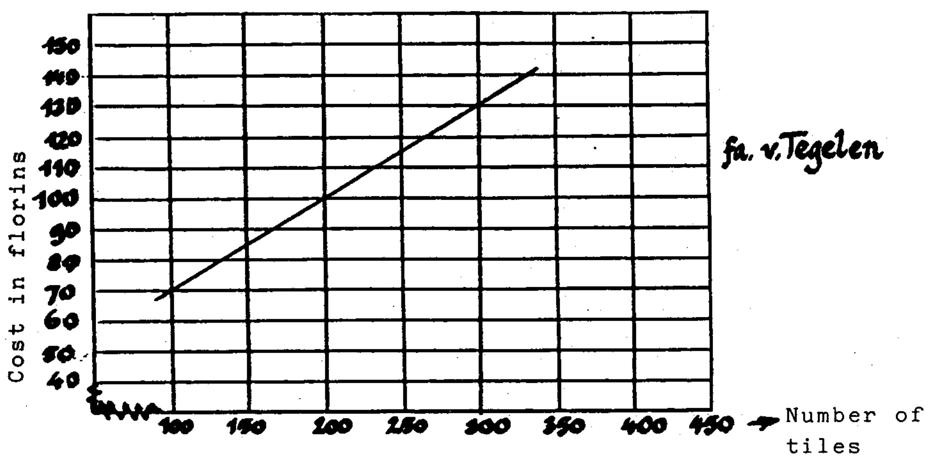
Leonie: 

Fig. 6



This graph shows how much to pay for a certain brand of tiles if brought home by Company Bouwhuis, dependent of the number bought.
The next is such a graph for the company Van Tegelen.



If you compare the graph you will see that they are not the same. The difference arises from the fact that one company does not charge the transport, whereas at the other the price per piece is lower.

A friend of yours intends to buy tiles. What would you advise her?