Estratto da

M. Barra e A. Zanardo (a cura di), *Atti degli incontri di logica matematica* Volume 5, Roma 6-9 aprile 1988.

Disponibile in rete su http://www.ailalogica.it

TEACHING LOGIC AND LOGICAL THINKING, TEACHING MATHEMATICS AND MATHEMATICAL THINKING, IN ENGLAND

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> *The chains by which the logicians imagine to be able to control the human mind seem to me of little value.*
>
> ***René Descartes***

Mi dispiace di non parlare italiano; ma posso emettere suoni che danno l'impressione che io lo parli. Il logico A. N. Whitehead ha detto:

> A un certo livello, il linguaggio non e altro che una serie di squittii.

I have done this to introduce awareness of language as one element of my talk. You will recall that the Greeks in general, and Aristotle in particular, was very interested in what separated humans from non-humans (or even Greeks from the barbarians on occasion) and his answer was language. He was also involved in discovering how talking about things got you to the Truth.

The structure of my talk is as follows. After introducing myself and my interests, I shall talk about the teaching of logic in schools in England and make some historical remarks about how this came about. Then I shall present a discussion of some recent work on proof in school mathematics. This in turn will raise some questions about use of mathematical language and also about being explicit or not with pupils, which I shall briefly explore.

My personal background is that I work with both adults and pupils, trying to encourage and support their mathematical thinking. The adults I meet are mainly either people studying beginning mathematics courses with the Open University, or teachers of other subjects retraining to be mathematics teachers. In either case, I meet adults who are coming to grips with mathematical and logical concerns which appear to be new for them. My particular interest is in language issues in mathematical education.

I was very pleased to be invited to come and talk to you today. But when I heard the requested topic, namely the teaching of logic in schools in England, my first thought was that this would be the shortest talk I have ever given. This is because there virtually is none now: logic, and certainly mathematical logic, is not taught in English schools - at least not explicitly! I could now either sit down and ask for questions, or go on to explain what I mean by this perhaps rather startling claim. I shall try to be provocative, so that the situation in England might allow you to reflect on your own situation and to clarify your beliefs.

Some History

In order to discuss the position of logic and its teaching in schools, I need to make some historical remarks about the curriculum. The mediaeval tradition which we share comprised firstly the *trivium* of grammar, logic and rhetoric and then later the *quadrivium* of arithmetic, geometry, music and astronomy. The *trivium* had to do with writing, thinking and speaking - correctly and with power - and was seen as a basis for all future work, not just mathematics.

The apparently necessary link between mathematics and logical thinking is not very old. It is true that Aristotle, when wishing to make a general logical point, would frequently turn to mathematics for an example. This presumes both that he felt that these examples would be familiar to his audience and that mathematical examples might provide the clear or immediate instances of what he wanted to point to. There were a lot of exciting things happening at the time in mathematics, particularly in Plato's Academy. For example, Aristotle returns time and again to the status of the incomensurability of the side and diagonal of a square. It is something you can prove by logic that you cannot prove any other way. John Fauvel (1987) has pointed out that the very notion of incommensurability and its proof must have occurred at the same time. One force of this example is thereby to validate the use of logic, to convince people of its value.

But Aristotle's aim for logic was much broader, namely how you found out about finding out about all things, not just mathematical things. This meta-aim is one which will recur throughout my talk, that of learning how to learn.

The purpose of Greek rhetoric was to teach people how to convince other people. One way of thinking about the debate between Aristotle and the sophists is whether or not you should believe the things you were trying to convince others about. (This same issue is alive in English education at the moment, for example in history with questions about empathy: "imagine you are a PLO terrorist and write from that perspective about the period 1970 - 1980").

In many English schools before the late1960s, in keeping with this tradition, it was in *English language* classes that logic was to be found, where syllogisms were studied and where logical thinking was encouraged. Learning explicitly how to think clearly was more the province of English teaching, rather than mathematics teaching. (This tradition continues at the University level where logic is more likely to be found in the department of philosophy than in that of mathematics.)

This discussion has introduced one of the themes that I wish to explore today, namely how are we best to encourage pupils to think about their own thinking, in order to be able to think better. For me, this is one of the central questions of mathematics education, and one which impinges directly on logic. Perhaps I can invite you to ask yourself **if** and **why** you think that explicit teaching about logic enables you think better? I will come back to this point later. (In the teaching of English, the same debate is currently very alive over whether explicit attention to syntactic concerns will allow greater control over the use of language as a whole.)

What is logic about? Professor Freudenthal's talk to this group explored the difference between logic as a subject, a body of knowledge - and logic as an attitude of mind. Justifications for teaching logic may also fall into roughly these two areas. One is that it is worth knowing in its own right about propositions, truth tables, sets and operations on them, and perhaps some Boolean algebra. The other is that knowing about these topics will (a) help pupils understand mathematics better or (b) help them to think better in general.

Mathematicians tend to think of logic as intrinsically and fundamentally mathematical. But this is very much a twentieth-century belief - historically, logic happens in another sphere of education. And was valued differently too: Cesare Cremonini, professor of philosophy at the University of Padua in the seventeenth century, who would have taught logic, was paid 2000 florins a year. His colleague, one Galileo Galilei, professor of mathematics at the same institution, was only paid 520 florins a year. No wonder Galileo wanted to be a professor of philosophy.

Even the nineteenth century is full of treatises on logic, all in prose, with not a symbol in sight, all concerned with right argument. One further question for you is why are twentieth-century schoolchildren presented with logic as something essentially mathematical? How has logic got lost from the rest of the curriculum?

The answer, for England, is relatively straightforward. Since the late 1960s, English teaching has been almost completely refocused on creative writing ,and the teaching of grammar and logic rather lapsed. This will be the fate for mathematics too, only twenty years later. (Though as I mentioned earlier, this year has seen this debate in language teaching recur, with the publication of the Kingman Report into the teaching of English in schools.)

However, there was a second historical tradition, one supported in the nineteenth-century curricula in Cambridge, whereby classical

Euclidean geometry was seen as a 'training for the mind', and this was one central justification for its being a compulsory study for *all* undergraduates. This justification was to be repeated by mathematics teachers in the grammar schools throughout much of the twentieth century. Thus, the claim goes, it is the study of mathematics itself that teaches you to think logically.
So the presumption was that by studying Euclidean geometry, you would tacitly learn about deductive reasoning, learn what a proof was and learn how to carry one out. The logical content would be absorbed 'by osmosis'. As an aside, let me say that this is one example of a dangerous false belief, that is that we learn from experience. What is minimally required is reflection on that experience - and this is much harder to encourage.

With the demise of Euclidean geometry from mathematics syllabuses also in the 1960s, what took its place? One thing that took its place was the explicit teaching of algebraic structures and deduction inside axiomatic systems. Here the idea was to make explicit the reasons for being able to make claims in terms of axioms. Another was the inclusion of certain topics on the borderline of mathematics and logic such as sets and operations on them, logical notation and some Boolean algebra. This curriculum reform, never widely successful, has recently given way to a change of a different sort.

Another grass-roots reform increasingly common since the 1960s had to do with pupils working on their own mathematics, rather than being taught someone else's. The content reform has almost gone now - what has replaced it now is the doctrine of problem solving. This values the individual mathematical activity of the pupil, and provides a mathematical counterpart (twenty years on) to the creative writing movement in English. One consequence of this is the virtual absence of presented proof from mathematics up to age 16. But before discussing why this is so, and how we approach trying to develop logical thinking in our pupils, I want to describe briefly a major change which is happening in secondary (11-18) education in England as I speak.

The English seem obsessed with public, written examinations and the questions in them are one of the prime determinants of what actually goes on in secondary schools. This is the first year of a change from the dual examination system at 16 of O-level and CSE to a single examination, the GCSE. GCSE mathematics has a considerable coursework element, that is pupil's own extensive investigative work in the school. This has come about in large part due to the Cockcroft report, a national enquiry into the teaching of mathematics in schools which was published in 1982. One particular change has been in the moving away from mathematics seen solely in content terms, such as decimals, solving quadratic equations and matrices, towards a more activity-based and process description, valuing estimating, modelling, generalising and so on.

The content revisions of the new GCSE leave out any mathematical logic. This indicates that the mathematical-logic-for-its-own-sake argument has been rejected. What happened, in fact, with these curriculum items was that they were taught in isolation, and little purpose for them was seen. Little more than the set notation was introduced, and the hoped-for unification did not occur. Textbooks had their compulsory chapter on sets and logic, but it was soon forgotten as very little was done with it.

The second argument I mentioned above, namely logic helping you to do mathematics better, has also been rejected, in favour of more specific attention on how to do mathematics itself. For although there is now no explicit teaching of logic, an increasing amount of attention is being paid to improving pupil awareness of the proceses of mathematical thinking.

However, there is a major tension here. The experience of twenty years of attempting to draw pupils' attention to mathematical processes is that one of the biggest dangers is that the processes are taken by the pupils as the new content to be learned. Instead of learning about triangles, they are learning about specialising and generalising. This is a perennial danger whatever kind of meta-teaching you try to undertake. Pupils will see it at the same level as triangles and quadratic equations - it is just more stuff to

be learnt. (See the article by Eric Love, 1988, for a further discussion of this point.)

The Question of Proof

As I mentioned above, the GCSE pays almost no attention to proof. Let me give you an example. Many of the coursework investigations are mathematical situations for the pupils to explore - to develop and present their own mathematics. A number of mathematical process objectives may be reached, such as:

- looking at special cases;
- generalising;
- conjecturing;
- working systematically;
- presenting information.

However, proofs for many of the results about the situations explored are not accessible to the age-range of pupils to whom they are offered. For example, explore the number of regions created by placing different numbers of dots equally spaced round a circle and joining all the diagonals. This may allow a rule to be conjectured (for example, by looking inductively at common differences in a table of values). The idea that mathematics is essentially about seeing *why* something is true, not just the *what* that is true, seems missing.

Since the mid-1970s, little work has been done in England on pupils' notions of proof. Alan Bell in 1976 wrote a Ph.D. on pupils' generalisation strategies and, more importantly, Imre Lakatos provided a description of mathematical activity. In his book Proofs and Refutations, Lakatos provides a collection of notions for thinking about the doing of mathematics, examining intimately the relations between statement of theorem and proof, the roles and types of counterexamples and how they are dealt with. His work is meta-mathematical in the broad sense of providing a way of talking **about** mathematics.

For example, he writes of lemma incorporation, an *ad hoc* device to save a conjecture from refutation by adding in an extra requirement, one that rules out the particular counterexample. I was working with a twelve-year-old boy who was coming up with his own generalisations about the area and perimeter of rectangles, after having examined the truth or falsity of examples such as 'for every rectangle with a given perimeter, there is another one with the same perimeter and a larger area'.
He offered the conjecture that 'the area is always smaller than the perimeter' which he verified on three numerical examples. When I offered him one for which his conjecture did not work, he looked at his examples and my counterexample, and modified his conjecture by adding 'when the sum of the two sides is less than 12'. This was a criterion that distinguished between his cases and the one I had proposed.

Lakatos' description helped me to understand and recognise the processes that this pupil was using and if you, or the teachers you work with, do not know Lakatos' work I can recommend it highly to you. (It is available in Italian.)

Another commonly-offered problem is that of finding the number of diagonals of a polygon if you know the number of sides. I have chosen this problem as it formed the basis of an extensive piece of work on types of proof reasoning carried out by Nicolas Balacheff (1988) in France. I know I was supposed to talk about England, but this study, more than any other, has given me insight into much of what I have seen in England, in the mathematical thinking of both pupils and adults.

What follows is but a very small part of his overall study. He has identified four different types of reasoning strategy which pupils employ when trying to justify why the generalisation they have come to for the above problem about diagonals is true. Balacheff calls their efforts 'proofs' because that is how they are seen by the pupils. His aim is to understand better the proof processes that these thirteen-year-old pupils employ.

The four types he details are as follows:

Naive empiricism:

This occurs when asserting something is true after verifying several cases. Balacheff gives the example of a pupil pair who conjecture that $f(n) = n/2$, because it works "with a square, an eight-sided one and a six-sided one, so there you are, it must always be divide by two".
In my own work with adults retraining to teach mathematics, I have noticed this very frequently as a style of reasoning. The question "when will I have done enough special cases to be sure?" also points to such an approach. Such a mode of thinking is very common in the everyday world, where there is no expectation of any other means of justification, no structure to draw on. How do people learn to move beyond this style of reasoning, or how do they learn that other styles are possible?

The crucial experiment:

Balacheff writes, "This comes from choosing a particular case, one that has not been looked at before in order to arrive at the generality, asserting that 'if it works here, it will always work'.
Here, the pupil has explicitly posed the problem of generality and resolves it by staking all on the outcome of a particular case that she recognises to be not too special. It is both a means for checking a result and a weapon in discussions about the validity between two pupils from the same pair. 'We'll try it with a 15-sided one, and then if it works for that, well then that means that it works for the others.' In fact, they actually carry this out on a ten-sided one, because the fifteen-sided one is too complex."

This style of thinking too I have regularly seen in adults, both Open University students at summer schools and teachers undertaking further education. One actually codified the principle as "if it is true for $n = 17$, then it is true!" The status of this try is different from the others in that it does not just add one more confirming instance, it is putting the proposition to the test. Again, the realisation that the case should not already have been looked at nor should it be 'too special' indicate some thought about ensuring the validity of the proceedings.

The generic example:

This involves making explicit the reasons why something is true by a showing on a particular example, but one that is there as a representative of its whole class. (For further discussion of this idea, see Mason and Pimm, 1984)
This style of thinking can be very hard to distinguish from looking at a particular case, since it is only in the mind that the differences occur.

The thought experiment:

This fourth type involves general arguments, frequently from the definitions, rather than operating on any particular case. The relations upon which the proof depends are indicated in some other way than the result of their use. Balacheff found very few instances of this approach in the spontaneous arguments of the pairs.

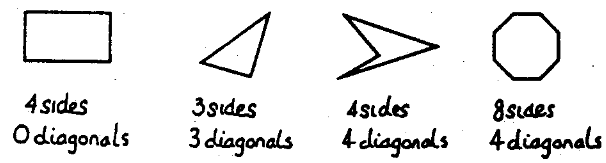
He asserts that there is a hierarchy among these types of proofs, and that there is a break between naive empiricism and the crucial experiment on the one hand, and the generic example and the thought experiment on the other. This divide can be characterised as passing from a truth asserted on the basis of a statement of fact to one of an assertion based on reasons. It is a change in the way of thinking about the problem.

What do we learn from this work? Firstly, that there are at least these different approaches to justifying generalities, ordered in this way. So again, they are of interest to the teacher to look out for in her pupils. How might the pupils themselves become aware of the fact that this is how they are arguing and, also, that there are other ways of arguing?

These four types of thinking are all of relevance to mathematics

teaching, and their characteristics should be identified. Too frequently, logic has taken a prescriptive view of identifying errors of reasoning, rather than actually examining how people actually do reason and why. It is only then that we might identify what it is about mathematics that is distinctive and calls for different thinking styles.

Let me give an example here. The came from a pupil I interviewed on the above problem involving diagonals posed in the following way. She responded to four particular cases as follows:



Can you make sense of these results? What do you think her concept of diagonal was?

In English, the word *diagonal* is grammatically an adjective, contrasting with vertical and horizontal. In mathematics, there is a shift and the word becomes a noun, *a diagonal.* In mathematics, however, diagonals do not need to be diagonal to the page. Her answer, that 'no, you cannot tell how many diagonals given the number of sides of a polygon, because it depends how it is placed', is perfectly correct given her concept of diagonal, one which agrees with the everyday language usage.

This provides one instance of many of the fact that there is a collection of particular uses that have developed for expressing mathematical ideas within the English language, and that there will be for Italian too. One technical term for this is register. Linguist Michael Halliday (1975, p.65) writes:

> A *register* is a set of meanings that is appropriate to a

particular function of language, together with the words and structures which express these meanings. ... We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is used for mathematical purposes. ... In order to express new meanings, it may be necessary to invent new words; but there are many different ways in which a language can add new meanings, and inventing new words is only one of them.

How much are mathematics teachers aware of the particular attributes of the register for their language? (For further discussion of this, see Pimm, 1987.)

The move from natural language:

Although logic has to do with thought, this is frequently only made manifest in language, whether spoken or written. One reason that I believe that using natural language examples to teach about logic fails is that logic does not describe either all or accurately the relations which hold in natural language. The logic of everyday conversation is rarely that of the the predicate calculus. (See, for example, Stubbs, 1986)

To give two small examples: as René Thom (1973) points out, to say that the flag is red and white does not allow the conclusion that the flag is red. Yet if the logician's 'and' is at work, it should. Secondly, to say 'I opened the door and he fell over' is to say something different from 'he fell over and I opened the door'. Temporal sequence and causality are frequently intended in such sentences.

In addition to the shift in grammatical status of certain terms, there is a shift in the meaning in the small connecting words, such as and, or, not, if ... then, or quantiative markers such as some, all and any when they move over to the mathematics register. David Tall (1977), at Warwick University, carried out some experiments with first-year mathematics undergraduates on their use of the terms some and all. They regularly marked propositions such as

'some rational numbers are real numbers' as false, because all rational numbers were real numbers. Some apparently entailed not all for these speakers.

Similar difficulties arise with the use of the word 'any'. Mathematicians use it in the sense of 'every': show that for any 2 x 2 matrix A of the form ..., $A^2 = A$. Many adult students regularly take one particular example of such a class of matrices and show it for that one. "Well it said to show it for any one, so I just picked one. I thought that was what it meant."

It was in my English lessons at school that my sixty-five-year-old teacher tried, by means of examples such as 'if you wish to go to Italy, you can get information from the tourist board', to teach me the illogicality of such utterances, because their contrapositive is false. He failed. The above is the way that I, and many other native English speakers, express the idea it contains. We block the 'logical' sense of the if ... then construction, and so are not bothered by the fact that the contrapositive is false.

Mathematicians and logicians, it sometimes seems to me, are imperialistic and wish to impose their usages on the world at large. When we fail to conform to their requirements, we are labelled illogical. It is only in the mathematics register that *every* use of the construction if ... then carries with it the force of logical implication. And it is this that pupils learning mathematics have to be aware of. It is only in logic that two negatives make a positive. Double negatives, such as 'I don't know nothing' are not logically wrong; they occur in certain dialects of English and not others. They also occur in standard French, for example: '**Je n'en sais rien**' has two negative markers. These words are not logical operators. Linguist Michael Stubbs (1983) has described this as 'the pseudo-algebraic view of language'.

The second difficulty with natural language examples is that attention is often paid to the content of the sentences rather than the form alone - and it is therefore knowledge of the world rather than linguistic knowledge which guides understanding. However, this can also act as a problem in the opposite direction. There are two ways of realising that 'if the function has a minimum, then f'(x) = 0 there' does not allow you to conclude that 'if f'(x) = 0 then you must have a minimum'. The first is particular and comes from understanding the mathematical situation, namely that there are counterexamples. The second is to argue from the logical form, without recourse to the mathematical content - that such a deduction is not always warranted - but it is hard to contend with the fact that nonetheless, the particular answer may also be correct in the instance you are looking at. The presence of logical errors does not necessarily invalidate the conclusion of a particular argument; that is content specific and depends on the circumstances. It invalidates the necessity of the argument, that is all. All it says is this argument is incomplete, I am not yet convinced.

How to encourage pupils to become more aware of their own thinking?

There are four direct quotations from Professor Freudenthal's talk that I wish to recite here.

> Thinking about one's own thoughts - reflective thinking - is a hard thing even for adults.

> Logic in the sense of formal logic means as little in school instruction as it does in life. So it is neither worth teaching or learning.

> ... logic cannot be taught as a subject matter.

> teachers .. should avail themselves of opportunities to make explicit the implicit logical behaviour.

I said at the outset that I was interested in assisting the mathematical thinking of others. Past thinking has been codified, formalised and frozen into mathematical 'knowledge' and it is this that tends to be taught. By looking at examples of mathematical arguments, regularities and styles of logical thinking can be observed and studied, giving rise to the academic subject of logic. One reason that it seems to me to be a bad idea to try to teach formal logic is that it is so far away from where I wish to be.

I wish to end with four brief suggestions for possible ways of encouraging pupils to become more aware of thieir own thinking.

> Become more aware of your own thinking. Talk about it in class.

> Ask questions to try to focus pupils' attention on their own thinking. Show that you value thinking in itself and not just the validity of the products of thinking.

> Engage in classroom activities which nmake manifest the effects of their own thinking (e.g. debugging in LOGO).

> 'Communication is not the only function of language'. Explore aspects of speaking and writing which allow you better access to and control over your own thinking. How to convince pupils of this?

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I am very grateful to my colleague John Fauvel for extensive discussions on the historical context of this talk.