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Dynamic logic and the logic of ability.

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A propositional "logic of ability" was proposed by Mark A. Brown [B]. In addition to the boolean connectives, it has unary connectives $\boxdot$ and $\Diamond$ whose intended interpretations are:

$\boxdot A$: a (fixed) agent can reliably effect A,

$\Diamond A$: the agent can bring it about that A *might* obtain.

More specifically, the agent, in the currently actual world, has available a set of *actions*, each of which leads to an undetermined one of a set of possible worlds; $\boxdot A$ [respectively, $\Diamond A$] is true in the actual world if A is true in all [respectively, some] of the worlds to which some action leads. (Brown used $\diamondsuit$ and ◈ instead of the present $\boxdot$ and $\Diamond$.)

The semantics is one of neighborhoods: a model is a triple $\mathcal{M} = \langle W,N,V\rangle$, where W is a non-empty set, N is a function from W to $\mathcal{P}(\mathcal{P}(W))$, and V is a function from the set of propositional variables to $\mathcal{P}(W)$. The truth-definition has

$$\mathcal{M} \models_s \boxdot A \Leftrightarrow (\exists K \in N(s))(\forall t \in K)(\mathcal{M} \models_t A),$$

$$\mathcal{M} \models_s \Diamond A \Leftrightarrow (\exists K \in N(s))(\exists t \in K)(\mathcal{M} \models_t A).$$

Notice that the "actions" available to the agent are not represented either in the syntax or in the formal semantics. It is natural to inquire into the relationship between this logic of

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abstracted abilities and logics which take the "actions" more seriously, namely variants of propositional dynamic logic (PDL). (A good reference for PDL is [G].) In this context, it is in order to extend Brown's logic slightly: since the agent will generally have available actions which lead to the empty set of outcomes (this as a matter of logic, if "tests" are permitted) we exclude these as ways of effecting things, by changing the $\boxdot$-clause of the truth-definition to

$$\mathcal{M} \models_s \boxdot A \Leftrightarrow (\exists K \in N(s))[K \neq \emptyset \ \& \ (\forall t \in K)(\mathcal{M} \models_t A)]$$

(hence $\boxdot A \to \Diamond A$ is valid).

In particular, Krister Segerberg [S1] proposed adding to PDL (without program variables or the "test" operator) a program-forming operator δ; if A is a formula then δA is supposed to be the action of "bringing about A" in the most general possible way. A model for the resulting logic is a triple $\mathcal{M} = \langle W,D,V \rangle$, where D is a function from $\mathcal{P}(W)$ to $\mathcal{P}(W \times W)$ whose role is to interpret δ. It is required that

(D1) $\langle s,t \rangle \in D(S) \Rightarrow t \in S$,

(D2) $(\forall t)(\langle s,t \rangle \in D(T) \Rightarrow t \in S)$
$\Rightarrow (\forall t)(\langle s,t \rangle \in D(T) \Rightarrow \langle s,t \rangle \in D(S))$.

By (D1) and (D2) respectively, the formulas

(δ1) $[\delta A]A$,

(δ2) $[\delta B]A \to (\langle \delta B \rangle C \to \langle \delta A \rangle C)$

are valid in every model. Thus by (δ1), if $\langle \delta A \rangle \top$ (that is, some execution of δA terminates) then δA is a way of reliably

effecting A, and by (δ2), δA is the most general such (at least, most general among actions of the form δB).

In [S2] Segerberg proposed adding another program-forming operator ϵ, with the intended interpretation that ϵA is δA if $\langle \delta A \rangle \top$, otherwise ϵA is some action which has some outcome in which A obtains if there is any such action, otherwise ϵA is "abort". The models are expanded to quadruples $\langle W,D,E,V \rangle$, satisfying (D1), (D2), and

(E1) $(\exists t)(\langle s,t \rangle \in E(S)) \Rightarrow (\exists t)(\langle s,t \rangle \in E(S) \ \& \ t \in S)$,

(E2) $(\exists t)(\langle s,t \rangle \in E(T) \ \& \ t \in S) \Rightarrow (\exists t)(\langle s,t \rangle \in E(S))$,

(DE) $(\exists t)(\langle s,t \rangle \in D(S))$
$\Rightarrow (\forall t)(\langle s,t \rangle \in D(S) \Leftrightarrow \langle s,t \rangle \in E(S))$.

(Observe that if $R \subseteq W \times W$ and $D(S) = E(S) = R \cap (W \times S)$ for all $S \subseteq W$, then (D1)-(DE) are satisfied.)

There is a natural interpretation of Brown's logic in Segerberg's δ-ϵ logic: $\boxdot A$ is interpreted as $\langle \delta A \rangle \top$, and $\Diamond A$ as $\langle \epsilon A \rangle \top$. It can be shown that if B is any thesis of Brown's logic of ability then the interpretation of B is a thesis of Segerberg's δ-ϵ logic. It is not, however, the case that the interpretation of B "says the same thing" about a model as B itself does.

To make this claim precise, define: if $\mathcal{M} = \langle W,D,E,V \rangle$ is a Segerberg model then the *derived* Brown model is $\mathcal{M}^B = \langle W,N,V \rangle$, where $N(s)$ comprises all sets of the form $K_{\alpha,s} =$

$\{t : \langle s,t\rangle \in \|\alpha\|\}$, where α is a program. [Given $\mathcal{M}$, the *intensions* $\|A\|^{\mathcal{M}} = \|A\| \subseteq W$ of formulas A and $\|\alpha\|^{\mathcal{M}} = \|\alpha\| \subseteq W \times W$ of programs α are how $\mathcal{M}$ interprets formulas as subsets of W and programs as binary relations on W. See the definition of S-model below for more detail.] Thus $\mathcal{M}^B$ represents the assumption that the "actions" available to the agent at each world are exactly the programs of Segerberg's variant of PDL, and the worlds to which they lead are as specified by $\mathcal{M}$; we have $\mathcal{M}^B \models_s \Diamond A$ if and only if, for some program α and some $t \in K_{\alpha,s}$, $\mathcal{M}^B \models_t A$.

Now programs are composable in Segerberg's logic, and in $\mathcal{M}$ $\|\alpha;\beta\| = \{\langle s,t\rangle : (\exists u)(\langle s,u\rangle \in \|\alpha\| \;\&\; \langle u,t\rangle \in \|\beta\|)\}$. Hence if $u \in K_{\alpha,s}$ and $t \in K_{\beta,u}$ then $t \in K_{\alpha;\beta,s}$, and thus $\Diamond\Diamond p \to \Diamond p$ is valid in $\mathcal{M}^B$. But the interpretation $\langle \varepsilon(\langle \varepsilon p\rangle\tau)\rangle\tau \to \langle \varepsilon p\rangle\tau$ of $\Diamond\Diamond p \to \Diamond p$ need not be valid in $\mathcal{M}$: if $W = \{s,t,u\}$, $R = \{\langle s,t\rangle,\langle t,u\rangle\}$, $D(S) = E(S) = R \cap (W \times S)$, and $V(p) = \{u\}$, then $\|\varepsilon p\| = \{\langle t,u\rangle\}$, $\|\langle \varepsilon p\rangle\tau\| = \{t\}$, $\|\varepsilon(\langle \varepsilon p\rangle\tau)\| = \langle s,t\rangle$, and $\|\langle \varepsilon(\langle \varepsilon p\rangle\tau)\rangle\tau\| = \{s\}$.

[The basic problem seems to be that just as δA is not really the most general way of bringing about A, only most general among actions of the form δB, so εA is only required to have an outcome in which A holds when some action of the form εB has such an outcome. Neither $[\alpha]A \to (\langle\alpha\rangle C \to \langle\delta A\rangle C)$ nor $\langle\alpha\rangle A \to \langle\varepsilon A\rangle\tau$ is valid, as the example above shows (take $\alpha = \delta\tau;\delta\tau$ and $A = C = p$).]

The main purpose of the present paper is to investigate another way (also suggested by Segerberg [S2]) of interpreting Brown's logic of ability in a logic of actions: add the operators $\Diamond$ and $\boxdot$ to some variant of PDL, with a semantics whose motivation is that the agent's "actions" are determined by the programs.

We begin with a very minimal variant of PDL--the only programs are the program variables. (This minimal PDL may be regarded as the logic of independent, unstructured actions.) The system S has countably infinite sets PropVar of proposition variables and ProgVar of program variables. The sets Fla and Prog of formulas and programs are defined by:

PropVar $\subseteq$ Fla,

$\bot \in$ Fla,

$A, B \in$ Fla $\Rightarrow$ $A \to B$, $\Diamond A$, $\boxdot A \in$ Fla,

ProgVar $\subseteq$ Prog,

$\alpha \in$ Prog, $A \in$ Fla $\Rightarrow$ $[\alpha]A \in$ Fla.

Usual abbreviations are used; in particular, $\Box A$, $\blacklozenge A$, and $\langle\alpha\rangle A$ abbreviate $\neg\Diamond\neg A$, $\neg\boxdot\neg A$, and $\neg[\alpha]\neg A$ respectively.

An S-model is a triple $\mathcal{M} = \langle W,P,V\rangle$, where W is a non-empty set, P maps ProgVar into $\mathcal{P}(W \times W)$, and V maps PropVar into $\mathcal{P}(W)$. Intensions $\|A\|^{\mathcal{M}} = \|A\|$ and $\|\alpha\|^{\mathcal{M}} = \|\alpha\|$ are defined by

$\|p\| = V(p)$, if $p \in PropVar$,

$\|\bot\| = \emptyset$,

$\|A \rightarrow B\| = (W - \|A\|) \cup \|B\|$,

$\|\Diamond A\| = \{s : (\exists\alpha \in Prog)(K_{\alpha,s} \cap \|A\| \neq \emptyset)\}$,

$\|\boxdot A\| = \{s : (\exists\alpha \in Prog)(\emptyset \neq K_{\alpha,s} \subseteq \|A\|)\}$,

$\|\pi\| = P(\pi)$, if $\pi \in ProgVar$,

$\|[\alpha]A\| = \{s : K_{\alpha,s} \subseteq \|A\|\}$,

where $K_{\alpha,s} = \{t : \langle s,t\rangle \in \|\alpha\|\}$. We sometimes write $\mathcal{M} \vDash_s A$ for $s \in \|A\|$; A is *valid* in $\mathcal{M}$ if $\|A\| = W$. Similarly, if $\mathcal{M}$ is a neighborhood model we sometimes write $\|A\|$ for $\{s : \mathcal{M} \vDash_s A\}$.

A *pseudo-model* is a quadruple $\mathcal{N} = \langle W,P,N,V\rangle$, where $\langle W,P,V\rangle$ is an S-model and N is a function from W to $\mathcal{P}(\mathcal{P}(W))$. Define intensions in a pseudo-model as above, except that

$\|\Diamond A\| = \{s : (\exists K \in N(s))(K \cap \|A\| \neq \emptyset)\}$,

$\|\boxdot A\| = \{s : (\exists K \in N(s))(\emptyset \neq K \subseteq \|A\|)\}$.

If $\mathcal{M} = \langle W,P,V\rangle$ is a model then the *derived* pseudo-model $\mathcal{M}^{ps} = \langle W,P,N,V\rangle$ is defined by setting $N(s) = \{K_{\alpha,s} : \alpha \in Prog\}$; clearly $\|A\|^{\mathcal{M}^{ps}} = \|A\|^{\mathcal{M}}$ and $\|\alpha\|^{\mathcal{M}^{ps}} = \|\alpha\|^{\mathcal{M}}$.

The axioms and rules of S are those of the classical propositional calculus, together with:

(C◇)	$\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$,
(V)	$\boxdot(A \vee B) \rightarrow (\boxdot A \vee \Diamond B)$,
(W)	$\Diamond\top \rightarrow (\Box A \rightarrow \boxdot A)$,
(D⊡)	$\boxdot A \rightarrow \Diamond A$,
(Inst◇)	$\langle\alpha\rangle A \rightarrow \Diamond A$,

(Inst⊡)	$(\langle\alpha\rangle\top \wedge [\alpha]A) \rightarrow \boxdot A$,
(K)	$[\alpha](A \wedge B) \equiv ([\alpha]A \wedge [\alpha]B)$,
(Nec)	from A, infer $[\alpha]$,
(RN□)	from A, infer $\Box A$,
(RM⊡)	from $A \rightarrow B$, infer $\boxdot A \rightarrow \boxdot B$.

Of these axiom schemes and rules, those which do not refer to programs constitute an axiomatization of Brown's system $\nabla D\boxdot$. By [B], a formula is provable in that system if and only if it is valid, in the sense of Brown's original truth-definition, in all neighborhood models $\langle W,N,V\rangle$ satisfying $\emptyset \notin N(s)$. If $\mathcal{M} = \langle W,N,V\rangle$ is any neighborhood model, let $\mathcal{M}^1 = \langle W,N^1,V\rangle$ be like $\mathcal{M}$ except that $N^1(s) = N(s) - \{\emptyset\}$. Then $\mathcal{M} \vDash_s A$ in the sense of our modified truth-definition if and only if $\mathcal{M}^1 \vDash_s A$ in the sense of Brown's original truth-definition. Hence $\nabla D\boxdot \vdash A$ if and only if A is valid in every neighborhood model--here and hereafter, we have in mind the modified truth-definition.

The axiom scheme (K) and rule (Nec) suffice for the fragment of PDL whose programs are just the program variables; indeed, this is merely the logic of a countable infinity of independent normal modalities. The axioms (Inst◇) and (Inst⊡) express existential generalization principles for the respective connectives.

Lemma 1. (Soundness) If $S \vdash A$ then A is valid in every S-model.

Proof. It is straightforward to check that the axioms are valid and that the rules preserve validity.

Lemma 2. Given any neighborhood model $\mathcal{M} = \langle W,N,V\rangle$, there is a neighborhood model $\mathcal{M}^1 = \langle W,N^1,V\rangle$ such that each $N^1(s)$ is non-empty and countable, and for every formula A, $\|A\|^{\mathcal{M}^1} = \|A\|^{\mathcal{M}}$.

Proof. "Define" $N^1(s)$ as follows. For each formula A such that $\mathcal{M} \vDash_s \Diamond A$ choose $K \in N(s)$ such that $K \cap \|A\| \neq \emptyset$ and put $K \in N^1(s)$, and for each A such that $\mathcal{M} \vDash_s \boxdot A$ choose $K \in N(s)$ such that $\emptyset \neq K \subseteq \|A\|$ and put $K \in N^1(s)$, and also put $\emptyset \in N^1(s)$. Then each $N^1(s)$ is non-empty and countable. One may prove by induction on A that $(\forall s \in W)(\mathcal{M}^1 \vDash_s A \Leftrightarrow \mathcal{M} \vDash_s A)$. Consider the case $A = \boxdot B$:

$$\mathcal{M} \vDash_s \boxdot B$$
$$\Leftrightarrow (\exists K \in N(s))(K \neq \emptyset \ \ \& \ \ (\forall t \in K)(\mathcal{M} \vDash_t B))$$
$$\Leftrightarrow (\exists K \in N^1(s))(K \neq \emptyset \ \ \& \ \ (\forall t \in K)(\mathcal{M} \vDash_t B))$$
$$\Leftrightarrow (\exists K \in N^1(s))(K \neq \emptyset \ \ \& \ \ (\forall t \in K)(\mathcal{M}^1 \vDash_t B)) \qquad [\text{I. H.}]$$
$$\Leftrightarrow \mathcal{M}^1 \vDash_s \boxdot B.$$

By Lemma 2, and Brown's completeness theorem, $\mathcal{T}D\boxdot$ is the logic of ability associated with the assumption that at each possible world the agent has available a non-empty countable set of actions. In other words, this assumption about the cardinalities of the sets of actions available to the agent has no effect on the logic of ability.

Theorem 3. S is a conservative extension of $\mathcal{T}D\boxdot$, that is, if A is a formula of $\mathcal{T}D\boxdot$ then $S \vdash A$ if and only if $\mathcal{T}D\boxdot \vdash A$.

Proof. Since the axioms and rules of S include those of $\mathcal{T}D\boxdot$, the implication from right to left is trivial. For the other direction, it suffices to show that if $\mathcal{M} = \langle W,N,V\rangle$ is any neighborhood model then there is an S-model $\mathcal{M}^1 = \langle W,P,V\rangle$ such that for every formula A of $\mathcal{T}D\boxdot$, $\|A\|^{\mathcal{M}^1} = \|A\|^{\mathcal{M}}$. Without loss of generality (by Lemma 2) we may assume that each $N(s)$ is non-empty and countable. For each s, let ϕ_s be a function mapping ProgVar onto $N(s)$, and let $P(\pi) = \{\langle s,t\rangle : t \in \phi_s(\pi)\}$; let $\mathcal{N}$ be the pseudo-model $\langle W,P,N,V\rangle$. Then by construction $\mathcal{N}$ is the derived pseudo-model of the S-model $\mathcal{M}^1 = \langle W,P,V\rangle$, so $\|A\|^{\mathcal{M}} = \|A\|^{\mathcal{M}^1}$ for all formulas A. And $\|A\|^{\mathcal{N}} = \|A\|^{\mathcal{M}}$ for all formulas A of $\mathcal{T}D\boxdot$, independently of the definition of P.

By Theorem 3, $\mathcal{T}D\boxdot$ is the logic of ability determined by a fixed non-empty countable set of independent actions: if there is a non-empty countable set of actions that are available to the agent at each possible world, and no assumptions are made concerning relations among the actions, then the principles geverning $\boxdot$ and $\Diamond$ are exactly those given by $\mathcal{T}D\boxdot$. This result differs from that implied by Lemma 2 by transposition of the quantifiers "$\exists$ set of actions" and "$\forall$ possible world". Trans-world identification of actions has no effect.

If A is a formula and Γ is a set of formulas, then $\Gamma \vdash A$ if $S \vdash (B_1 \wedge \ldots \wedge B_n) \rightarrow A$ for some $B_1,\ldots,B_n \in \Gamma$; Γ is *consistent* if $\Gamma \nvdash \bot$; $\Diamond[\Gamma] = \{\Diamond A : A \in \Gamma\}$, $\Diamond^{-1}[\Gamma] = \{A : \Diamond A \in \Gamma\}$, and similarly for other modal operators.

The *canonical* pseudo-model $\mathcal{N} = \langle W,P,N,V\rangle$ for S is defined by:

W is the set of maximal consistent sets of formulas,

$\langle s,t\rangle \in P(\pi) \Leftrightarrow [\pi]^{-1}[s] \subseteq t$,

$K \in N(s) \Leftrightarrow \Diamond[\cup K] \subseteq s \;\&\; \boxdot[\cap K] \subseteq s$,

$s \in V(p) \Leftrightarrow p \in s$.

Lemma 4. In the canonical pseudo-model, $s \in \|A\| \Leftrightarrow A \in s$.

Proof. By induction on A. The cases $A = \bot$ and $A = B\rightarrow C$ are trivial. The case $A = [\alpha]B$ is quite standard. Since $\alpha = \pi \in$ ProgVar, if $A = [\pi]B \in s$ then $(\forall t)(\langle s,t\rangle \in P(\pi) = \|\pi\| \Rightarrow B \in t)$ by definition of P, so that $s \in \|[\pi]B\| = \|A\|$; we must prove the converse. Suppose $A \notin s$; let $x = [\alpha]^{-1}[s] \cup \{\neg B\}$. Then x is consistent: otherwise $S \vdash C\rightarrow B$ for some $C \in [\alpha]^{-1}[s]$ ($[\alpha]^{-1}[s]$ is closed under conjunction by (K)), so $S \vdash [\alpha]C \rightarrow [\alpha]B$ and $A = [\alpha]B \in s$, a contradiction. Since x is consistent, there is a $t \in W$ such that $t \supseteq x$. Then $\langle s,t\rangle \in \|\alpha\|$ and $B \notin t$, so by the induction hypothesis $t \notin \|B\|$, and by the definition of $\|[\alpha]B\|$, $s \notin \|A\|$.

The cases $A = \Diamond B$ and $A = \boxdot B$ are handled much as in [B]. Note first a few facts about the canonical pseudo-model.

Note 1. If $s \in W$ then

$\Box^{-1}[s] \vdash A \Rightarrow \Box A \in s$,

$\Box^{-1}[s] \vdash C\rightarrow A \Rightarrow \dot\Diamond C \rightarrow \dot\Diamond A \in s$.

For the first, suppose $S \vdash B\rightarrow A$ where $\Box B \in s$ ($\Box^{-1}[s]$ is closed under conjunction by $(C\Diamond)$ and $(RN\Box)$); then by $(RN\Box)$ $S \vdash \Box B \rightarrow \Box A$. By maximality, $\Box A \in s$. For the second, suppose $S \vdash B\wedge C \rightarrow A$ where $\Box B \in s$. Then $S \vdash \neg A \rightarrow (\neg B\vee\neg C)$ so by $(RM\boxdot)$ $S \vdash \boxdot\neg A \rightarrow \boxdot(\neg B\vee\neg C)$; by (V), $S \vdash \boxdot\neg A \rightarrow (\Diamond\neg B \vee \boxdot\neg C)$, that is, $S \vdash \Box B \rightarrow (\dot\Diamond C \rightarrow \dot\Diamond A)$.

Note 2. If $s \in W$ and $K \subseteq W$ then

$\Diamond[\cup K] \subseteq s \Leftrightarrow \Box^{-1}[s] \subseteq \cap K$,

$\boxdot[\cap K] \subseteq s \Leftrightarrow \dot\Diamond^{-1}[s] \subseteq \cup K$.

To verify the second:

$\boxdot[\cap K] \subseteq s$

$\Leftrightarrow (\forall A)[(\forall t\in K)(A \in t) \Rightarrow \boxdot A \in s]$

$\Leftrightarrow (\forall A)[\dot\Diamond\neg A \in s \Rightarrow (\exists t\in K)(\neg A \in t)]$

$\Leftrightarrow (\forall A)[\dot\Diamond A \in s \Leftrightarrow (\exists t\in K)(A \in t)]$

$\Leftrightarrow \dot\Diamond^{-1}[s] \subseteq \cup K$.

The first is proved similarly.

Note 3. If $s \in W$ and $K \in N(s)$ then $K \neq \emptyset$. For $S \vdash \Box\top$ by (RN), and $S \vdash \Box\top \rightarrow \dot\Diamond\top$ by $(D\boxdot)$, so $\top \in \dot\Diamond^{-1}[s] \subseteq \cup K$.

Note 4. If $s,t \in W$ and $\Box^{-1}[s] \subseteq t$ then $t \in K$ for some $K \in N(s)$. For let $K_C = \{u \in W : \Box^{-1}[s] \subseteq u \ \& \ C \in u\}$ for each $C \in \Diamond^{-1}[s]$, and let $K = \{t\} \cup \bigcup\{K_C : C \in \Diamond^{-1}[s]\}$. Clearly $t \in K$; I claim $K \in N(s)$. By construction $\Box^{-1}[s] \subseteq \cap K$, so it suffices to show that $\Diamond^{-1}[s] \subseteq \cup K$. Suppose the contrary, say $\Diamond C \in s$ but $C \notin \cup K$. Then $C \notin \Box^{-1}[s]$ ($K \neq \emptyset$, so $\Box^{-1}[s] \subseteq \cap K \subseteq \cup K$) so $\Diamond\neg C \in s$ and $\Diamond\top \in s$; by (W), $\Box\neg C \to \boxdot\neg C \in s$. But $K_C = \emptyset$ (otherwise $C \in \cup K_C \subseteq \cup K$) so $\Box^{-1}[s] \vdash \neg C$ and $\Box\neg C \in s$. Hence $\boxdot\neg C \in s$, so $\Diamond C \notin s$, a contradiction.

Note 5. If $s \in W$ then $\Box A \in s \iff (\forall K \in N(s))(A \in \cap K)$. The "$\Rightarrow$" part follows from Note 2 above--if $K \in N(s)$ then $\Box^{-1}[s] \subseteq \cap K$. Suppose $A \in \cap K$ for every $K \in N(s)$. Then $\Box^{-1}[s] \vdash A$ (for suppose $u \in W$ and $u \supseteq \Box^{-1}[s]$; by Note 4 there is a $K \in N(s)$ such that $u \in K$, so by hypothesis $A \in \cap K$, so $A \in u$) and by Note 1, $\Box A \in s$.

Note 6. If $s \in W$ then $\Diamond A \in s \iff (\forall K \in N(s))(A \in \cup K)$. Again the "$\Rightarrow$" part follows from Note 2. Suppose $\Diamond A \notin s$, so $\neg\Diamond A \in s$. We construct $K \in N(s)$ such that $(\forall t \in K)(\neg A \in t)$. Namely, for each $C \in \Diamond^{-1}[s]$ let $K_C =$ $\{u \in s : \Box^{-1}[s] \subseteq u \ \& \ \neg A \in u \ \& \ C \in u\}$, and let $K =$ $\bigcup\{K_C : C \in \Diamond^{-1}[s]\}$. Then $\Box^{-1}[s] \subseteq \cap K$ by construction, and it suffices to show that $\Diamond^{-1}[s] \subseteq \cup K$. If not, let C be such that $\Diamond C \in s$ but $C \notin \cup K$. Then $C \notin \cup K_C$, so $K_C = \emptyset$, so $\Box^{-1}[s] \cup \{C\} \vdash A$, so $\Diamond C \to \Diamond A \in s$ by Note 1. Since $\Diamond C \in s$, we have $\Diamond A \in s$ contrary to hypothesis.

We can now complete the proof of Lemma 4:

$\mathcal{N} \models_s \boxdot A$

$\iff (\exists K \in N(s))[K \neq \emptyset \ \& \ (\forall t \in K)(\mathcal{N} \models_t A)]$ [Def $\models$]

$\iff (\exists K \in N(s))(\forall t \in K)(\mathcal{N} \models_t A)$ [Note 3]

$\iff (\exists K \in N(s))(\forall t \in K)(A \in t)$ [I. H.]

$\iff$ not $(\forall K \in N(s))(\exists t \in K)(\neg A \in t)$

$\iff$ not $(\forall K \in N(s))(\neg A \in \cup K)$

$\iff \Diamond\neg A \notin s$ [Note 6]

$\iff \boxdot A \in s$;

$\mathcal{N} \models_s \Diamond A$

$\iff (\exists K \in N(s))(\exists t \in K)(\mathcal{N} \models_t A)$

$\iff (\exists K \in N(s))(\exists t \in K)(A \in t)$

$\iff$ not $(\forall K \in N(s))(\forall t \in K)(\neg A \in t)$

$\iff$ not $(\forall K \in N(s))(\neg A \in \cap K)$

$\iff \Box\neg A \notin s$ [Note 5]

$\iff \Diamond A \in s$.

Theorem 5. (Completeness, Finite Model Property). If A is valid in all finite S-models, then $S \vdash A$.

Proof. Suppose $S \nvdash A$. Let $\mathcal{N} = \langle W,P,N,V \rangle$ be the canonical pseudo-model, and let Γ consist of all the subformulas of A. Define $\approx$ on W by

$$s \approx t \iff (\forall B \in \Gamma)(\mathcal{N} \models_s B \iff \mathcal{N} \models_t B).$$

Let $[s] = \{t : s \approx t\}$, and if $K \subseteq W$ let $K^\Gamma = \{[s] : s \in K\}$. Let $\mathcal{N}^\Gamma = \langle W^\Gamma, P^\Gamma, N^\Gamma, V^\Gamma \rangle$, where

$W^\Gamma = \{[s] : s \in W\}$,

$P^\Gamma(\pi) = \{<[s],[t]> : <s,t> \in P(\pi)\}$ if π occurs in A,

$N^\Gamma(s) = \{K^\Gamma : (\exists s' \approx s)(K \in N(s'))\}$.

(The definition of $P^\Gamma(\pi)$, where π does not occur in A, is postponed.)

Then we have: for all $B \in \Gamma$ and $s \in W$, $\mathcal{N}^\Gamma \models_{[s]} B$ if and only if $\mathcal{N} \models_s B$. The proof is by induction on B. For example, if $\mathcal{N}^\Gamma \models_{[s]} \boxdot C$ then for some $s' \approx s$ and $K \in N(s')$ we have $K^\Gamma \neq \emptyset$ and $\mathcal{N}^\Gamma \models_{[t]} C$ for all $[t] \in K^\Gamma$. Hence $K \neq \emptyset$ and by the induction hypothesis $\mathcal{N} \models_t C$ for all $t \in K$, so that $\mathcal{N} \models_{s'} \boxdot C$. Since $s' \approx s$ and $\boxdot C \in \Gamma$, $\mathcal{N} \models_s \boxdot C$. Conversely, if $\mathcal{N} \models_s \boxdot C$ then $\mathcal{N} \models_t C$ for all $t \in K$, where $\emptyset \neq K \in N(s)$. Then $\mathcal{N}^\Gamma \models_{[t]} C$ for all $[t] \in K^\Gamma$, and $\emptyset \neq K^\Gamma \in N^\Gamma([s])$, so $\mathcal{N}^\Gamma \models_{[s]} \boxdot C$.

Now for each pair $<x,L>$ such that $x \in W^\Gamma$ and $L \in N^\Gamma(x)$, choose a distinct $\pi \in$ ProgVar not occurring in A, and define $P^\Gamma(\pi) = \{<x,y> : y \in L\}$. Since W^Γ is finite, only finitely many π's are required; set $P^\Gamma(\pi) = \emptyset$ for the others. Let $\mathcal{M} = <W^\Gamma, P^\Gamma, V^\Gamma>$.

Then for every $B \in \Gamma$ and $x \in W^\Gamma$, $\mathcal{M} \models_x B$ if and only if $\mathcal{N}^\Gamma \models_x B$. Again the proof is by induction on B; the only non-trivial cases are $\diamond C$ and $\boxdot C$. If $\mathcal{M} \models_x \diamond C$ then $\mathcal{M} \models_x <\pi>C$ for some $\pi \in$ ProgVar, so $\mathcal{M} \models_y C$ for some y such that $<x,y> \in P^\Gamma(\pi)$, and by the induction hypothesis $\mathcal{N}^\Gamma \models_y C$. If π occurs in A then (by definition of $P^\Gamma(\pi)$) $<s,t> \in P(\pi)$ for some $s \in x$ and $t \in y$, and $\mathcal{N} \models_t C$ so $\mathcal{N} \models_s <\pi>C$. By (Inst$\diamond$), $\mathcal{N} \models_s \diamond C$, so $\mathcal{N}^\Gamma \models_x \diamond C$. If π does not occur in A then (again by definition of $P^\Gamma(\pi)$) $y \in L$ for some $L \in N^\Gamma(x)$, so $\mathcal{N}^\Gamma \models_x \diamond C$. Conversely, if $\mathcal{N}^\Gamma \models_x \diamond C$ then for some $y \in L \in N^\Gamma(x)$ we have $\mathcal{N}^\Gamma \models_y C$. By the induction hypothesis, $\mathcal{M} \models_y C$. Since $P^\Gamma(x) = \{<x,y> : y \in L\}$ for some $\pi \in$ ProgVar, we have $\mathcal{M} \models_x \diamond C$. The proof for $\boxdot C$ is similar.

Corollary. $\mathcal{VD}\boxdot$ has the finite model property. S and $\mathcal{VD}\boxdot$ are decidable.

Proof. By Theorems 3 and 5.

Thus the only additional logical principles required for adjoining Brown's logic of ability to the logic of independent, unstructured actions are the axioms (Inst$\diamond$) and (Inst$\boxdot$). The same remains true when disjunction of actions is permitted.

Let the language and semantics of S^+ be defined like those of S, but with the additional clauses

$\alpha,\beta \in$ Prog $\Rightarrow$ $\alpha+\beta \in$ Prog,

$\|\alpha+\beta\| = \|\alpha\| \cup \|\beta\|$.

Thus an S^+-model is just an S-model $\mathcal{M} = <W,P,V>$, but with the intensions defined differently. We often write Prog^+, $\mathcal{M}^+$, $\| \|^+$, etc., to emphasize the association with S^+. The intuitive meaning of $\alpha+\beta$ is "either do α or do β".

Lemma 6. Let $\mathcal{M}$ be an S-model. Then for every program α and formula A of S, $\|\alpha\|^+ = \|\alpha\|$ and $\|A\|^+ = \|A\|$.

Proof. The only programs in question are $\pi \in$ ProgVar, and $\|\pi\|^+ = P(\pi) = \|\pi\|$. For formulas, proceed by induction. Note that if $\alpha \in \text{Prog}^+$ then $\alpha = \pi_1+\ldots+\pi_n$ (with parentheses inserted in some manner) for some $\pi_1,\ldots,\pi_n \in$ ProgVar, so $\|\alpha\|^+ = \|\pi_1\|^+ \cup \ldots \cup \|\pi_n\|^+ = \|\pi_1\| \cup \ldots \cup \|\pi_n\|$. Then

$s \in \|\Diamond C\|^+$

$\Leftrightarrow (\exists\alpha \in \text{Prog})(\exists t)(\langle s,t\rangle \in \|\alpha\|^+ \ \& \ t \in \|C\|^+)$

$\Leftrightarrow (\exists\pi \in \text{ProgVar})(\exists t)(\langle s,t\rangle \in \|\pi\| \ \& \ t \in \|C\|)$

$\Leftrightarrow s \in \|\Diamond C\|$;

similarly, $\|\Box C\|^+ = \|\Box C\|$.

The axioms and rules of S^+ are described by the same schemes as those of S (of course, the schemes now have more instances), together with

(Un) $[\alpha+\beta]A \equiv [\alpha]A \wedge [\beta]A$,

a standard axiom of PDL.

It is easily seen that every thesis of S^+ is valid in every S^+-model. It follows that S^+ is a conservative extension of S: if A is an S-formula provable in S^+ then A is valid in every S^+-model, hence by Lemma 6 in every S-model, and so is provable in S. Hence S^+ is a conservative extension of $\mathcal{V}D\Box$. Moreover, using (Un), every S^+-formula is provably (in S^+)

equivalent to an S-formula, so we obtain a completeness and finite model property theorem for S^+: an S^+-formula is provable in S^+ if and only if it is valid in every finite S^+-model.

Thus $\mathcal{V}D\Box$ is also the logic of ability associated with the logic of disjoinable actions, and the only new principles required for the amalgamation of the two logics are (Inst$\Diamond$) and (Inst$\Box$). But $\mathcal{V}D\Box$ is *not* the logic of ability associated with the logic of composable actions.

Let the language and semantics of $S^;$ be defined like those of S, but with the additional clauses

$\alpha,\beta \in \text{Prog} \Rightarrow \alpha;\beta \in \text{Prog}$,

$\|\alpha;\beta\| = \|\alpha\| \circ \|\beta\|$

$= \{\langle s,t\rangle : (\exists u)(\langle s,u\rangle \in \|\alpha\| \ \& \ \langle u,t\rangle \in \|\beta\|\}$.

Again, an $S^;$-model is just an S-model $\mathcal{M} = \langle W,P,V\rangle$, but we often write $\mathcal{M}^;$, $\| \ \|^;$, $\text{Prog}^;$, etc. The intuitive meaning of $\alpha;\beta$ is "do α, then do β".

The analog of Lemma 6 is false: let $W = \{s,t,u\}$, $V(p) = \{u\}$, and $P(\pi) = \{\langle s,t\rangle,\langle t,u\rangle\}$ for all $\pi \in$ ProgVar; then $s \in \|\langle\pi;\pi\rangle p\|^;$ so $s \in \|\Diamond p\|^;$, but $s \notin \|\Diamond p\|$.

Let the axioms and rules of the system $S^;$ be those of S, together with

(Comp) $[\alpha;\beta]A \equiv [\alpha][\beta]A$

(4$\Diamond$) $\Diamond\Diamond A \rightarrow \Diamond A$.

It is easy to prove soundness (if $S^{;} \vdash A$ then A is valid in every $S^{;}$-model) and to define the canonical pseudo-model $\mathcal{N}^{;}$ and prove the fundamental lemma ($\|\alpha\|^{;} = \{\langle s,t\rangle : (\forall A)([\alpha]A \in s \Rightarrow A \in t)\}$ and $\|A\|^{;} = \{s : A \in s\}$), but I have not been able to convert this into a completeness theorem. (If one tries to adapt the proof of Theorem 5, one cannot prove that if $\diamond C \in \Gamma$, $x \in W^{\Gamma}$, and $\mathcal{M} \models_x \diamond C$ then $\mathcal{N}^{\Gamma} \models_x \diamond C$; if $\mathcal{M} \models_x \diamond C$ then $\mathcal{M} \models_x \langle\alpha\rangle C$ for some $\alpha \in Prog^{;}$, but α may be an arbitrary composition of program variables.) I am inclined to believe that $S^{;}$ is not complete.

In any case, the logic of ability associated with the logic of composable actions contains at least the principal (4◇), which is not provable in $\mathcal{V}D\boxdot$ (by the counter-example to the analog of Lemma 6). I am inclined to believe that it contains more. It does not contain

(4⊡) $\boxdot\boxdot A \rightarrow \boxdot A$,

For let

$W = \{s, t_1, t_2, u, v\}$,

$V(p) = \{u\}$,

$P(\pi_0) = \{\langle s,t_1\rangle, \langle s,t_2\rangle\}$,

$P(\pi_1) = \{\langle t_1,u\rangle, \langle t_2,v\rangle\}$,

$P(\pi_2) = \{\langle t_1,v\rangle, \langle t_2,u\rangle\}$,

$P(\pi) = \emptyset$ for $\pi \notin \{\pi_0,\pi_1,\pi_2\}$;

then $\mathcal{M}^{;} \models_s \langle\pi_0\rangle\top \wedge [\pi_0]\boxdot p$ so $\mathcal{M}^{;} \models_s \boxdot\boxdot p$, but $\mathcal{M}^{;} \models_s \neg\boxdot p$. (This example is essentially due to Brown [B]; I cannot explain his

concurrent assertion that (4⊡) "amounts to an acceptance of composition of actions".)

One problem in characterizing the logic of ability associated with composable actions is to characterize (up to isomorphism) the derived neighborhood models $\mathcal{M}^{\beta} = \langle W,N,V\rangle$ of models $\mathcal{M}^{;} = \langle W,P,V\rangle$ (where the N is that of the derived pseudo-model $\mathcal{M}^{ps} = \langle W,P,N,V\rangle$, that is, $N(s) = \{K_{\alpha,s} : \alpha \in Prog^{;}\}$). Define: a *cover* of K is a family $\langle K_u : u \in K\rangle$ such that each $K_u \in N(u)$. We have $K_{\alpha;\beta,s} = \{t : (\exists u)(\langle s,u\rangle \in \|\alpha\|^{;}\ \&\ \langle u,t\rangle \in \|\beta\|^{;})\} = \cup\{K_{\beta,u} : u \in K_{\alpha,s}\}$. Hence if $K \in N(s)$ then there is at least one cover of K whose union is in $N(s)$. If $K = \emptyset$ then the empty cover will do, and otherwise every neighborhood of every point in K is a component of a cover of K whose union is in $N(s)$: if $K = K_{\alpha,s}$, $u_0 \in K$, and $K_{u_0} = K_{\beta,u_0} \in N(u_0)$ then we may take $K_u = K_{\beta,u}$ for all $u \in K$. I do not know whether this condition is sufficient. Certainly, not every cover of every $K \in N(s)$ need have union in $N(s)$--in the example showing the invalidity of (4⊡), let $K = K_{\pi_0,s} = \{t_1,t_2\}$ and let the cover consist of K_{π_1,t_1} and K_{π_2,t_2}.

Finally, I offer a couple of comments on the general issue of combining Brown's logic of ability with logics of action. It is not at all clear that the program-forming operators of PDL are the right ones. Certainly we want to include composition of actions (so we can tie our shoes). Given this, if the test A?

is permitted, for any formula A, then $\boxdot A$ and $\Diamond A$ are equivalent: $\boxdot A \rightarrow \Diamond A$ is valid in any case, and if $\mathcal{M} \vDash_S \langle\alpha\rangle A$ then $\mathcal{M} \vDash_S \langle\alpha;A?\rangle A$ & $[\alpha;A?]A$ so $\mathcal{M} \vDash_S \boxdot A$. The canonical example of $\Diamond$ without $\boxdot$ is hitting the bull's-eye at darts; the "$\alpha;A?$" action corresponds to "taking back" each throw until one succeeds. Tests should probably be eliminated. In a system using propositional constants or specific atomic formulas, rather than propositional variables, tests might reasonably be restricted to the atomic formulas (or equivalently, to boolean formulas). This course is not suitable for propositional variables since substitutivity is lost: $\Diamond p \equiv \boxdot p$ is valid, but not $\Diamond A \equiv \boxdot A$ in general.

We do need constructs like "if A then α else β" (in order reliably to effect the state in which a pull-chain lamp is on) and "repeat α until A" (for sawing a board), which in PDL are defined using tests and non-deterministic disjunction and iteration (as $(A?;\alpha)+((\neg A)?;\beta)$ and $\alpha;(\neg A?;\alpha)^*;A?$ respectively). I would propose adopting these as primitive program-forming operators, and dispensing with iteration as well as tests. It seems to me that human actions are never of the form "repeat α some random number of times"; the closest we come is "repeat α until you feel like stopping". Though disjunction of actions seems technically innocuous, I would propose deleting it on the grounds that it introduces non-determinism. It is reasonable that a given action may have many different outcomes, but this is because of factors in the world beyond our control,

not because the action itself non-deterministically selects one of two alternatives. Note that actions like "(flip a coin) ; (if it's heads then α else β)" would still be available.

References.

[B] Mark A. Brown, *On the logic of ability*, J. Philos. Logic 17(1988), 1-26.

[G] Robert Goldblatt, "Logics of Time and Computation", Lecture Notes No. 7, C. S. L. I., Stanford University, 1987.

[S1] Krister Segerberg, *Bringing it about*, J. Philos. Logic, to appear.

[S2] Krister Segerberg, *Reflexions on a paper by Brown*, Auckland Philosophy Papers, University of Auckland, 1989.

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