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Non-standard truth values and modalities

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Introduction

In my paper Reyes [6], I developed a topos-theoretic approach to reference and modality. This approach, based on the notion of a locally connected topos $\mathcal{E}$ over a base topos $\mathcal{S}$ led straightforwardly to a higher-order many-sorted system of modal logic, HMAO, and a topos-theoretic semantics for this formal system. If the base topos is the category of sets, *Sets*, the axiom of excluded middle turns out to be valid for modal sentences and we obtain an extension of HMAO, namely the system HIBM. Since the writing of that paper a more general topos semantics, based on the notion of an open topos $\mathcal{E}$ over a base topos $\mathcal{S}$, has been introduced in Lavendhomme, Lucas and Reyes[4] to study several metamathematical properties of the first-order fragment IBM of HIBM, such as soundness and completeness with respect to that semantics.

This approach uses topos-theory rather freely and thus it is hard to understand for people who are not yet acquainted with that theory. After the writing of that paper, I discovered a simpler alternative approach based on non-standard truth values. This new approach, which arose in discussions with John Macnamara and Marie Reyes on the possibility of applying HIBM to learning theory and literary theory, reduces "non-standard" prerequisites to the rudiments of category theory. The reader who is interested in these applications may consult Macnamara and Reyes [5] and Reyes, M. [7]

The approach developed in this paper is actually equivalent to the original topos-theoretic one provided that one restricts attention to kinds only, leaving variable sets out of the picture (cf. Appendix). Although simpler than the original, I find it less cogent.

1 Preorders and coverage systems

From now on, let S be the category of sets. The knowledgeable reader will observe that all that we shall do (with the exceptions that will be mention explicitely at the relevant places) can be generalized from the category of sets to an arbitrary topos. Let $\leq$ be a *pre-order* on a set P, ie, a binary relation on P satisfying the following conditions:

1. $x \leq x$

2. $x \leq y \wedge y \leq z \Longrightarrow x \leq z$

I shall make the usual abuse of language and use "P" for both the structure $(P, \leq)$ and the underlying set P. By a *coverage system* on P, we understand a couple (P, Cov), where Cov is a function that associates with each $p \in P$ a set $Cov(p)$ of subsets of $\downarrow(p) = \{q \in P : q \leq p\}$ satisfying the following conditions:

1. $\{p\} \in Cov(p)$

2. $A \in Cov(p) \wedge q \leq p \Longrightarrow \exists B \in Cov(q)\ \forall a \in A\ \exists b \in B\ b \leq a$

3. $A \in Cov(p) \Longrightarrow \bigcup\{B \in \Omega^P : \exists a \in A\ B \in Cov(a)\}$

We refer to (ii) as the "stability condition" and to (iii) as the "locality condition".

I now define $\Omega(1)$ to be the set of downwards closed subsets $K \subseteq P$ satisfying the following "glueing" condition:

$$A \in Cov(p) \wedge A \subseteq K \Longrightarrow p \in K$$

The elements of $\Omega(1)$ will be called *non-standard truth values* and $\Omega(1)$ itself, the *set of non- standard truth values.*

Proposition 1.1 *$\Omega(1)$ is a complete Heyting algebra*

Proof. First a definition: if K is a downwards closed subset of P, we let $cl(K) = \{p \in P : \exists A \in Cov(p) A \subseteq K\}$. It is clear that $0 = \emptyset$ and $1 = P$. If K is a set of downwards closes subsets of P, then so is $\bigcup \mathcal{K}$. Then $\bigvee \mathcal{K}$ $= cl(\bigcup \mathcal{K})$. On the other hand, if K_1 and K_2 are two elements in $\mathcal{K}$, then $K_1 \wedge K_2 = K_1 \cap K_2$.

Example 1. Let $\leq$ be just ordinary equality on a set I. In this case there is exactly one coverage on I, namely $Cov(i) = \{i\}$ and $\Omega(1)$ is just the complete Boolean algebra of all the subsets of I.

Example 2. Let $\leq$ be a pre-order on a set P and let $Cov(p) = \{p\}$. This is again a coverage on the pre-order P and $\Omega(1)$ is just the set of downwards closed subsets of P. The Heyting operations are the set-theoretical union, finite intersection and implication given by $K_1 \Longrightarrow K_2 = \bigcup\{K \in \Omega(1) : K_1 \wedge K \subseteq K_2\}$

Example 3. Let P be a tree and let p be a node. A set A of nodes is a *bar* for p if every path of P containing p also contains a node of A. For each $p \in P$ we let $Cov(p)$ be the set of downwards closed bars below p (we represent the tree as growing downwards). It is easy to check that this is a coverage that we shall call the *Beth coverage* on P.

We shall see that the first of these examples corresponds to the well-known possible worlds semantics. On the other hand, the second and the third correspond to the Kripke and Beth semantics for intuitionistic logic, respectively.

2 Languages and their interpretations

In this section we introduce the language of modal type theory and interpret it in S with respect to a coverage system (P, Cov).

We define *sorts* and *terms* by recursion as follows:

Sorts

a) *Basic sorts* are *sorts*: passenger, person, reading, rivers,... (whose intended interpretation will be sets passengers, persons, readings, rivers, ...)

b) *prop* is a *sort* (whose intended interpretation is $\Omega(1)$, the set of non-standard truth-values)

c) If X, Y are *sorts*, so are $X \times Y$ and Y^X

d) Nothing else is a *sort*.

Terms (of a given sort) are defined by recursion as follows (where $t : X$ is an abreviation for "t is a term of sort X"):

a) Basic ***constant terms*** $c \in Con_X$ are ***terms*** of sort X: e.g., $John \in Con_{prop(prop^{person})}$ (whose intended interpretation is the set of predicates that John has); or again run_{man} of sort $prop^{man}$ (whose intended interpretation is the corresponding predicate as applied to men). Other sorted constants may be, for instance, ***is a son of*** of sort $prop^{man \times man}$ (whose intended interpretation is the obvious relation between men). We also have sorted constants for "underlying" maps between the interpretation of the sorts. These are needed, for instance, to account for the fact that "a man is an animal". In this case, we postulate a constant $u_{man,animal}$ of sort $prop^{man \times animal}$ whose intended interpretation is the function which associates with a man "its underlying animal". This is the precise expression, in our many-sorted language, of what natural language expresses imprecisely by the above sentence "a man is an animal".

b) If $x \in Var_X$, then x is a ***term*** of sort X, where Var_X is an infinite set, for each sort X

c) If $t : X$ and $s : Y$, then $< t, s >: X \times Y$

d) If $x \in Var_X$ and $t : Y$, then $\lambda x t : Y^X$

e) If $t : Y^X$ and $s : X$ then $t(s) : Y$

f) $\top$ and $\perp$ are terms of sort *prop*

g) If $t, s : X$, then $t = s$ is a term of sort *prop*

h) If ϕ, ψ are terms of sort *prop*, then $\phi \,\Delta\, \psi$ is a term of sort *prop*, where $\Delta \in \{\wedge, \vee, \rightarrow\}$

i) If ϕ is a term of sort *prop* and $x \in Var_X$, then $\exists x \phi$ and $\forall x \phi$ are terms of sort *prop*

j) If ϕ is a term of sort *prop*, then so are $\Box\phi$ and $\Diamond\phi$

k) Nothing else is a ***term***.

Formulas are defined to be terms of sort *prop*

If ϕ is a formula, we let $\neg\phi \equiv \phi \rightarrow \perp$.

We shall assume that we have defined the usual notions like "substitution of a variable by a term", "free variable of a term or a formula", "a term being free for a variable in a term or a formula", etc.

An ***interpretation*** of L in S relative to a coverage system (P, Cov) is a function $|...|$ which associates with the sort *prop* the set of non-standard truth values $\Omega(1)$ and with each sort s a set $|s|$ of S and with each constant of sort s an element of $|s|$. Given such an interpretation, we can interpret terms in the usual way as set-theoretical functions between the interpretations of the sorts and formulas as functions whose target is $\Omega(1)$. To interpret formulas it is convenient to introduce the ***forcing*** relation

$$p \Vdash \phi(x_1, ..., x_n)[a_1, ..., a_n]$$

between $p \in P$, ϕ a formula of L and $(a_1, ..., a_n) \in S_1 \times ... \times S_n$,where the S are the interpretation of the sorts of the variables x.

We proceed as in the Kripke's forcing but with the following modifications for $\vee$ and $\exists$ and new clauses for modal operators:

1. $p \Vdash (\phi \vee \psi)[a_1, \ldots, a_n]$ iff there is an $X \in Cov(p)$ such that for all $q \in X$ ($q \Vdash \phi[a_1, \ldots, a_n]$ or $q \Vdash \psi[a_1, \ldots, a_n]$)

2. $p \Vdash \exists x \phi[a_1, \ldots, a_n]$ iff there is an $X \in Cov(p)$ such that for all $q \in X$ there is $a \in S$ such that

$$q \Vdash \phi[a_1, \ldots, a_n, a]$$

3. $p \Vdash \Diamond\phi[a_1, \ldots, a_n]$ iff $\exists q \; q \Vdash \phi[a_1, \ldots, a_n]$

4. $p \Vdash \Box\phi[a_1, \ldots, a_n]$ iff $\forall q \; q \Vdash \phi[a_1, \ldots, a_n]$

The following expresses the functoriality of the forcing relation and is proved by induction on formulas in the usual way:

Proposition 2.1 *If* $U \Vdash \phi[a_1, \ldots, a_n]$ *and* $V \leq U$, *then* $V \Vdash \phi[a_1, \ldots, a_n]$.

We can finally define the basic notion of ***validity*** of a sentence, namely a formula without free variables. A sentence σ is ***valid*** in S relative to the coverage system (P, Cov) if and only if for every $p \in P$ $p \Vdash \sigma[\;]$, where $[\;]$ is the empty sequence. We shall write $(S, (P, Cov)) \models \sigma$. More generally, we say that a formula is ***valid*** if and only if its universal closure, which is a sentence, is valid.

3 Formal system and Soundness theorem

In this section, we shall describe a formal system HIBM (for "higher order intuitionistic with Boolean modalities") based on Gentzen's sequents. These expressions, following Boileau and Joyal [2], will be of the form $\Gamma \vdash_{\vec{X}} \phi$, where Γ is a finite set of formulas of the language of modal higher order theory already described, ϕ a single formula and $\vec{X}$ a finite sequence of variables containing all the free variables of Γ and ϕ. We shall assume that these expressions satisfy the following rules. This system follows, in part, Lambek and Scott [3, page 134] which in turn is based on Gentzen's work.

1. *Structural rules*

1.1 $p \vdash_{\vec{X}} p$

1.2 $$\frac{\Gamma \vdash_{\vec{X}} p \qquad \Gamma \cup \{p\} \vdash_{\vec{X}} q}{\Gamma \vdash_{\vec{X}} q}$$

1.3 $$\frac{\Gamma \vdash_{\vec{X}} q}{\Gamma \cup \{p\} \vdash_{\vec{X}} q}$$

1.4 $$\frac{\Gamma \vdash_{\vec{X}} q}{\Gamma \vdash_{\vec{X},y} q}$$

1.5 $$\frac{\Gamma \vdash_{\vec{X},y} \phi}{\Gamma[t/y] \vdash_{\vec{X}} \phi[t/y]}$$

where t is free for y in ϕ and Γ

2. *Logical rules*

2.1 $p \vdash_{\vec{X}} \top$ and $\bot \vdash_{\vec{X}} p$

2.2 $r \vdash_{\vec{X}} p \wedge q$ iff $r \vdash_{\vec{X}} p$ and $r \vdash_{\vec{X}} q$
$p \vee q \vdash_{\vec{X}} r$ iff $p \vdash_{\vec{X}} r$ and $q \vdash_{\vec{X}} r$

2.3 $p \vdash_{\vec{X}} q \to r$ iff $p \wedge q \vdash_{\vec{X}} r$

2.4 $p \vdash_{\vec{X}} \forall x \phi$ iff $p \vdash_{\vec{X},x} \phi$
$\exists x \phi \vdash_{\vec{X}} p$ iff $\phi \vdash_{\vec{X},x} p$

provided that x is not a variable in $\vec{X}$.

3. *Identity rules*

3.1 $\vdash_{\vec{X}} t = t$

3.2 $t = s \vdash_{\vec{X}} \tau[t/x] = \tau[s/x]$

provided that t and s are free for x in τ

3.3 $t = s \vdash_{\vec{X}} \phi[t/x] \to \phi[s/x]$

provided that t and s are free for x in ϕ

4. *Rules on special symbols*

4.1 $<a, b> = <c, d> \vdash_{\vec{X}} a = c$
$<a, b> = <c, d> \vdash_{\vec{X}} b = d$

4.2 $$\frac{\Gamma, \quad z = <x, y> \vdash_{\vec{X},x,y,z} \phi}{\Gamma \vdash_{\vec{X},z} \phi}$$

provided that x and y are not free in Γ or ϕ

5. *Rules for the λ -calculus*

5.1 $\vdash_{\vec{X}} \lambda x t(x) = t$

provided that x is not a variable in $\vec{X}$

5.2 $\vdash_{\vec{X}} \lambda x \phi(t) = \phi[t/x]$

provided that t is free for x in ϕ

5.3 $$\frac{\Gamma \vdash_{\vec{X},x} t = s}{\Gamma \vdash_{\vec{X}} \lambda x t = \lambda x s}$$

6. *Rules for modal operators*

6.1 $\Box \phi \vdash_{\vec{X}} \phi \qquad \phi \vdash_{\vec{X}} \Diamond \phi$

6.2 $\Box \phi \vdash_{\vec{X}} \Box \Box \phi \qquad \Diamond \Diamond \phi \vdash_{\vec{X}} \Diamond \phi$

6.3 $\phi \vdash_{\vec{X}} \Box \Diamond \phi \qquad \Diamond \Box \phi \vdash_{\vec{X}} \phi$

6.4 $x = y \vdash_{\vec{X}} \Box (x = y)$

6.5 $$\frac{\phi \vdash_{\vec{X}} \psi}{\Box \phi \vdash_{\vec{X}} \Box \psi}$$

6.6 $$\frac{\phi \vdash_{\vec{X}} \psi}{\Diamond\phi \vdash_{\vec{X}} \Diamond\psi}$$

6.7 $\vdash_{\vec{X}} \Diamond\phi \leftrightarrow \neg\Box\neg\phi$

6.8 $\vdash_{\vec{X}} \Box\phi \vee \neg\Box\phi$

This completes our system. We have not tried to describe it in the simplest or most economical manner. In fact, a more economical system seems possible (see [3]).

In order to formulate a soundness theorem, we need the following definition: $\Gamma \models_{\vec{X}} \phi$ if and only if

$$\forall p \in P\, \forall(a_1, \ldots, a_n) \in S_1 \times \ldots \times S_n\; p \Vdash \Gamma[a_1, \ldots, a_n] \Rightarrow p \Vdash \phi[a_1, \ldots, a_n]$$

where $\vec{X}$ is a finite sequence of free variables of sorts $S_1, \ldots, S_n$ containing the free variables of ϕ and Γ, and $p \Vdash \Gamma[a_1, \ldots, a_n]$ iff $p \Vdash \gamma[a_1, \ldots, a_n]$ for all $\gamma \in \Gamma$.

Theorem 3.1 (Soundness)
If $\Gamma \vdash_{\vec{X}} \phi$, then $\Gamma \models_{\vec{X}} \phi$.

Proof.(In sketch) Induction on proofs. For most axioms and rules of inference this is quite straightforward, the only tricky verifications being those connected with substitutions of terms for which a substitution lemma has to be stated and proved.

To illustrate this proof, we verify it for 6.8

We show that given p, $p \Vdash \Box\phi \vee \neg\Box\phi$. By definition of forcing, we must then show

$$p \Vdash \Box\phi \text{ or } p \Vdash \neg\Box\phi$$

equivalently

$$\forall q\; q \Vdash \phi \text{ or } \forall r\; \leq p r \not\Vdash \Box\phi$$

equivalently

$$\forall q\; q \Vdash \phi \text{ or } \forall r\; \leq P \neg(\forall q\; q \Vdash \phi)$$

equivalently

$$\forall q\; q \Vdash \phi \text{ or } \forall r\; \leq p \exists q\; q \not\Vdash \phi$$

and finally

$$\forall q\; q \Vdash \phi \text{ or } \exists q\; q \not\Vdash \phi$$

which is true.

Remark 3.2 *This proof uses the fact that S is a Boolean topos.* If S is not Boolean, soundness theorems go through for the formal system HMAO, which is HIBM minus axiom 6.8.

4 Completeness theorems

In this section, we state a completeness theorem for the first- order fragment IBM of HIBM as well as another for one of its extensions, namely IBM together with the axiom of constant domain (stated below). These theorems are related to examples 2 and 3 of section 1 and use, in an essential way, the Booleanness of S. Proofs may be found in Lavendhomme, Lucas and Reyes [4].

Theorem 4.1 *Let L be a denumerable first-order modal language (with both $\Box$ and $\Diamond$), let Γ be a set of sentences and let σ be a sentence such that $\Gamma \not\vdash \sigma$ in the system IBM together with the following axiom of constant domain: $\forall x\, (\sigma \vee \phi(x)) \longleftrightarrow \sigma \vee \forall x\, \phi(x)$, where x is not free in σ. Then there is a pre-order P and an interpretation of L relative to P with the trivial coverage system such that all sentences of Γ are valid, although σ is not valid.*

Theorem 4.2 *Let L be a denumerable first-order modal language (with both $\Box$ and $\Diamond$), let Γ be a set of sentences and let σ be a sentence such that $\Gamma \not\vdash \sigma$ in the system IBM. Then there is a tree T, a Beth coverage Cov on T and an interpretation of L relative to (T, Cov) such that all sentences of Γ are valid, although σ is not valid.*

Of course, there is a completeness theorem related to example 1, but this reduces to the well-known completeness theorem of the system $S5$ for the possible worlds semantics.

The first theorem characterizes thus the modal logic of presheaves (or Kripke trees), whereas the second characterizes the modal logic of Beth trees with bars.

The *proofs* of these theorems are somewhat different. Whereas the first is a rather straightforward application of the Henkin method, this is not so for the second. In fact, the proof is rather roundabout and proceeds in 2 steps: we introduce a modification of Beth semantics, the graft semantics together with a modification of IBM: the system $IS4$. Then we prove completeness of this new system relative to the graft semantics, essentially by proving a cut-elimination for this system. For details, the reader is referred to [4].

Remark 4.3 *In Reyes* [6], ***a category of kinds with coincidence relations is introduced to account for opaqueness of natural languages,*** ie, for the "lack of substitutivity of equals for equals". The main idea is that members of a given kind may *coincide* at a given situation without being equal. At the syntactical level, this means that we have a further primitive relation symbol $\asymp$ together with a series of new axioms and rules of inference. For simplicity sake, we left coincidence relations out of the picture in this paper. Correspondingly, we considered languages without a relation symbol for this coincidence relation. Nevertheless, Theorem 4.3 goes through for this case too. In fact this follows from Reyes [6, Theorem 6, 2.4].

Appendix

In this appendix, we prove the assertion of the Introduction, namely that this approach with sets and non-standard truth-values is actually equivalent to the topos-theoretical one, provided that we restrict ourselves to kinds, ie, constant sets of the topos in question.

Let $\Gamma : \mathcal{E} \longrightarrow \mathcal{S}$ be a locally connected, bounded topos. This means that $\mathcal{E} = Sh(\mathcal{C})$ for a site $\mathcal{C}$ in $\mathcal{S}$ for some site $\mathcal{C}$ in $\mathcal{S}$ such that constant presheaves are sheaves. Equivalently, this means that the left adjoint Δ to Γ has itself an $\mathcal{S}-indexed$ left adjoint. In particular, Δ preserves exponentials (cf. Barr [1]). From the site $\mathcal{C}$ we construct a pre-order P together with a coverage system Cov as follows: P is the set of objects of $\mathcal{C}$ and for $C, C' \in P, C \leq C'$ *iff* $\exists f : C \longrightarrow C' \in \mathcal{C}$. Furthermore, a family $\{C_i\}_{i\in I} \in Cov(C)$ iff there is a family $\{f_i : C_i \longrightarrow C\}$ which is a covering family in the site $\mathcal{C}$. It is easy to check that (P, Cov) is indeed a coverage system.

Furthermore, recalling (from Reyes [6], say) that $\Gamma(\Omega)$ = set of closed sieves, we check at once that $\Gamma(\Omega) = \Omega(1)$.

Let L be a higher-order many-sorted modal language as discussed in Reyes [6] and let $|...|$ be an interpretation (of the basic sorts and basic constants) of L in $\mathcal{S}$ with respect to (P, Cov).

We now obtain an interpretation in $\Gamma : \mathcal{E} \longrightarrow \mathcal{S}$ as follows:

1. $\|X\| = \Delta|X|$
2. $\|c\| = \Delta|c|$ for basic constants $c : X$

We may now state the main result of this appendix:

Theorem 4.4 *For every formula ϕ of L and every $C \in \mathcal{C}$,*

$$C \Vdash \phi[a_1, ..., a_n] \quad iff \quad C \Vdash^* \phi[a_1, ..., a_n]$$

where the first forcing relation refers to the interpretation in the topos $\mathcal{E} \longrightarrow \mathcal{S}$ and the second forcing refers to the interpretation in $\mathcal{S}$ with respect to (P, Cov).

Corollary 4.5 *For every sentence σ of L,*

$$\mathcal{E} \longrightarrow \mathcal{S} \models \sigma \quad iff \quad (\mathcal{S}, \Omega(1)) \models \sigma$$

We first prove the following

Lemma 4.6 *If $t : X$ is any term of L such that $FV(t) \subseteq \{x_1, ..., x_n\}$ and $\vec{x} = (x_1, ..., x_n)$, then*

$$\|\vec{x} : t\| = \Delta|\vec{x} : t|$$

Proof. By induction on t. For instance, if t is a constant c of sort X, this is just the definition of the interpretation: $\|c\| = \Delta|c|$. We shall do in detail only two cases of Reyes[6]), the others being similar or simpler.

Assume that t is of the form λxs. But this follows from the following easily proved remark: let tr(...) stand for the exponential transpose of the morphism (...) in either of the toposes $\mathcal{E}$ or $\mathcal{S}$. If $F : A \longrightarrow Y^X$ is a morphism in $\mathcal{S}$, then $\Delta tr(F) = tr(\Delta F)$ (The proof uses of course the fact that Δ preserves exponentials). To finish the proof of this case, we assume (induction hypothesis) that $\|x\vec{x} : s\| = \Delta|x\vec{x} : s|$ and argue as follows: $|\vec{x} : \lambda xs| = tr(|x\vec{x} : s|)$. Hence, $\Delta|\vec{x} : \lambda xs| = tr(\Delta|x\vec{x} : s|)$, by the remark. But this is equal to $tr(\|x\vec{x} : s\|)$ by induction hypothesis, which in turn equals, by definition, $\|\vec{x} : s\|$.

Assume now that t is of the form $\forall y\phi$. It is enough to show that

$$|\vec{x} : \forall y\phi| = tr(\forall_\pi(\epsilon_\Omega \circ \|(x,y) : \phi\|))$$

where π is the obvious projection and $tr(...)$ is the transpose of $(...)$. Indeed, we then apply the functor Δ.

But C belongs to the right hand side iff $(\forall_\pi(\epsilon_\Omega \circ \|(x,y) : \phi\|))_C(a) = T_C$, by Reyes[6, Proposition 4, section 2.2]. By the definition of $\forall_\pi$, this holds precisely when $\forall C' \longrightarrow C\ \forall b\ (\epsilon_\Omega \circ \|(x,y) : \phi\|))_{C'}(a,b) = T_{C'}$. Using loc.cit. once again, this condition may be written as $\forall C' \longrightarrow C\ \forall b\ C' \in tr(\epsilon_\Omega \circ \|(x,y) : \phi\|))(a,b)$. The induction hypothesis on ϕ and a simple computation of transpose shows that the last condition is equivalent to $\forall C' \longrightarrow C\ \forall b\ C' \in (|(x,y) : \phi|))(a,b)$. Using the definition of $\Vdash^*$, we may rewrite the condition as $\forall C' \longrightarrow C\ \forall b\ C' \Vdash^* \phi(a,b)$, which by the very definition of this forcing is equivalent to $C \Vdash^* \forall y\phi[a]$. But this is equivalent to $C \in |\vec{x} : \forall y\phi|(a)$, by definition of $|...|$.

We now turn to the

Proof (of Theorem 4.3). Assume that $C \Vdash \phi[a]$. By definition of $\Vdash$, this is equivalent to $(\epsilon_\Omega \circ \|x : \phi\|))_C(a) = T_C$. By Reyes[6, Proposition 4, section 2.2], this holds precisely when $C \in tr(\epsilon_\Omega \circ \|x : \phi\|))_C(a)$. By induction hypothesis this, in turn, holds precisely when $C \in |x : \phi|(a)$. This completes the proof (recall the definition of $|...|$ in terms of non-standard forcing).

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