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A very simple proof of Ono's theorem for S_n^+, $n \geq 1$. (Abstract)

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The systems S_n^+, $n \geq 1$.

In [2], Ono introduces the logics S_n^+ $(n \geq 1)$ defined so:

$S_n^+ = \text{Q-LC} + \mathbf{P}_n$, where

Q-LC is the predicate intuitionistic logic plus the axiom $(\alpha \to \beta \vee \beta \to \alpha)$ and $\mathbf{P}_n =$ $\forall x_1(P_1(x_1) \vee (P_1(x_1) \to \forall x_2(P_2(x_2) \vee (P_2(x_2) \to \ldots \to \forall x_n(P_n(x_n) \vee \neg P_n(x_n)) \ldots))))$.

($P_1, \ldots, P_n$ are unary predicates.)

Among other properties of these logics, it is shown that

Ono's theorem S_n^+ is characterized by the class of all linear Kripke frames of height not greater than n and with nested domains.

We shall give a short and simple proof of this result by using the method of diagrams, see [1]. Let L be the language of S_n^+, $n \geq 1$.

Let α be a closed formula of L and suppose that $S_r^+ \not\vdash \alpha$, $r \geq 1$. Let $n = \mu s(S_s^+ \not\vdash \alpha)$.

If $n = 1$, then, since $S_1^+ \vdash \forall x(P_1(x) \vee \neg P_1(x))$, we can easily construct a saturated diagram Δ whose support contains only one point, say 1, and such that $\langle 1, \alpha^- \rangle \in \Delta$, hence the theorem follows.

If $n > 1$, then $S_n^+ \not\vdash \mathbf{P}_{n-1} \vee \alpha$, since $S_n^+ + \mathbf{P}_{n-1} \vdash \alpha$ and $S_n^+ \not\vdash \alpha$.

The following diagram Δ, with support $\{1, \ldots, n\}$, is S_n^+-coherent:

$\Delta = \{\langle 1, \alpha^- \rangle, \langle 1, P_1(c_1)^- \rangle, \langle 2, P_1(c_1)^+ \rangle \langle 2, P_2(c_2)^- \rangle, \ldots, \langle n-1, P_{n-1}(c_{n-1})^+ \rangle \langle n-1, P_n(c_n)^- \rangle, \langle n, P_n(c_n)^+ \rangle\}$, where c_i is a constant of C_i, $i = 1, \ldots, n$.

It is important to notice that for any two consecutive points i and i+1, $i = 1, \ldots, n-1$, in Δ, there is a sentence α such that $\{\langle i, \alpha^- \rangle, \langle i+1, \alpha^+ \rangle\} \subseteq \Delta$. Any diagram satisfying this condition is said to contain n strongly distict points and that α <u>separates</u> i from i+1. Observe that axiom $\mathbf{P}_n$ is equivalent to:

(*) $\forall x_1(\top \to P_1(x_1) \vee \forall x_2 (P_1(x_1) \to (P_2(x_2) \vee \ldots \vee \forall x_n (P_{n-1}(x_{n-1}) \to P_n(x_n) \vee (P_n(x_n) \to \bot)) \ldots))))$ which is the formula of the coherence.

Hence any diagram containing a subdiagram Σ composed of n + 1 strongly distinct points is not S_n^+-coherent. In fact from (*) follows that

$S_n^+ \vdash \forall \vec{x}_1 [\Sigma_1^+(\vec{x}_1) \to [\Sigma_1^-(\vec{x}_1) \vee \ldots \vee \forall \vec{x}_n [\Sigma_n^+(\vec{x}_1, \ldots, \vec{x}_n) \to \Sigma_n^-(\vec{x}_1, \ldots, \vec{x}_n) \to$ $\forall \vec{x}_n [\Sigma_n^+(\vec{x}_1, \ldots, \vec{x}_{n+1}) \to \Sigma_n^-(\vec{x}_1, \ldots, \vec{x}_{n+1})]] \ldots]]$.

Consider the diagram Δ we started with, all that is left to show is that Δ can be extended to a saturated diagram Γ such that $Supp(\Gamma) = Supp(\Delta)$. In the construction of a saturated diagram the moments we need to add a new element to the support of the diagram built up to that stage are when either $\langle r, (\alpha \to \beta)^- \rangle$ or $\langle r, \forall x \alpha(x)^- \rangle$ is added to it, $r = 1, \ldots, n$. So let us consider the following cases.

Let $\Delta \subseteq \Gamma$, $Supp(\Delta) = Supp(\Gamma)$ and $\langle r, (\alpha \to \beta)^- \rangle \in \Gamma$, $r = 1, \ldots, n$. Suppose, by reductio, that there is no s, $s = r, \ldots, n$, such that $\Gamma \cup \{\langle s, \alpha^+ \rangle, \langle s, \beta^- \rangle\}$ is S_n^+-coherent. Among the various cases, let us examine only that in which for some s, $s = r, \ldots, n-1$, $\Gamma' = \Gamma \cup \{\langle s, \alpha^- \rangle, \langle s+1, \alpha^+ \rangle, \langle s+1, \beta^+ \rangle\}$ is S_n^+-coherent. Then, for some w, $s < w < s+1$, $\Gamma' \cup \{\langle w, \alpha^+ \rangle, \langle w, \beta^- \rangle\}$ is S_n^+-coherent. But this is impossible because it contains n+1 strongly distinct points; to wit α separates s from w and β separates w from w+1.

Let $\Delta \subseteq \Gamma$, $Supp(\Delta) = Supp(\Gamma)$ and $\langle r, \forall x \alpha(x)^- \rangle \in \Gamma$, $r = 1, \ldots, n$. Suppose, by reductio, that there is no s, $r \leq s \leq n$ such that $\Gamma \cup \{\langle s, \alpha(c)^- \rangle\}$ is S_n^+-coherent, for some constant c of L_S. Let us examine the case in which for some s, $s = r, \ldots, n-1$, $\Gamma' = \Gamma \cup \{\langle s, \forall x \alpha(x)^- \rangle, \langle s+1, \forall x \alpha(x)^+ \rangle\}$ is S_n^+-coherent and for all $c \in L_S$, $\Gamma' \cup \{\langle s, \alpha(c)^- \rangle\}$ is not S_n^+-coherent. Then, $\Sigma = \Gamma' \cup \{\langle s, \exists x(\alpha(x) \to \forall x \alpha(x))^- \rangle\}$ is S_n^+-coherent. Take any rational number w, $s < w < s+1$, then $\Sigma \cup \{\langle w, \alpha(d)^- \rangle\}$ is S_n^+-coherent for some constant d of C_W.

$\Sigma \cup \{\langle w, \alpha(d)^- \rangle\} \cup \{\langle s, \exists x(\alpha(x) \to \forall x \alpha(x))^+ \rangle\}$ can not be S_n^+-coherent, because it contains n+1 strongly distict points. To wit, $\exists x(\alpha(x) \to \forall x \alpha(x))$ separates s from w and $\forall x \alpha(x)$ separates w from s+1. Hence $\Sigma \cup \{\langle w, \alpha(d)^- \rangle\} \cup \{\langle w, \exists x(\alpha(x) \to \forall x \alpha(x))^- \rangle\}$ is S_n^+-coherent and so $\Sigma' = \Sigma \cup \{\langle w, (\alpha(d) \to \forall x \alpha(x))^- \rangle\}$ is S_n^+-coherent. It follows that $\Sigma'' = \Sigma' \cup \{\langle v, \alpha(d)^+ \rangle\} \cup \{\langle v, \forall x \alpha(x)^- \rangle\}$ is S_n^+-coherent, for some v, $w < v < s+1$. But then Σ'' includes a subdiagram whose support is $\{1, \ldots, s-1, w, v, s+1, \ldots, n\}$ and contains n+1 strongly distinct points. To wit, $\alpha(d)$ separates w from v, $\forall x \alpha(x)$ separates v from s+1 and the formula that separates s-1 from w is the formula that separates s-1 from s in Δ.

It follows that we can get a saturated diagram Γ, $\Delta \subseteq \Gamma$, based on $\langle 1, \ldots, n \rangle$ and such that $\langle 1, \alpha^- \rangle \in \Delta$. A Kripke model with nested domains based on the frame $\mathcal{F} = \langle \{1, \ldots, n\}, \leq \rangle$ is easily obtainable.

REFERENCES

[1] CORSI, Giovanna, 'Completeness theorem for Dummett's LC quantified and some of its extensions', sent to Studia Logica.

[2] ONO, Hiroakira, 'On finite linear intermediate predicate logics', Studia Logica, 4 (1988), pp.81-89.

OSSERVAZIONI SUL TEOREMA DI SOLOVAY NELL'AMBITO DELLA FORMULAZIONE ALLA GENTZEN DELL'ARITMETICA

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1. INTRODUZIONE : E' noto che il teorema di Solovay stabilisce il seguente rapporto fra l'aritmetica PA (PRA) e il sistema modale G :

se $A(p_1, \ldots p_n)$ è formula modale e $\{\varphi\}$ è l'insieme delle interpretazioni del linguaggio modale nell'Aritmetica allora

$$\vdash_{PA} A^{\varphi} \quad \text{per ogni } \varphi \implies \vdash_{G} A$$

Ricordiamo che G è il sistema modale esprimibile in termini di sequenti come :

$$PC + \frac{X, \Box X, \Box B \vdash B}{\Box X \vdash \Box B} GLR$$

(dove X insieme di formule, B formula);

inoltre per interpretazione φ del linguaggio proposizionale nell'Aritmetica intendiamo una applicazione :

$$\varphi : \{\text{lettere proposizionali}\} \longrightarrow \{\text{formule di PA}\}$$

tale che :
$$\varphi(\sim A) \equiv \sim \varphi(A)$$
$$\varphi(A \wedge B) \equiv \varphi(A) \wedge \varphi(B)$$
$$\varphi(\Box A) \equiv Pr(\varphi(A))$$

(scriviamo anche A^{φ} per $\varphi(A)$) ;

nella prospettiva di una riconduzione nell'ambito della proof - theory del teorema di Solovay si propone qui una indagine sulle prove di sequenti del tipo S^{φ} nell'Aritmetica Ricorsiva Primitiva PRA ,con la regola di induzione ristretta alle formule atomiche .

2. RISULTATI :

Concentriamo la nostra attenzione sulla classe di interpretazioni del tipo :

$$p_i \longmapsto B(p_i)$$

dove $B(p_i)$ è combinazione booleana di formule della forma $Pr(h_i)$, h_i godeliano di formula.

Si nota innanzitutto che se S è un sequente modale e vale

$$\vdash_{PRA} S^{\varphi} \quad \text{allora vale} \quad \vdash_{PRA} S_i^{\varphi}$$

$i = 1,\ldots.t$, con S_i della forma:

$$Pr_{PRA}(h_1),\ldots.Pr_{PRA}(h_m) \vdash Pr_{PRA}(d_1),\ldots.Pr_{PRA}(d_n)$$

h_i , d_j godeliani, che può scriversi più esplicitamente:

$$\exists x\,(X(x, h_1) = o),\ldots \exists x\,(X(x, h_m) = o) \vdash$$
$$\vdash \exists x\,(X(x, d_1) = o),\ldots \exists x\,(X(x, d_n) = o)$$

dove $X(\ ,\ )$ è funzione caratteristica del predicato $Prov_{PRA}(\ ,\ )$.

Indicheremo con T un sequente di PRA di questo tipo.

Si provano:

PROPOSIZIONE : Data in PRA una prova $\mathscr{P}$ del sequente:

$$X(a_1, h_1) = o,\ldots.X(a_m, h_m) = o \vdash$$
$$\vdash \exists x\,(X(x,d_1) = o),\ldots \exists x\,(X(x,d_n) = o)$$

a_i variabili libere distinte,

allora esiste $d \in \{d_1, \ldots d_n\}$

tale che è provabile in PRA il sequente :

$$X(a_1,h_1) = o, \ldots X(a_m,h_m) = o \vdash$$

$$\vdash X(t(a_1, \ldots a_m), d) = o$$

$t(a_1, \ldots a_m)$ termine arbitrario che può contenere $a_1, \ldots a_m$

PROPOSIZIONE : Sia data in PRA una prova $\mathscr{P}$ del sequente:

$$X(a_1,h_1) = o,\ldots,X(a\ \ ,h_m) = o \vdash$$
$$\vdash X(t(a_1,\ldots a_m), d) = o$$

allora:

1) Possiamo ritenere eliminata ogni induzione in $\mathscr{P}$ che introduca nella formula principale destra un termine chiuso, o un termine aperto le cui variabili $b_1,\ldots.b_k$ sono diverse da $a_1,\ldots a_m$

2) Possiamo ritenere eliminabile ogni induzione in $\mathscr{P}$ la cui formula principale destra è esplicita

3) Possiamo ritenere eliminabile ogni induzione in $\mathscr{P}$ la cui formula principale sinistra è esplicita

DEFINIZIONE : Diciamo cascata una prova in PRA costituita solo da :

- Assiomi
- Tagli atomici
- Induzioni atomiche implicite introducenti termini aperti

Abbiamo allora :

COROLLARIO : Un enunciato della logica della provabilità in PRA è la quantificazione esistenziale del sequente finale di una cascata .