DEDUZ. NATURALE AL PRIM'ORDINE

$$\mathrm{I}\wedge\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}\Gamma'\\ \vdots\\ \psi\end{array}}{\varphi\wedge\psi}
\qquad
\mathrm{E}\wedge\mathrm{Sx}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\wedge\psi\end{array}}{\varphi}
\qquad
\mathrm{E}\wedge\mathrm{Dx}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\wedge\psi\end{array}}{\psi}$$

$$\mathrm{I}\vee\mathrm{Sx}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}}{\varphi\vee\psi}
\qquad
\mathrm{I}\vee\mathrm{Dx}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \psi\end{array}}{\varphi\vee\psi}
\qquad
\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}\Gamma,[\varphi]^1\\ \vdots\\ \chi\end{array}\quad \begin{array}{c}\Gamma,[\psi]^1\\ \vdots\\ \chi\end{array}}{\chi}\ \mathrm{E}\vee_1$$

$$\mathrm{E}{\to}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}\Gamma'\\ \vdots\\ \varphi\to\psi\end{array}}{\psi}
\qquad
\mathrm{I}{\to}_1\ \dfrac{\begin{array}{c}\Gamma,[\varphi]^1\\ \vdots\\ \psi\end{array}}{\varphi\to\psi}$$

$$\mathrm{I}\neg_1\ \dfrac{\begin{array}{c}\Gamma,[\varphi]^1\\ \vdots\\ \bot\end{array}}{\neg\varphi}
\qquad
\mathrm{RA}_1\ \dfrac{\begin{array}{c}\Gamma,[\neg\varphi]^1\\ \vdots\\ \bot\end{array}}{\varphi}
\qquad
\mathrm{E}\neg\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}\Gamma'\\ \vdots\\ \neg\varphi\end{array}}{\bot}$$

$$\mathrm{I}{\leftrightarrow}\ \dfrac{\begin{array}{c}[\psi]^1\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}[\varphi]^1\\ \vdots\\ \psi\end{array}}{\varphi\leftrightarrow\psi}
\qquad
\mathrm{E}{\leftrightarrow}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \psi\end{array}\quad \begin{array}{c}\Gamma'\\ \vdots\\ \varphi\leftrightarrow\psi\end{array}}{\varphi}
\qquad
\mathrm{E}{\leftrightarrow}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}\Gamma'\\ \vdots\\ \varphi\leftrightarrow\psi\end{array}}{\psi}$$

Tabella 2: Regole della deduzione naturale

$$\mathrm{I}\forall\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi\end{array}}{\forall x\varphi}$$

con la prescrizione che tale regola può essere utilizzata solo quando x non compare libera nelle ipotesi da cui φ dipende.

$$\mathrm{E}\forall\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \forall x\varphi\end{array}}{\varphi[t/x]}$$

per ogni termine t sostituibile per x in φ.

$$\mathrm{I}\exists\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \varphi(t)\end{array}}{\exists x\varphi[x/t]}$$

dove t è un termine sostituibile per x in φ.

$$\mathrm{E}\exists_1\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \exists x\varphi\end{array}\quad \begin{array}{c}[\varphi]^1,\Gamma\\ \vdots\\ \psi\end{array}}{\psi}$$

Con la prescrizione che tale regola può essere applicata solo se la variabile x non compare libera in ψ né nelle ipotesi da cui ψ dipende oltre a φ stessa (in cui ovviamente può apparire libera).

1) PRODURRE DERIVAZ CON DEDUZ. NATURALE

a) $\vdash \forall x \neg (F(x) \wedge \neg F(x))$

$$[F(x) \wedge \neg F(x)]^1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[F(x) \wedge \neg F(x)]^1}{F(x)}\, E\wedge_{sx} \qquad \dfrac{[F(x) \wedge \neg F(x)]^1}{\neg F(x)}\, E\wedge_{dx}}{\bot}\, E\neg}{\neg (F(x) \wedge \neg F(x))}\, I\neg_1}{\forall x \neg (F(x) \wedge \neg F(x))}\, I\forall$$

b) $R(a), \forall x (R(x) \rightarrow S(x)) \vdash \exists x S(x)$ $\quad$ a simbolo di cost.

$$R(a) \qquad \forall x (R(x) \rightarrow S(x))$$

$$\dfrac{\dfrac{R(a) \qquad \dfrac{\forall x (R(x) \rightarrow S(x))}{R(a) \rightarrow S(a)}\, E\forall}{S(a)}\, E\rightarrow}{\exists x S(x)}\, I\exists$$

c) $\exists x R(x),\ \forall x (R(x) \to S(x)) \vdash \exists x S(x)$

$$
\dfrac{\exists x R(x) \qquad \dfrac{\dfrac{\dfrac{\forall x (R(x) \to S(x))}{R(x) \to S(x)}\,E\forall \qquad [R(x)]^1}{S(x)}\,E\to}{\exists x S(x)}\,I\exists}{\exists x S(x)}\,E\exists_1
$$

d) $\forall x R(x) \vdash \forall y R(y)$

$$
\dfrac{\dfrac{\forall x R(x)}{R(y)}\,E\forall}{\forall y R(y)}\,I\forall
$$

e) $\exists x R(x) \vdash \exists y R(y)$

$$\dfrac{\exists x R(x) \qquad \dfrac{[R(x)]^{1}}{\exists y R(y)}\,\mathrm{I}\exists}{\exists y R(y)}\,\mathrm{E}\exists_1$$

f) $\neg(\exists x(\neg R(x))) \vdash \forall x R(x)$

$$\dfrac{\dfrac{\dfrac{\neg(\exists x(\neg R(x))) \qquad \dfrac{[\neg R(x)]^{1}}{\exists x(\neg R(x))}\,\mathrm{I}\exists}{\bot}\,\mathrm{E}\neg}{R(x)}\,\mathrm{RA}_1}{\forall x R(x)}\,\mathrm{I}\forall$$

g) $\neg(\forall x\, R(x)) \vdash \exists x\, (\neg R(x))$

$\neg(\forall x (R(x))), [\neg \exists x (\neg R(x))]^1, [\neg R(x)]^2$

$\neg(\forall x\, R(x)) \quad [R(x)]^1$

$\overline{\qquad\qquad}\ E\neg$

$\bot$

$\overline{\qquad\qquad}\ RA_2$

$R(x)$

$I\forall\ \overline{\qquad\qquad}$

$\forall x (R(x))$

$E\neg\ \overline{\qquad\qquad}$

$\bot$

$RA_1\ \overline{\qquad\qquad}$

$\exists x (\neg R(x))$

$\overline{\qquad\qquad}\ I\forall$

$\forall x\, R(x)$

$E\neg\ \overline{\qquad\qquad}$

$\bot$

$I\neg_1\ \overline{\qquad\qquad}$

$\neg R(x)$

$I\exists\ \overline{\qquad\qquad}$

$\exists x (\neg R(x))$

h) $\exists x (\neg R(x)) \vdash \neg(\forall x\, R(x))$

$[\forall x\, R(x)]^2, [\neg R(x)]^1$

$\exists x (\neg R(x))$

$\overline{\qquad\qquad}\ E\forall$

$R(x)$

$\overline{\qquad\qquad}$

$\bot$

$\overline{\qquad\qquad}\ I\neg_2$

$\neg(\forall x\, R(x))$

$E\exists_1\ \overline{\qquad\qquad}$

$\neg(\forall x\, R(x))$

i) $\exists x \exists y\, R(x,y) \vdash \exists y \exists x\, R(x,y)$

$$[R(x,y)]^2$$

$$[\exists y\, R(x,y)]^1$$

$$\exists x \exists y\, R(x,y)$$

$$\dfrac{[R(x,y)]^2}{\exists x R(x,y)}\ I\exists$$

$$\dfrac{\exists x R(x,y)}{\exists y \exists x R(x,y)}\ I\exists$$

$$\dfrac{\exists y\exists x R(x,y)}{\exists y \exists x R(x,y)}\ E\exists_2$$

$$E\exists_1\ \dfrac{}{\exists y \exists x R(x,y)}$$

j) $\exists x (P \to R(x)) \vdash P \to (\exists x R(x))$

$$[P \to R(x)]^1,\ [P]^2$$

$$\exists x (P \to R(x))$$

$$\dfrac{}{R(x)}\ E\to$$

$$\dfrac{R(x)}{\exists x R(x)}\ I\exists$$

$$\dfrac{\exists x R(x)}{P \to (\exists x R(x))}\ I\to_2$$

$$E\exists_1\ \dfrac{}{P \to (\exists x R(x))}$$

ULTRAPRODOTTI: $\mathcal{I}$ LINGUAGGIO AL PRIM'ORDINE

- $\{\mathcal{Q}_i\}_{i\in I}$ $\mathcal{L}$-STRUTTURE
- $\mathcal{F} \subseteq \mathcal{P}(I)$ ULTRAFILTRO

$\rightsquigarrow$ NUOVA $\mathcal{L}$-STRUTTURA $\prod \mathcal{Q}_i / \sim_{\mathcal{F}}$ T.C.

(LE PROPRIETÀ DI FILTRO GARANTISCONO CHE $\sim_{\mathcal{F}}$ SIA UNA CONGRUENZA SU $\prod \mathcal{Q}_i$) $\rightarrow$

$$\boxed{\begin{gathered}(a_i)_{i\in I} \sim_{\mathcal{F}} (b_i)_{i\in I} \\ \text{SSE} \\ \{i\in I \mid a_i = b_i\} \in \mathcal{F}\end{gathered}}$$

<u>OSS.</u> SE $\mathcal{F}$ ULTRAFILTRO PRINCIPALE, i.e.

$\mathcal{F} = \{j\}\uparrow = \{S \subseteq I \mid j \in S\}$ PER UN QUALCHE $j \in I$, ALLORA

$$\boxed{\begin{gathered}(a_i)_{i\in I} \sim_{\mathcal{F}} (b_i)_{i\in I} \\ \text{SSE} \\ j \in \{i\in I \mid a_i = b_i\} \\ \text{SSE} \\ a_j = b_j\end{gathered}}$$

DUNQUE

$$\prod \mathcal{Q}_i / \sim_{\mathcal{F}} \xrightarrow{\pi_j} \mathcal{Q}_j$$

$$(a_i)_{i\in I} \longmapsto a_j$$

È ISOMORFISMO

Teorema 4.66 (Łoš)

Sia $(\mathfrak{A}_i \mid i \in I)$ *una famiglia di strutture nello stesso linguaggio del prim'ordine* $\mathcal{L}$ *e sia* $\mathcal{F}$ *un ultrafiltro su* I. *Sia* $\mathfrak{A} := \prod_{i \in I} \mathfrak{A}_i / \mathcal{F}$ *l'ultraprodotto della* $\mathfrak{A}_i$. *Per ogni* $\mathcal{L}$*-formula e per ogni famiglia di interpretazioni* $\{v_i \mid i \in I\}$, *con* $v_i \colon \mathsf{Var} \to A_i$ *per* $i \in I$, *si ha*

$$\mathfrak{A}, \prod v_i \models \varphi \text{ se, soltanto se, } \{i \in I \mid \mathfrak{A}_i, v_i \models \varphi\} \in \mathcal{F}.$$

COROLLARIO $\mathfrak{A}$ $\mathcal{L}$-STRUTT., φ $\mathcal{L}$-FORM.
I INSIEME, $\mathcal{F} \subseteq \mathcal{P}(I)$ ULTRAF

$$\mathfrak{A}^I/_{\sim_\mathcal{F}} \models \varphi \quad \text{SSE} \quad \mathfrak{A} \models \varphi$$

(CIOÈ $\mathfrak{A}^I/_{\sim_\mathcal{F}} \equiv \mathfrak{A}$)

2) $\exists$ ULTRAPOTENZA INFINITA DI UNA STRUTTURA DI CARDINALITÀ 2 ?

$\mathfrak{A}$ $\qquad$ $A = \{0, 1\}$

I INSIEME $\qquad$ $\mathfrak{A}^I / \sim_{\mathcal{F}}$

$\mathcal{F} \subseteq \mathcal{P}(I)$

ULTRAF.

ALMENO 2: $\quad \exists x_1 \exists x_2 (\neg (x_1 = x_2))$

AL PIÙ 2: $\quad \forall x_1 \forall x_2 \forall x_3 ((x_1 = x_2) \vee (x_1 = x_3) \vee \vee (x_2 = x_3))$

Teorema 4.69 (Compattezza della logica del prim'ordine)

Un insieme Γ di formule del prim'ordine ha un modello se, e soltanto se, ogni suo sottoinsieme finito ha un modello.

3) LA CLASSE DEI GRUPPI FINITI NON È ASSIOMATIZZ. AL PRIM'ORDINE

SOL.

Γ GRUPPI FINITI

φ_n : "CI SONO ALMENO n ELEM."

$$\Gamma' = \Gamma \cup \{\varphi_n \mid n \in \mathbb{N}\}$$

$\Delta \subseteq \Gamma'$ FINITO

$$N = \max\{n \mid \varphi_n \in \Delta\}$$

$$\mathbb{Z}/N\mathbb{Z} \models \Delta$$

$\exists\ G$ t.c. $G \models \Gamma'$

$G \models \Gamma$

$G \models \varphi_n \ \forall n \in \mathbb{N}$

4) LA CLASSE DEI GRUPPI DI TORSIONE NON È ASSIOM. AL PRIM'ORDINE

SOL. Γ GRUPPI DI TORSIONE

PER OGNI $x \in G$ ESISTE $n \geq 1$ T.C. $x^n = 1$

$$T_n : \quad \neg(x^n = 1)$$

$$\Gamma' = \Gamma \cup \{T_n \mid n \in \mathbb{N}\}$$

$\Delta \subseteq \Gamma'$ FINITO

$$N = \max\{n \in \mathbb{N} \mid T_n \in \Delta\}$$

g GEN.

$$\mathbb{Z}/(N+1)\mathbb{Z} \models \Delta$$

$$v(x) = g$$

COMP. $\Rightarrow$ $\exists\ G, v$ t.c. $G, v \models \Gamma'$

$G, v \models \Gamma$

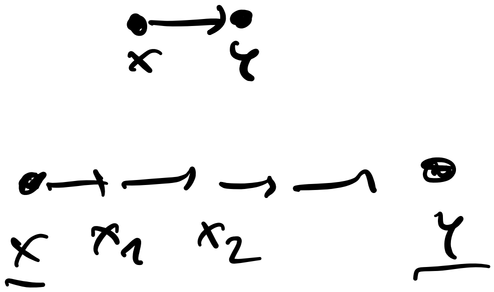
$G, v \models \{T_n\}$

$v(x) \in G$

5) LA CLASSE DEI GRAFI DIRETTI CONNESSI NON È ASSIOMATIZZ. AL PRIM'ORDINE

SOL. Γ GRAFI DIRETTI CONNESSI

R



$C_n \; \forall x_1 \ldots \forall x_n \left(\neg\left(R(x,x_1) \wedge R(x_1,x_2) \ldots \wedge x_n = y\right)\right)$

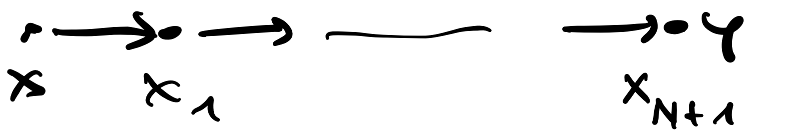
$C_0 := \neg(x=y)$

$D_n := \bigwedge_{k \leq n} C_n$

$\Gamma' = \Gamma \cup \{D_n \mid n \in \mathbb{N}\}$

$\Delta \subseteq \Gamma'$ FINITO

$N = \max\{n \in \mathbb{N} \mid D_n \in \Delta\}$

 $\models \Delta$

$\Rightarrow \exists$ G GRAFO t.c. $G, \nu \models \Gamma'$

ν VAL. $\nu(x)$ $\nu(y)$

ES. DIMOSTRARE φ SODD. E $\not\models \varphi$

1) $\varphi := (\forall x \exists y R(x,y)) \wedge \neg \forall x P(x)$

$A = \{0,1\}$

- $R^{a} = \{(0,0),(1,0)\}$ $\models \varphi$
  $P^{a} = \{0\}$

- A
  $R^{a'}$
  $P^{a'} = \{0,1\}$ $\not\models \varphi$

b) $(\forall x P(x)) \vee (\forall x \neg P(x))$

- $A =$ ______ $\models \varphi$
  $P^{a} = A$

- $A = \{0,1\}$ $\not\models \varphi$

$P^{a'} = \{0\}$

c) $P(c) \to \neg P(c) \equiv \neg P(c)$ $\quad c$ COST.

- $A = \{0, 1\}$
  $P^{a} = \{0\}$ $\quad c^{a} = 1$ $\quad \models \varphi$

- $c^{a'} = 0$

d) $(\forall x \exists y R(x,y)) \to (\exists y \forall x R(x,y))$

- $A = \{*\}$
  $R^{a} = \emptyset$ $\quad$ OPPURE $\quad R^{a} = \{(*, *)\}$
  $\models \varphi$

- $A' = \{0, 1\}$
  $R^{a'} = \{(0,0), (1,1)\}$ $\quad \not\models \varphi$

e) $\forall x\,(P(x) \vee Q(x)) \to (\forall x\, P(x) \vee \forall x\, Q(x))$

$A = \ldots$

- $P^{a} = A \quad Q^{a} = - \qquad \models \varphi$

- $A = \{0, 1\}$

  $P^{a} = \{0\}, \quad Q^{a} = \{1\} \qquad \not\models \varphi$