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TEORIA

SINTASSI

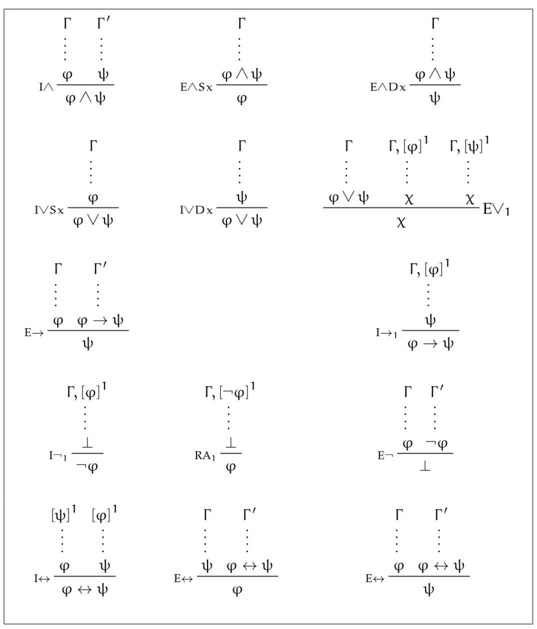
LINGUAGGIO:

- VARIABILI $\mathcal{L} = \{p, q, \ldots\}$
- CONNETTIVI $\{\wedge, \vee, \neg, \rightarrow, \leftrightarrow\}$
- SIMBOLI AUS. $\{ (,) \}$

$\rightsquigarrow$ $Form_{\mathcal{L}}$ FORMULE (BEN FORMATE)

DERIVAZIONI (DEDUZIONE NATURALE):

$\Gamma \subseteq Form_{\mathcal{L}}$, $\varphi \in Form_{\mathcal{L}}$



$\Gamma \vdash \varphi$ SE ESISTE DERIVAZ. DI φ A PARTIRE DA Γ

SEMANTICA

VALUTAZIONE:

$\mathcal{L} \xrightarrow{\forall v} \{0, 1\}$

$\mathcal{L} \searrow Form_{\mathcal{L}}$ "

$\exists! \tilde{v}$ CHE RISPETTI LE TAVOLE DI VERITÀ ($Form_{\mathcal{L}} \dashrightarrow \{0,1\}$)

CONSEGUENZA LOGICA:

$\Gamma \subseteq Form_{\mathcal{L}}$, $\varphi \in Form_{\mathcal{L}}$

$\Gamma \models \varphi$ SE, $\forall v$, $\tilde{v}(\psi) = 1 \ \forall \psi \in \Gamma$
$\Downarrow$
$\tilde{v}(\varphi) = 1$

COERENZA

$\Gamma \subseteq Form_{\mathcal{L}}$ è

COERENTE SSE $\Gamma \nvdash \bot$

MASSIMALE

DATO $\Gamma \subseteq Form_{\mathcal{L}}$ COER.
$\exists\, \Gamma \subseteq M \subseteq Form_{\mathcal{L}}$ COER. MASSIMALE

COMPATTEZZA

$\Gamma \vdash \varphi$ SSE $\exists\, \Delta \subseteq \Gamma$ FINITO T.C. $\Delta \vdash \varphi$
(DUNQUE, Γ COER. SSE FINITAM. COER.)

SODDISFACIBILITÀ

$\Gamma \subseteq Form_{\mathcal{L}}$ è

SODDISFACIBILE SSE $\exists\, v$ T.C. $\tilde{v}(\psi) = 1$ $\forall \psi \in \Gamma$

(EQUIV. $\Gamma \nvDash \bot$)

MASSIMALE:

DATO $\Gamma \subseteq Form_{\mathcal{L}}$ SODD.
$\exists\, \Gamma \subseteq M \subseteq Form_{\mathcal{L}}$ SODD. MASSIMALE

COMPATTEZZA

SODD. $\Leftrightarrow$ FINITAM. SODD.

ADEGUATEZZA

$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$

COMPLETEZZA

$\Gamma \vdash \varphi \Leftarrow \Gamma \vDash \varphi$

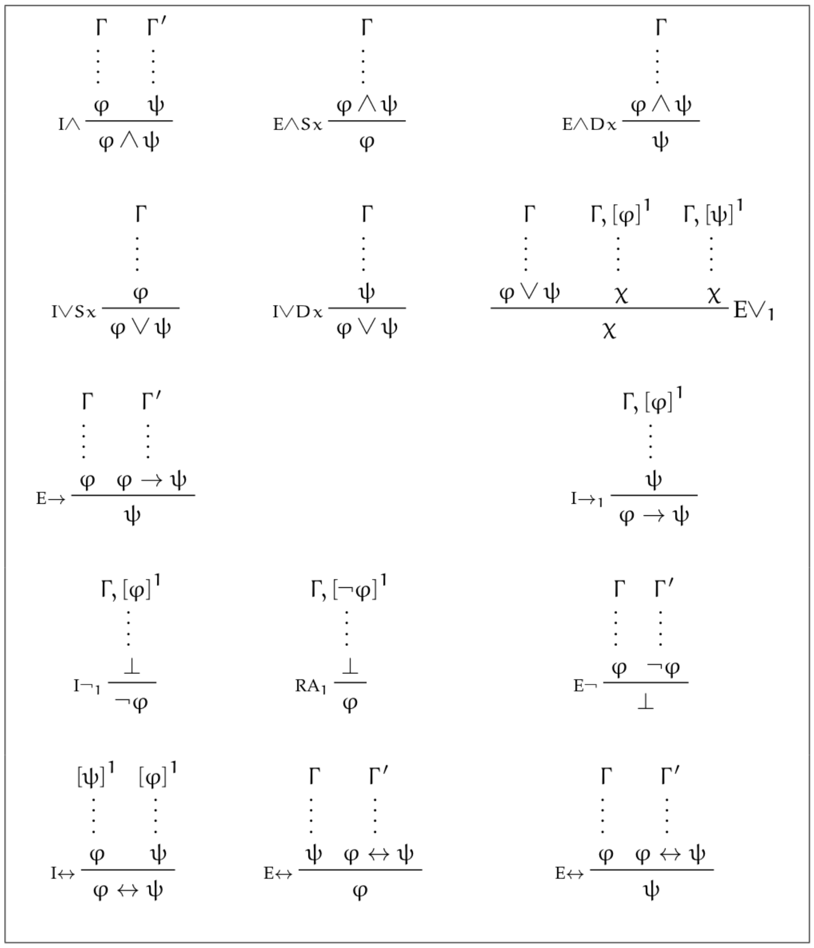
EQUIV.

Γ COER. $\Leftrightarrow$ Γ SODD.

ESERCIZI

Ded. NATURALE

- $\neg(a \vee b) \vdash \neg a \wedge \neg b$
- $\neg a \wedge \neg b \vdash \neg(a \vee b)$
- $\neg(a \wedge b) \vdash \neg a \vee \neg b$



$\neg(a \vee b), [a]^1$ $\qquad\qquad$ $\neg(a \vee b), [b]^2$

$$\dfrac{\dfrac{\dfrac{a \vee b}{\bot}}{\neg a}\, I\neg_1 \qquad \dfrac{\dfrac{a \vee b}{\bot}}{\neg b}\, I\neg_2}{\neg a \wedge \neg b}\, I\wedge$$

$\neg a \wedge \neg b,\ [a \vee b]^1 \quad \neg a \wedge \neg b \quad [a]^2 \quad \neg a \wedge \neg b \quad [b]^2$

$\neg a \qquad \neg b$

$\bot \qquad \bot$

$E\vee_2$ ———

$\bot$

$I\neg_1$ ———

$\neg(a \vee b)$

$\neg(a \wedge b),\ [\neg(\neg a \vee \neg b)]^1 \quad [\neg a]^2 \quad \neg(a \wedge b),\ [\neg(\neg a \vee \neg b)]^1$

$I\vee$ ——— $\neg a \vee \neg b$

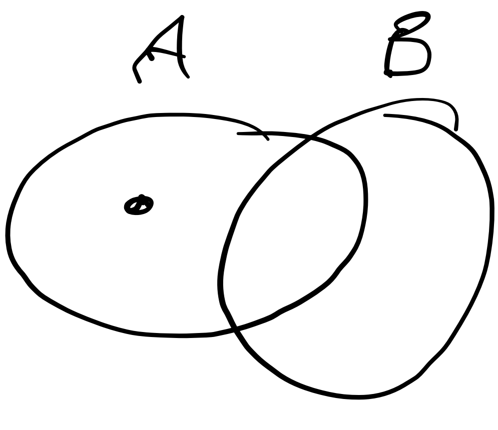
RA_2 $\dfrac{\bot}{a}$ $\quad \dfrac{\bot}{b}$

$I\wedge$ ——— $a \wedge b$

$E\neg$ $\neg(a \wedge b)$ ———

RA_1 $\bot$ ———

$\neg a \vee \neg b$



- $\vdash \varphi \to (\psi \to \varphi)$
- $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
- $\neg\psi \to \neg\varphi \vdash \varphi \to \psi$
- $\vdash \neg\neg\varphi \leftrightarrow \varphi$

$$\mathrm{I}\wedge\ \dfrac{\begin{array}{cc}\Gamma & \Gamma' \\ \vdots & \vdots \\ \varphi & \psi\end{array}}{\varphi\wedge\psi} \qquad \mathrm{E}\wedge\mathrm{Sx}\ \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \varphi\wedge\psi\end{array}}{\varphi} \qquad \mathrm{E}\wedge\mathrm{Dx}\ \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \varphi\wedge\psi\end{array}}{\psi}$$

$$\mathrm{I}\vee\mathrm{Sx}\ \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \varphi\end{array}}{\varphi\vee\psi} \qquad \mathrm{I}\vee\mathrm{Dx}\ \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \psi\end{array}}{\varphi\vee\psi} \qquad \dfrac{\begin{array}{ccc}\Gamma & \Gamma,[\varphi]^1 & \Gamma,[\psi]^1 \\ \vdots & \vdots & \vdots \\ \varphi\vee\psi & \chi & \chi\end{array}}{\chi}\ \mathrm{E}\vee_1$$

$$\mathrm{E}{\to}\ \dfrac{\begin{array}{cc}\Gamma & \Gamma' \\ \vdots & \vdots \\ \varphi & \varphi\to\psi\end{array}}{\psi} \qquad \mathrm{I}{\to}_1\ \dfrac{\begin{array}{c}\Gamma,[\varphi]^1 \\ \vdots \\ \psi\end{array}}{\varphi\to\psi}$$

$$\mathrm{I}\neg_1\ \dfrac{\begin{array}{c}\Gamma,[\varphi]^1 \\ \vdots \\ \bot\end{array}}{\neg\varphi} \qquad \mathrm{RA}_1\ \dfrac{\begin{array}{c}\Gamma,[\neg\varphi]^1 \\ \vdots \\ \bot\end{array}}{\varphi} \qquad \mathrm{E}\neg\ \dfrac{\begin{array}{cc}\Gamma & \Gamma' \\ \vdots & \vdots \\ \varphi & \neg\varphi\end{array}}{\bot}$$

$$\mathrm{I}{\leftrightarrow}\ \dfrac{\begin{array}{cc}[\psi]^1 & [\varphi]^1 \\ \vdots & \vdots \\ \varphi & \psi\end{array}}{\varphi\leftrightarrow\psi} \qquad \mathrm{E}{\leftrightarrow}\ \dfrac{\begin{array}{cc}\Gamma & \Gamma' \\ \vdots & \vdots \\ \psi & \varphi\leftrightarrow\psi\end{array}}{\varphi} \qquad \mathrm{E}{\leftrightarrow}\ \dfrac{\begin{array}{cc}\Gamma & \Gamma' \\ \vdots & \vdots \\ \varphi & \varphi\leftrightarrow\psi\end{array}}{\psi}$$

$[\varphi]^1, [\chi]^2$

$$\mathrm{I}{\to}_1\ \dfrac{\mathrm{I}{\to}_2\ \dfrac{\varphi^1}{\psi\to\varphi}}{\varphi \to (\psi\to\varphi)}$$

$$[\varphi \to \psi]^1,\ [\neg\psi]^2,\ [\varphi]^3$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to \psi]^1 \quad [\varphi]^3}{\psi}\,E{\to}_{1,3} \qquad \neg\psi^2}{\bot}\,E\neg}{\neg\varphi}\,I\neg_3}{\neg\psi \to \neg\varphi}\,I{\to}_2}{(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)}\,I{\to}_1$$

$$\neg\psi \to \neg\varphi,\ [\varphi]^1,\ [\neg\psi]^2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi \to \neg\varphi \quad [\neg\psi]^2}{\neg\varphi}\,E{\to} \qquad \dfrac{}{\varphi}}{\bot}\,E\neg}{\psi}\,RA_2}{\varphi \to \psi}\,I{\to}_1$$

$$\dfrac{\dfrac{\dfrac{\neg\neg\varphi \quad [\neg\varphi]^1}{\bot}\,E\neg}{\varphi}\,RA_1 \qquad \dfrac{\dfrac{\varphi \quad [\neg\varphi]^2}{\bot}\,E\neg}{\neg\neg\varphi}\,I\neg_2}{\neg\neg\varphi \leftrightarrow \varphi}\,I\leftrightarrow$$

- $\varphi \wedge \psi \vdash \neg(\neg\varphi \vee \neg\psi)$

$$\dfrac{\dfrac{\varphi \wedge \psi \quad [\neg\varphi \vee \neg\psi]^1 \qquad \dfrac{\dfrac{\varphi\wedge\psi}{\varphi}\,E\wedge_s \quad [\neg\varphi]^2}{\bot}\,E\neg \qquad \dfrac{\dfrac{\varphi\wedge\psi}{\psi}\,E\wedge_d \quad [\neg\psi]^2}{\bot}\,E\neg}{\bot}\,E\vee_2}{\neg(\neg\varphi \vee \neg\psi)}\,I\neg_1$$

- $\bot \vdash \varphi$

$$\bot, [\neg\varphi]^1$$

$$RA_1\ \dfrac{\bot}{\varphi}$$

- $\vdash \varphi \to (\neg\varphi \to \chi)$

$$[\varphi]^1, [\neg\varphi]^2, [\neg\chi]^3$$

$$I{\to}_1\ \dfrac{I{\to}_2\ \dfrac{RA_3\ \dfrac{E\neg\ \dfrac{}{\bot}}{\chi}}{\neg\varphi \to \chi}}{\varphi \to (\neg\varphi \to \chi)}$$

- $p \wedge q \vdash p \vee q$

$$p \land q$$

$$\frac{}{p}\ E\land$$

$$\frac{p}{p \lor q}\ I\lor$$

- $p \lor q \vdash q \lor p$

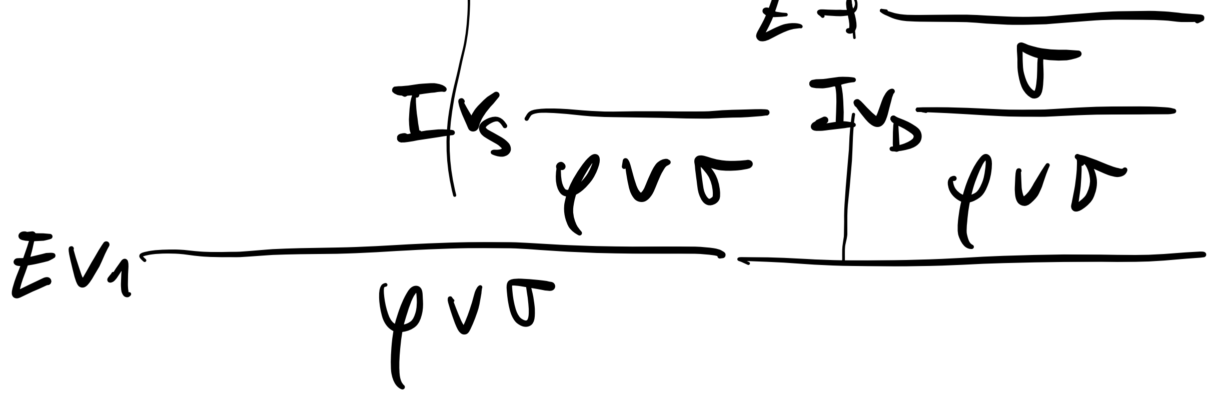
$$p \lor q \qquad [p]^1 \qquad [q]$$

$$\frac{}{q \lor p}\ I\lor_D \qquad \frac{}{q \lor p}\ I\lor_S$$

$$\frac{}{q \lor p}\ E\lor_1$$

- $\varphi \lor \chi,\ \chi \to \sigma \vdash \varphi \lor \sigma$

$$\varphi \lor \chi \qquad \chi \to \sigma \qquad [\varphi]^1 \qquad \chi \to \sigma \qquad [\chi]^1 \qquad \chi \to \sigma$$



Definizione 3.30

Un'**algebra di Boole** è un reticolo complementato, limitato e distributivo. Alternativamente, possiamo definire un'algebra di Boole come una struttura del tipo $\langle A, \wedge^2, \vee^2, \neg^1, 0, 1\rangle$, che soddisfi i seguenti assiomi:

$$(x \vee y) \vee z = x \vee (y \vee z) \quad \text{e} \quad (x \wedge y) \wedge z = x \wedge (y \wedge z), \tag{BA1}$$
$$x \vee y = y \vee x \quad \text{e} \quad x \wedge y = y \wedge x, \tag{BA2}$$
$$(x \wedge y) \vee y = y \quad \text{e} \quad (x \vee y) \wedge y = y, \tag{BA3}$$
$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \quad \text{e} \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z), \tag{BA4}$$
$$x \wedge \neg x = 0 \quad \text{e} \quad x \vee \neg x = 1 \tag{BA5}$$

Si noti, che sebbene nessuna delle equazioni qui sopra dica esplicitamente che 0 e1 sono rispettivamente il minimo e il massimo dell'algebra, come vedremo nella Proposizione 3.33, ciò è una conseguenza di questi assiomi.

$x \in B$ $\quad$ $y \in B$ t.c.

$x \wedge y = 0 \quad x \vee y = 1$

B Boole $\Rightarrow \exists X$ insieme e

$B \hookrightarrow \mathcal{P}(X)$ omomorfismo

- $n = p_1 \cdots p_k \in \mathbb{N}$ n SQUAREFREE

$$D = \{m \in \mathbb{N} \mid m \mid n\}$$

$m \leq m'$ sse $m \mid m'$

$m, m' \in D$

$\inf(m, m') = MCD(m, m')$

$\sup(m, m') = mcm(m, m')$

$\min = 1$, $MAX = n$

$m \in D$ $\neg m = \frac{n}{m}$ $n = p_1 \cdots p_k$

$m \wedge \neg m = 1$

$m \vee \neg m = n$

D, X

$D \subseteq \mathcal{P}(X)$

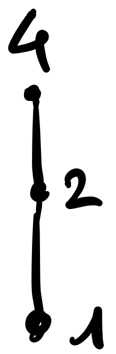
$$X = \{p_1, \dots, p_k\}$$

$\wedge^2, \vee^2, \neg^1, 0^0, 1^0$ $\qquad$ $\cap, \cup, {}^c, X, \emptyset$

$$D \longrightarrow \mathcal{P}(X)$$

$$D \ni m \longmapsto \{p \in X \mid p \mid m\}$$

$$D = \{m \in \mathbb{N} \mid m \mid 4\} = \{1, 2, 4\}$$



COMPL. DI 2

y t.c.

$$y \wedge 2 = 1 \Rightarrow y = 1$$

$$y \vee 2 = 4$$

X insieme infinito

$$\mathcal{B} := \{Y \subseteq X \mid Y \text{ finito o } Y \text{ cofinito}\}$$

$$\mathcal{B} \subseteq \mathcal{P}(X)$$