Small advances in the algebraic proof theory of substructural logics

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Introduction

This talk will be about some novel algebraic characterisations of cut-elimination and reductive cut-elimination and their relationships.

It will be argued that in a suitable subsystem of **FL**, extensions with any set of acyclic structural rules preserves cut-elimination if, and only if, they preserve reductive cut-elimination.

This allow to modestly extend some recent result relating (modular) cut-elimination and McNeille completion.

Cut-elimination = semantic propagation

In [1] the author gives both a semantic and a syntactic characterisation of (left) additive structural rules which preserve cut-elimination when added to the Full Lambek Calculus (**FL**).

The property that such rules must enjoy is called *semantic* (resp. *syntactic*) *propagation*.

The key construction to prove that such requirement is sufficient to the cut-elimination of the resulting calculus, dates back to some works by M. Okada.

[1] K. Terui.
Which structural rules admit cut elimination? An algebraic criterion.
Journal of Symbolic Logic, 72(3):738, 2007.

Reductive cut-elimination = propagation

Shortly later a similar approach was proposed in [2] to study a wider class of sequent calculi and to extend the method to unrestricted structural rules and logical rules for connectives. In this work the rules that preserve a stronger form of cut-elimination, called *reductive cut-elimination*, are again characterised both syntactically and semantically.

So in a sense [2] widely generalises [1], but on the other hand it only handles rules with the stronger property of preserving *reductive* cut-elimination.

[2] A. Ciabattoni and K. Terui.
Towards a semantic characterization of cut-elimination.
Studia Logica, 82(1):95–119, 2006.

Reductive cut-elimination

Definition

An occurrence of (CUT) in a derivation is said to be reducible if one of the following holds:

1. Both cut formulae are the principal formulae of logical rules.
2. One of the two cut formulae is a context formula of a rule other than (CUT).
3. One of the two premises is an identity axiom.

We say that a simple sequent calculus $\mathcal{L}$ admits reductive cut-elimination if whenever a sequent S_0 is derivable in $\mathcal{L}$ from a set A of non-logical axioms, S_0 has a derivation in $\mathcal{L}$ from A without any reducible cuts.

An example

Notice that in general reductive cut-elimination is strictly stronger than cut-elimination

This can be checked by considering the sequent calculus consisting only of the rules $(\star, r)$ and $(\star, l)$ below:

$$\frac{\Theta \Rightarrow X \quad \Theta \Rightarrow Y}{\Theta \Rightarrow X \star Y}\ (\star, r) \qquad \frac{\Theta_l, X, Y, \Theta_r \Rightarrow \Xi}{\Theta_l, X \star Y, \Theta_r \Rightarrow \Xi}\ (\star, l)$$

Modular cut-elimination

Definition

A set S of sequents is said to be elementary if S consists of atomic formulas and is closed under cuts.
In other words if S contains $\Sigma \Rightarrow p$ and $\Gamma, p, \Delta \Rightarrow \Pi$, it also contains $\Gamma, \Sigma, \Delta \Rightarrow \Pi$.

Definition

A sequent calculus admits modular cut-elimination if for any elementary set S and a sequent s, if s is derivable from S, then it is also derivable from S without using (CUT).

Modular and reductive cut-elimination coincide in any extension of **FL** with structural rules (It can be checked just using the syntactic characterisations mentioned above)

Main aim

The main aim of our study is to show that:

(Claim) For any set of structural rues $\mathcal{R}$, in the extension **FL**$(\mathcal{R})$, reductive cut-elimination is equivalent to cut-elimination.

Some important motivation for this study can be found in [3], where a strong connection between *modular* cut-elimination and McNeille completion is unveiled.

[3] A. Ciabattoni , N. Galatos and K. Terui.
Algebraic proof-theory for substructural logics: cut-elimination and completions.
Submitted, 2009.

Main aim

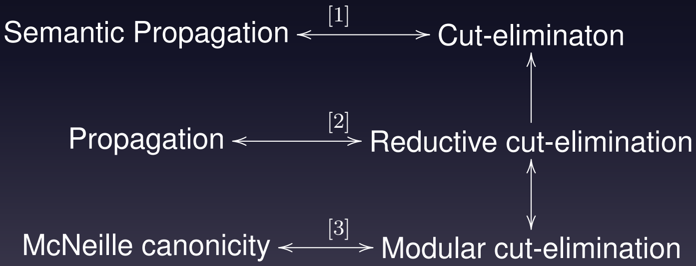
At the moment we are able to prove the claim only for a restricted class of structural rules (namely *acyclic*) in a weaker version of **FL** where the right-hand side of the sequents must contain exactly one formula.

Such restriction is known to characterise *Minimal Logic*.

Let us call **mFL** the sequent calculus of **FL** where each sequent has exactly one formula on the right side.

A short recap

In any extension of **FL** the following logical relations hold



Structural rules

For $0 \leq i \leq n$, let $\Theta_i, \Gamma_i, \Xi_i$ be sequences of (variables denoting) formulae.

Definition

By a structural rule we mean a rule (R) of the form

$$\frac{\Gamma_1, \Theta_1 \vdash \Xi_1 \ldots \Gamma_n, \Theta_n \vdash \Xi_n}{\Gamma_0, \Theta_0 \vdash \Xi_0} \ (R)$$

such that any formula appearing in $\Theta_1, ..., \Theta_n$ also appears in Θ_0 (*non erasing condition*).

Additive structural rules and axiomatic forms

If (R) can be written also in the form

$$\frac{\Gamma, \Theta_1 \vdash \Xi \ldots \Gamma, \Theta_n \vdash \Xi}{\Gamma, \Theta_0 \vdash \Xi} \ (R)$$

then it is said to be additive.

Definition

Given an additive structural rule $R = (\Theta_0 \lhd \Theta_1; ...; \Theta_n)$, its axiomatic form is defined by: $\widehat{R} := *\Theta_0 \to *\Theta_1 \vee ... \vee *\Theta_n$

Lemma ([1])

*An instance $R[\varphi]$ of a structural rule R is derivable from $\widehat{R[\varphi]}$ in **FL** and vice-versa.*

Lattive-valued interpretations of sequents

Let us define an interpretation of sequents into (bounded) residuated lattices.

Let $\mathbf{L} = \langle L, \vee, \wedge, /, \backslash, \cdot, 1 \rangle$ be a residuated lattice, then a valuation is a function which sends each propositional variable in an element of **L**. As usual any valuation can be extended to the whole set of formulas.

A formula φ is said to be true (or valid) under a valuation f, if $f(\varphi) \geq 1$. Given a set $X \subseteq \mathbf{L}$ we call X-valuation a valuation ranging inside X. A formula is X-valid if it is true for any X-valuation.

Semantic propagation

Let us use the following notation

$$\prod(X) := \{x_1 \cdots x_n \mid x_1, \ldots, x_n \in X\}$$

$$\coprod(X) := \{\bigvee Y \mid Y \subseteq X\}$$

Definition ([1])

A set $\mathcal{R}$ of structural rules satisfies the semantic propagation if in any residuated lattice $\mathbf{L}$ and for any $X \subseteq L$ if all formulae in $\widehat{\mathcal{R}}$ are X-valid then they are also $\coprod\prod(X)$-valid.

Pre-phase structures

Let $\langle A^*, \cdot, 1\rangle$ be the free monoid generated by A and, for any $X, Y \subseteq A^*$, let

$$X \bullet Y = \{x \cdot y \mid x \in X,\ \ y \in Y\},$$
$$X \setminus\!\!\setminus Y = \{y \in A \mid \forall x \in X,\quad x \cdot y \in Y\},$$
$$X \ /\!\!/ Y = \{y \in A \mid \forall x \in X,\quad y \cdot x \in Y\}.$$

A pre-phase structure is a triple $\mathbf{P} = \langle A, B, \perp\rangle$ such that $B \subseteq \mathscr{P}(A^*)$ and $\perp \subseteq A^*$.

A closed set is a subset of A^* of the form

$$\bigcap_{i \in I} (\{y_i\} \setminus\!\!\setminus Q_i \ /\!\!/ \{z_i\}) \text{ with } y_i, z_i \in A^*$$

Pre-phase structures

Let $C_{\mathbf{P}}$ be the operator assigning to any $X \subseteq A^*$ the smallest closed set containing X. Then $C_{\mathbf{P}}$ is a closure operator compatible with $\bullet$. Let

$$\bigvee \mathcal{X} = C(\bigcup \mathcal{X})$$
$$X \circ Y = C(X \bullet Y)$$

The set of closed sets $\mathcal{C}_{\mathbf{P}}$ forms a complete residuated lattice

$$\langle \mathcal{C}_{\mathbf{P}}, \cap, \bigvee, \circ, \backslash\!\!\backslash, /\!\!/, C_{\mathbf{P}}(\{1\}) \rangle.$$

The following lemma is a special case of a general result in [1].

Lemma

Let $\mathcal{C}'(A) = \{C_{\mathbf{P}}(\{y\}) \mid y \in A\}$, *then* $\mathcal{C}_{\mathbf{P}} = \coprod \prod (C'(A))$

Phase-valued interpretations of sequents I

Definition

Let $\mathbf{P} = \langle A, B, \bot \rangle$ be a pre-phase structure and $\Theta \vdash \Xi$ be a meta-sequent in which the meta-variables range among $X_1, \ldots, X_n$. Then given a valuation of $X_i \mapsto [\![X_i]\!] \in C_{\mathbf{P}}$, we extend it to $\Theta \vdash \Xi$ as follow:

$$[\![\Theta]\!] = \begin{cases} [\![X_{i_1}]\!] \circ \cdots \circ [\![X_{i_k}]\!] & \text{if } \Theta \equiv X_{i_1}, \ldots, X_{i_k} \\ \mathbf{1} = C_P(\{1\}) & \text{if } \Theta \text{ is empty.} \end{cases}$$

$$[\![\Xi]\!] = \begin{cases} [\![X_i]\!] & \text{if } \Xi \equiv X_i \\ \bot & \text{if } \Xi \text{ is empty.} \end{cases}$$

Definition

We say that $[\![\ ]\!]$ satisfies $\Theta \vdash \Xi$ if $[\![\Theta]\!] \subseteq [\![\Xi]\!]$. Given a structural rules (R) we say that it is valid in $\mathbf{P}$ if whenever $[\![\ ]\!]$ satisfies the premises of (R) it also satisfies the conclusion.

Phase-valued interpretations of sequents II

Definition

Let $\mathbf{P} = \langle \mathcal{A}, \mathcal{B}, \perp \rangle$ and $\Theta \vdash \Xi$ as above and let Ξ be either Y or empty. Then given two functions $|\ |$ and $[\ ]$, which associate $X_i \mapsto |X_i| \in \mathcal{A}$ and $Y \mapsto [Y] \in \mathcal{B}$, we interpret $\Theta \vdash \Xi$ as follow:

$$|\Theta| = \begin{cases} |X_{i_1}| \cdots |X_{i_k}| \in \mathcal{A}^* & \text{if } \Theta \equiv X_{i_1}, \ldots, X_{i_k} \\ 1 & \text{if } \Theta \text{ is empty.} \end{cases}$$

$$[\Xi] = \begin{cases} [X_i] & \text{if } \Xi \equiv X_i \\ \perp & \text{if } \Xi \text{ is empty.} \end{cases}$$

Definition

We say that $|\ |$ and $[\ ]$ pre-satisfy $\Theta \vdash \Xi$ if $|\Theta| \in [\Xi]$. Given a structural rules (R) we say that it is pre-valid in $\mathbf{P}$ if whenever $|\ |$ and $[\ ]$ satisfy the premises of (R) they also satisfies the conclusion.

Propagation

Definition

A structural rule is propagating if it is *valid* in all pre-phase structures in which is *pre-valid*.

Definition

Given a structural rule

$$\frac{\Theta_1 \vdash \Xi_1 \quad \cdots \quad \Theta_n \vdash \Xi_n}{\Theta_0 \vdash \Xi_0} \; (R)$$

we build its dependency graph $D(R)$ as follows:

- The vertices of $D(R)$ are the (variables for) formulas occurring in the premises $\Theta_1 \vdash \Xi_1, \cdots, \Theta_n \vdash \Xi_n$ (without distinguishing occurrences).
- There is a directed edge $\alpha \longrightarrow \beta$ in $D(R)$ if and only if there is a premise $\Theta_i \vdash \Xi_i$ such that α occurs in Θ_i and $\beta = \Xi_i$.

A structural rule (R) is said to be acyclic if $D(R)$ is acyclic.

Two Lemmas

The proof of the (restricted) claim is based on two lemmas, one purely syntactical and one purely semantical.

Syntactical Lemma

There exists a procedure to transform every acyclic structural rule (R) in **mFL** in an equivalent additive structural rule (called the additive form of (R)).

Semantical Lemma

Let (R) be an additive structural rule. If (R) has the semantic propagation property then it is propagating.

Theorem

*Let **mFL**$(\mathcal{R})$ be the calculus obtained by adding a set of acyclic rules $\mathcal{R}$ to **mFL**, then its additive form $\mathcal{R}'$ is such that the following are equivalent:*

(i) ***mFL**$(\mathcal{R}')$ enjoys cut-elimination;*

(ii) ***mFL**$(\mathcal{R}')$ enjoys reductive cut-elimination;*

(iii) *The quasi-equational translations of the rules in $\mathcal{R}'$ are McNeille canonical.*

Proof.

(ii) $\Rightarrow$ *(i)* is straightforward.
Recall that, by [1], if **mFL**$(\mathcal{R}')$ has cut-elimination then $(\mathcal{R}')$ has the semantic propagation property. So by *(i)* and the Semantical Lemma $(\mathcal{R}')$ is propagating. Finally, by the characterisation in [2] we get *(ii)*.
The equivalence between *(ii)* and *(iii)* can be found in [3]. □

As far as acyclic rules are concerned in **mFL**, we have

Semantic Propagation ═══ Cut-eliminaton

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Propagation ‖ Reductive cut-elimination

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McNeille canonicity ═══ Modular cut-elimination

References

[1] K. Terui.
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[2] A. Ciabattoni and K. Terui.
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