Forcing in Łukasiewicz logic

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Overview

History of forcing

- P. COHEN The independence of the continuum hypothesis. *Proc. Natl. Acad. Sci. U.S.A* 50: 1143-1148 (1963).
- A. ROBINSON, Forcing in model theory. In *Actes du Congrès International des Mathématiciens* (Nice, 1970), Tome 1: 245-250. Gauthier-Villars, Paris, 1971.
- A. ROBINSON, Infinite forcing in model theory. In *Proceedings of the Second Scandinavian Logic Symposium* (Oslo, 1970): 317-340. Studies in Logic and the Foundations of Mathematics, Vol. 63, North-Holland, Amsterdam, 1971.
- H. J. KEISLER, Forcing and the omitting types theories, *Studies in Model Theory*, MAA Studies in Math., Buffalo, N.Y., 8: 96-133, 1973.
- J.-C. LABLANIQUE, Propriétés de consistance et forcing, *Ann. Sci. Univ. Clermont, Ser. Math.*, Fasc. 18: 37-45, 1979.

Motivations

The aim of our work is to generalize the classical model-theoretical notion of forcing to the infinite-valued Łukasiewicz predicate logic.

Łukasiewicz predicate logic is not complete w.r.t. standard models and, its set of standard tautologies is in Π_2.
The Lindenbaum algebra of Łukasiewicz logic is not semi-simple.

In introducing our notions we will follow the lines of Robinson and Keisler.

Łukasiewicz propositional logic

The language of **Łukasiewicz propositional logic** Ł_∞ is defined from a countable set *Var* of propositional variables $p_1, p_2, \dots, p_n, \dots,$ and two binary connectives $\to$ and $\neg$.
Ł_∞ has the following axiomatization:

- $\varphi \to (\psi \to \varphi)$;
- $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$;
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

where φ, ψ and χ are formulas. Modus ponens is the only rule of inference. The notions of proof and theorem are defined as usual.

MV-algebras

A **MV-algebra** is structures $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ satisfying the following equations:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- $x \oplus y = y \oplus x$,
- $x \oplus 0 = x$,
- $x \oplus 0^* = 0^*$,
- $x^{**} = x$,
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Other operations are definable as follows:

$$x \to y = x^* \oplus y \qquad \text{and} \qquad x \odot y = (x^* \oplus y^*)^*.$$

MV-algebras form the equivalent algebraic semantics of the propositional Łukasiewicz logic, in the sense of Blok and Pigozzi.

Łukasiewicz predicate logic

The following are the axioms of **Łukasiewicz predicate logic** (PŁ$_\infty$):

1. the axioms of ∞-valued propositional Łukasiewicz calculus L_∞;
2. $\forall x\varphi \to \varphi(t)$, where the term t is substitutable for x in φ;
3. $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is not free in φ;
4. $(\varphi \to \exists x\psi) \to \exists x(\varphi \to \psi)$, where x is not free in φ.

PŁ$_\infty$ has two *rules of inference*:

- Modus ponens (m.p.): from φ and $\varphi \to \psi$, derive ψ;
- Generalization (G): from φ, derive $\forall x\varphi$.

The semantic of PŁ$_\infty$

Let L be an MV-algebra. An ***L*-structure** of the language PŁ$_\infty$ has the form $\mathfrak{A} = \langle A, (P^{\mathfrak{A}})_P, (c^{\mathfrak{A}})_C \rangle$ where

- A is a non-empty set (the universe of the structure);
- for any n-ary predicate P of PŁ$_\infty$, $P^{\mathfrak{A}} : A^n \to L$ is an n-ary L-relation on A;
- for any constant c of PŁ$_\infty$, $c^{\mathfrak{A}}$ is an element of A.

The notions of evaluations, tautology, etc. are defined as usual.

Forcing properties

Let PŁ$_\infty$(C) be the language of PŁ$_\infty$, to which we add an infinite set C of new constants. Let E be set of sentences of PŁ$_\infty(C)$ and At the set of atomic sentences of PŁ$_\infty(C)$.

Definition

A **forcing property** is a structure of the form $\mathbf{P} =< P, \leq, 0, f >$ such that the following properties hold:

(i) $(P, \leq, 0)$ is a poset with a first element 0;

(ii) Every well-orderd subset of P has an upper bound;

(iii) $f : P \times At \to [0, 1]$ is a function such that for all $p, q \in P$ and $\varphi \in At$ we have $p \leq q \Longrightarrow f(p, \varphi) \leq f(q, \varphi)$.

The elements of P are called **conditions**.

Finite forcing

Definition

Let $< P, \leq, 0, f >$ be a forcing property. For any $p \in P$ and any formula φ we define the real number $[\varphi]_p \in [0,1]$ by induction on the complexity of φ:

1. if $\varphi \in At$ then $[\varphi]_p = f(p, \varphi)$;
2. if $\varphi = \neg\psi$ then $[\varphi]_p = \bigwedge_{p \leq q} \left[\psi\right]_q^*$;
3. if $\varphi = \psi \to \chi$ then $[\varphi]_p = \bigwedge_{p \leq q}([\psi]_q \to [\chi]_p)$;
4. if $\varphi = \exists x \psi(x)$ then $[\varphi]_p = \bigvee_{c \in C}[\psi(c)]_p$.

The real number $[\varphi]_p$ is called the **forcing value** of φ at p.

Some properties of finite forcing

For any forcing property P, $p \in P$ and for any sentence φ, ψ or $\forall x \chi(x)$ of PŁ$_\infty(C)$ we have :

1. If $p \leq q$ then $[\varphi]_p \leq [\varphi]_q$
2. $[\neg\neg\varphi]_p = \bigwedge_{p \leq q} \bigvee_{q \leq v} [\varphi]_v$;
3. $[\varphi]_p \leq [\neg\neg\varphi]_p$.
4. $[\neg\varphi]_p = [\neg\neg\neg\varphi]_p$.
5. $[\forall x \chi(x)]_p = \bigwedge_{p \leq q} \bigwedge_{c \in C} \bigvee_{q \leq r} [\chi(c)]_r$.
6. $[\varphi \to \psi]_p = [\neg\varphi]_p \oplus [\psi]_p$;
7. $[\varphi \oplus \psi]_p = [\neg\neg\varphi]_p \oplus [\psi]_p$;

Generic sets

Definition

A non-empty subset G of P is called **generic** if the following conditions hold

If $p \in G$ and $q \leq p$ then $q \in G$,

For any $p, g \in G$ there exists $v \in G$ such that $p, g \leq v$;

For any $\varphi \in E$ there exists $p \in G$ such that $[\varphi]_p \oplus [\neg\varphi]_p = 1$.

Definition

Given a forcing property $\langle P, \leq, 0, f \rangle$, a model $\mathfrak{A}$ is **generated by** a generic set G if for all $\varphi \in E$ and $p \in G$ we have $[\varphi]_p \leq \|\varphi\|_{\mathfrak{A}}$. A model $\mathfrak{A}$ is **generic** for $p \in P$ if it is generated by a generic subset G which contains p. $\mathfrak{A}$ is generic if it is generic for 0.

Generic model theorem

Theorem

Let $< P, \leq, 0, f >$ be a forcing property and $p \in P$. Then there exists a generic model for p.

Sketch of the proof.

For any $p \in P$ build by stages a generic set G such that $p \in G$, proving that the condition $[\varphi]_q \oplus [\neg\varphi]_q < 1$ must fail for some $q \geq p_n$

Build a structure starting form the constants in the language and define an evaluation by $e(\varphi) = \bigvee_{p \in G} [\varphi]_p$. Such an enumerable model is generated by G. $\square$

Generic model theorem

Corollary

If p belongs to some generic set G which has a maximum g, then there exists $\mathfrak{M}$, generic model for p, such that $[\varphi]_g = \|\varphi\|_{\mathfrak{M}}$

Corollary

For any $\varphi \in E$ *and* $p \in P$ *we have*

$$[\neg\neg\varphi]_p = \bigwedge\{\|\varphi\|_{\mathfrak{M}} \mid \mathfrak{M} \textit{ is a generic structure for } p\}.$$

Infinite forcing

Henceforth all structures will be assumed to be members of a fixed inductive class Σ.

Definition

For any structure $\mathfrak{A}$ and for any sentence φ of $\mathrm{PŁ}_\infty(\mathfrak{A})$ we shall define by induction the real number $[\varphi]_\mathfrak{A} \in [0,1]$:

1. If φ is an atomic sentence then $[\varphi]_\mathfrak{A} = ||\varphi||_\mathfrak{A}$;
2. If $\varphi = \neg\psi$ then $[\varphi]_\mathfrak{A} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} [\psi]^*_\mathfrak{B}$;
3. If $\varphi = \psi \to \chi$ then $[\varphi]_\mathfrak{A} = \bigwedge_{\mathfrak{A}\subseteq\mathfrak{B}} ([\psi]_\mathfrak{B} \to [\chi]_\mathfrak{A})$;
4. If $\varphi = \exists x \psi(x)$ then $[\varphi]_\mathfrak{A} = \bigvee_{a\in\mathfrak{A}} [\psi(a)]_\mathfrak{A}$.

$[\varphi]_\mathfrak{A}$ will be called the **forcing value** of φ in $\mathfrak{A}$.

An example

A natural question is whether $[\varphi]_{\mathfrak{A}} = 1$ for any formal theorem φ of PŁ$_\infty$. The following example shows that the answer is negative: Let us consider a language of PŁ$_\infty$ with a unique unary predicate symbol R. We define two standard structures $\mathfrak{A}$ and $\mathfrak{B}$ by putting

$$\begin{array}{llll} \mathfrak{A} = \{a, b\}, & R^{\mathfrak{A}}(a) = 1/2, & R^{\mathfrak{A}}(b) = 1/3 & \\ \mathfrak{B} = \{a, b, c\}, & R^{\mathfrak{B}}(a) = 1/2, & R^{\mathfrak{B}}(b) = 1/3, & R^{\mathfrak{B}}(c) = 1. \end{array}$$

An example

Of course $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. Let us take $\Sigma = \{\mathfrak{A}, \mathfrak{B}\}$ and consider the following sentence of $\mathrm{P}Ł_\infty$

$$\exists x R(x) \to \exists x R(x).$$

This sentence is a formal theorem of $\mathrm{P}Ł_\infty$ (identity principle), but:

$$[\exists x R(x)]_{\mathfrak{A}} = [R(a)]_{\mathfrak{A}} \vee [R(b)]_{\mathfrak{A}} = \max(1/2, 1/3) = 1/2$$
$$[\exists x R(x)]_{\mathfrak{B}} = [R(a)]_{\mathfrak{B}} \vee [R(b)]_{\mathfrak{B}} \vee [R(c)]_{\mathfrak{B}} = \max(1/2, 1/3, 1) = 1.$$

and

$$[\exists x R(x) \to \exists x R(x)]_{\mathfrak{A}} = [\exists x R(x)]_{\mathfrak{B}} \to [\exists x R(x)]_{\mathfrak{A}} = 1 \to 1/2 = 1/2.$$

Properties of infinite forcing

For any structure $\mathfrak{A}$ and for any sentences φ, ψ and $\forall x \chi(x)$ of $\text{PŁ}_\infty(\mathfrak{A})$ the following hold:

1. If $\mathfrak{A} \subseteq \mathfrak{B}$ then $[\varphi]_\mathfrak{A} \leq [\varphi]_\mathfrak{B}$.
2. $[\neg\neg\varphi]_\mathfrak{A} = \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \bigvee_{\mathfrak{B} \subseteq \mathfrak{C}} [\varphi]_\mathfrak{C}$;
3. $[\varphi]_\mathfrak{A} \leq [\neg\neg\varphi]_\mathfrak{A}$.
4. $[\varphi \to \psi]_\mathfrak{A} = [\neg\varphi]_\mathfrak{A} \oplus [\psi]_\mathfrak{A}$;
5. $[\varphi \oplus \psi]_\mathfrak{A} = [\neg\neg\varphi]_\mathfrak{A} \oplus [\psi]_\mathfrak{A}$;
6. $[\forall x \chi(x)]_\mathfrak{A} = \bigwedge_{\mathfrak{A} \subseteq \mathfrak{B}} \bigwedge_{b \in \mathfrak{B}} \bigvee_{\mathfrak{B} \subseteq \mathfrak{C}} [\chi(b)]_\mathfrak{C}$.
7. $[\varphi]_\mathfrak{A} \odot [\neg\varphi]_\mathfrak{A} = 0$.

Generic structures

The following result characterizes the members $\mathfrak{A}$ of Σ for which $[\ ]_{\mathfrak{A}}$ and $\|\ \|_{\mathfrak{A}}$ coincide.

Proposition

For any $\mathfrak{A} \in \Sigma$ the following assertions are equivalent:

(1) $\|\varphi\|_{\mathfrak{A}} = [\varphi]_{\mathfrak{A}}$, *for all sentences φ of $PŁ_{\infty}(\mathfrak{A})$;*

(2) $\|\varphi\|_{\mathfrak{A}} = [\neg\neg\varphi]_{\mathfrak{A}}$, *for all sentences φ of $PŁ_{\infty}(\mathfrak{A})$;*

(3) $[\varphi]_{\mathfrak{A}} \oplus [\neg\varphi]_{\mathfrak{A}} = 1$, *for all sentences φ of $PŁ_{\infty}(\mathfrak{A})$;*

(4) $[\neg\varphi]_{\mathfrak{A}} = [\varphi]^{*}_{\mathfrak{A}}$, *for all sentences φ of $PŁ_{\infty}(\mathfrak{A})$.*

Generic structures

Definition

A structure $\mathfrak{A} \in \Sigma$ which satisfies the equivalent conditions of the proposition above will be called **Σ-generic**.

Theorem

Any structure $\mathfrak{A} \in \Sigma$ is a substructure of a Σ-generic structure.

Theorem

Any Σ-generic structure $\mathfrak{A}$ is Σ-existentially-complete.

Characterization of generic structures

Let use denote by $\mathfrak{G}_\Sigma$ the class of Σ-generic structures.

Proposition

$\mathfrak{G}_\Sigma$ *is an inductive class.*

Theorem

$\mathfrak{G}_\Sigma$ *is the unique subclass of* Σ *satisfying the following properties:*

(1) *it is model-consistent with* Σ*;*

(2) *it is model-complete;*

(3) *it is maximal with respect to* (1) *and* (2)*.*