# Advances in the theory of fixed points in many-valued logics

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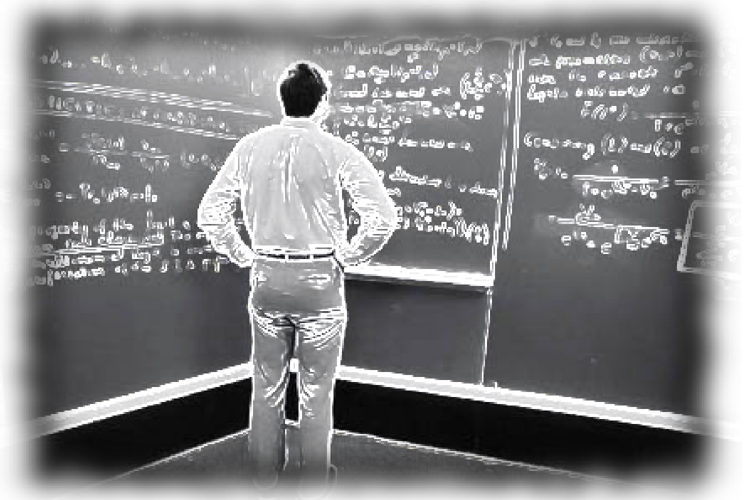
# This is a work in progress!!



# Overview

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# Motivations

Why introducing fixed points in many-valued logic?

- From a logical perspective, the high number of important results found in the cases of first order logic with fixed points as well as in modal logic with fixed points ($\mu$-calculus).
- From the point of view of applications, fixed points stand at the heart of computer science. Having a formal system to threat them in a "fuzzy" way may rise new topics in approximate reasoning.
- From the algebraic side, fixed points enrich the structures of the algebraic semantics of the logic under study, leading to new and interesting structures.

# Methods

How to give a "meaning" to the newly introduced fixed points ?

- The non-trivial part, when introducing fixed points in a logical systems, comes when looking for a semantics of such fixed points.
- The classical approach is based on Tarski fixed point Theorem (see First order logic with fixed points or $\mu$-calculus).
- In many-valued logic it is possible to use a new approach, based on the continuity of the semantic objects.

# Methods

### Theorem (Brouwer, 1909)

*Every continuous function from the closed unit cube* $[0, 1]^n$ *to itself has a fixed point.*

With this approach the families of formulas which have fixed points are different, compared to the classical cases where one has to restrict to formulas on which the variable under the scope of $\mu$ only appears positively.

# Łukasiewicz logic

Łukasiewicz logic can be presented as an axiomatic system in Hilbert's style or as a hyper-sequents calculus. For this talk it is sufficient to think of Łukasiewicz logic as follows.

The Łukasiewicz connectives are $\oplus, \neg, \bot$ and their interpretation in the real interval $[0, 1]$ is given by:

$$\neg x := 1 - x \qquad \text{and} \qquad x \oplus y := \min\{1, x + y\}$$

Also the derived connectives $x \to y := \neg x \oplus y$ and $x \odot y := \neg(\neg x \oplus \neg y)$ naturally acquire their semantics in $[0, 1]$.

Łukasiewicz logic is the logic which is complete w.r.t. the above connectives.

# MV algebras

A presented above it is clear that Łukasiewicz logic is an algebrisable logic, the equivalent algebraic semantics is given by the class of MV algebras.

## Definition

An MV algebra $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ is a commutative monoid $\mathcal{A} = \langle A, \oplus, 0 \rangle$ with an involution $(\neg\neg x = x)$ such that for all $x, y \in A$,

$$x \oplus \neg 0 = \neg 0$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

# The standard MV algebra

So $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$ where the operation are defined as

$$x \oplus y := \min\{1, x+y\} \qquad \text{and} \qquad \neg x := 1 - x,$$

is an MV algebra, called the standard MV algebra.

## Theorem (Chang 1958)

*The algebra* $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$ *generates the variety of MV algebras.*

# Notable properties of the class of MV algebras

Despite being such a simple equational class, MV algebras are connected to important fields of Mathematics.

## Definition

An ordered group is called lattice ordered ($\ell$-group, for short) if its order is a lattice. An $\ell$-group $G$ is called unital ($\ell u$-group, for short) if there exist an element $u \in G$ (called the strong unit) such that for any $0 \leq x \in G$ there exists a natural number $n$ such that $\underbrace{u \oplus \ldots \oplus u}_{n \text{ times}} \geq x$

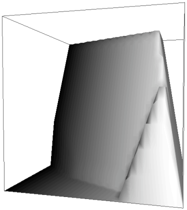
## Theorem (Mundici 1986)

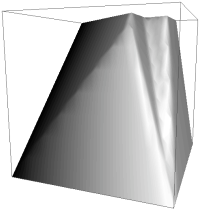
*There exists a categorical equivalence between MV algebras and abelian $\ell u$-groups.*

# Notable properties of the class of MV algerbas (cont'd)

## Theorem (McNaughton 1951)

*The free MV algebra over m generators is isomorphic to the algebra of continuous piece-wise linear functions with integer coefficients from* $[0,1]^m$ *to* $[0,1]$*, where the MV operations are defined pointwise.*



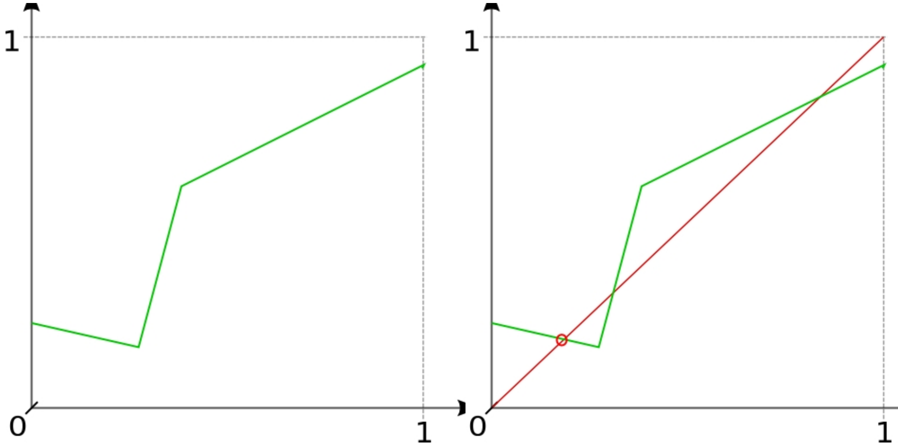


# Adding fixed points

Having in mind the functional representation, we want to endow Łukasiewicz logic with an operator $\mu_x$ on terms such that if $t(x, \bar{y})$ is a term then $\mu_x t(x, \bar{y})$ is also a term, giving for every tuple $\bar{a}$ the fixed point of the function $t(x, \bar{a})$.

Unfortunately the function giving the fixed point of a formula does not need to be continuous in the remaining variables, whence, in principle we can not allow nested occurences of $\mu$.

# Discontinuity of $\mu$-functions



# $\mu$MV-algebra

## Definition

A $\mu$MV-algebra

$$\mathcal{A} = \langle A, \oplus, \neg, \{\mu x_{t(x,\bar{y})}\}_{t \in Term}, 0 \rangle$$

is a structure such that $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ is a MV algebra, endowed with a function $\mu x_{t(x,\bar{y})}$ for any term $t(x, \bar{y})$ in the language of MV algebras, such that it satisfies the following conditions.

1. $\mu x_{t(x,\bar{y})}(\bar{y}) = t(\mu x_{t(x,\bar{y})}, \bar{y})$
2. If $t(s, \bar{y}) = s(\bar{y})$ then $\mu x_{t(x,\bar{y})} \leq s(\bar{y})$

# Some results about $\mu$MV-algebra

A number of results on the variety of $\mu$MV is known.

### Theorem

*The $\mu$MV algebra $\langle [0,1], \oplus, \neg, \{\mu x_{t(x,\bar{y})}\}_{t \in Term}, 0 \rangle$ generates the variety of $\mu$MV algebras.*

### Theorem

*The free $\mu$MV algebra over a finite number of generator n is given by all the piecewise linear functions with rational coefficients form $[0,1]^n$ to $[0,1]$.*

# The system $\mu$Ł

## Definition

Let $\mu$Ł be an expansion of Łukasiewicz with an operator $\mu_x$ for any variable $x$. The axioms of $\mu$Ł are the following:

(i) All axioms of Łukasiewicz propositional logic,

(ii) $\mu_x\varphi(x, \bar{y}) \leftrightarrow \varphi(\mu_x\varphi(x, \bar{y}), \bar{y})$,

(iii) If $\varphi(z, \bar{y}) \leftrightarrow z$ then $\mu_x\varphi(x, \bar{y}) \rightarrow z$.

# Interpreting $\mu$Ł

Let us see how to give a safe interpretation to the system $\mu$Ł in an expansion of the standard MV-algebra.

Let Sub($\varphi$) be the set of sub-formulae of $\varphi$.
The set Sub($\varphi$) can be linearly ordered by stipulating that sub-formulae of smaller complexity occur earlier and, in case of sub-formulae of the same complexity, the ones which are leftmost in the formula have priority.
We will refer to the $i^{th}$ sub-formula of $\varphi$, indicated by $\varphi^{(i)}$, in this sense.

# Interpreting $\mu$Ł

Given any formula $\varphi$ of $\mu$Ł, we define its interpretation $\varphi^{\Phi}$ in the set $\{f : [0,1]^n \longrightarrow [0,1] \mid n \in \mathbb{N}\}$ in two steps.
First, if $|\mathrm{Sub}(\varphi)| = n$, we consider the function $f_\varphi : [0,1]^n \longrightarrow [0,1]^n$, which sends the vector $(w_{\varphi(1)}, \ldots, w_{\varphi(n)})$ into $(w'_{\varphi(1)}, \ldots, w'_{\varphi(n)})$.
The elements $w'$'s are defined inductively on the complexity of the sub-formulae:

$$w'_{\psi \oplus \xi} := w_\psi \oplus w_\xi$$
$$w'_{\neg\psi} := \neg w_\psi$$
$$w'_{\mu_x \psi} := w_\psi$$
$$w'_x := \begin{cases} w_{\mu_x \psi} & \text{if } x \text{ is bounded by } \mu \text{ in } \psi \\ w_x & \text{otherwise} \end{cases}$$

# Interpreting $\mu$Ł

The function $f_\varphi$ is obviously continuous, hence, by Brouwer Theorem it has fixed points.

We can consider the least fixed points in the order which takes into account only the last component.

As second step, we notice that, if $y$ is not bound by $\mu$ then $f_\varphi$ is the identity on the component $w_y$. Thus, any choice of $w_y$ in $[0, 1]$ gives rise to a different fixed point.

Let $\omega f_\varphi[a_1/w_i]$ indicate the minimum among the fixed points of $f_\varphi$ whose $i^{th}$ component is $a_1$.

# Interpreting $\mu$Ł

Let $y_1, \ldots, y_m$ be the free variables in $\varphi$, and let them be equal to $\varphi^{(i_1)}, \ldots, \varphi^{(i_m)}$, respectively.
We define $\varphi^{\phi}$ as the function which sends the tuple $(a_1, \ldots, a_m) \in [0,1]^m$ into the last component of $\omega f_{\varphi}[a_1/w_{i_1}, \ldots, a_m/w_{i_m}]$.

## Proposition

Let $\varphi$ be a formula of $\mu$Ł without occurrences of $\mu$ then $\varphi^{\phi}$ is the McNaughton function associated to $\varphi$.

# An Example

## Example

Let $\varphi := \mu_x y \oplus \neg x$ then there are 5 subformulas of $\varphi$.
So we need to consider the function $f_\varphi : [0,1]^5 \longrightarrow [0,1]^5$ which sends:

$$(w_y, \quad w_x, \quad w_{\neg x}, \quad w_{y \oplus \neg x}, \quad w_{\mu_x y \oplus \neg x})$$

$$\downarrow$$

$$( \quad w_y, \quad w_{\mu_x y \oplus \neg x}, \quad \neg w_x, \quad w_y \underline{\oplus} w_{\neg x}, \quad w_{y \oplus \neg x}).$$

# Example cont.'d

So a fixed point of $f_\varphi$ is a vector which satisfies the following linear system:

$$\begin{cases} w_y = w_y \\ w_x = w_{\mu_x y \oplus \neg x} \\ w_{\neg x} = \underline{\neg} w_x \\ w_{y \oplus \neg x} = w_y \underline{\oplus} w_{\neg x} \\ w_{\mu_x y \oplus \neg x} = w_{y \oplus \neg x} \end{cases}$$

which can be reduced to

$$\begin{cases} w_y = w_y \\ w_x = w_y \underline{\oplus} \underline{\neg} w_x \end{cases} .$$

The system has one degree of freedom and, after some easy calculation, we have $\varphi^\phi(a) = \frac{a+1}{2}$

# Interpreting $\mu$Ł

Let us indicate by $f(x \rightsquigarrow g)$ the function obtained by substituting all occurrences of $x$ in $f$ with the function $g$.

## Lemma

*Let $x$ be a variable which appears only free in $\varphi$ then*

$$\varphi^{\Phi}[x \rightsquigarrow (\mu_x\varphi)^{\Phi}] = (\mu_x\varphi)^{\Phi}$$

# Algebraic semantics

In order to prove the standard completeness of the calculus we need to introduce a new class of algebras.

## Definition

Let *Term* be the set of MV terms, we define inductively the set $\mu Term$.

- If $t(x) \in Term$ then $\mu x_{t(x)} \in \mu Term$,
- If $t(x) \in \mu Term$ and $x$ in not bound by $\mu x$ then $\mu x_{t(x)} \in \mu Term$
- $\mu Term$ is the smallest set with the above properties.

# $\mu^+$ MV algebras

## Definition

A $\mu^+$ MV algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, 1, \mu Term \rangle$ such that $\mathcal{A} = \langle A, \oplus, \neg, 1 \rangle$ is an MV-algebra and

1. $\mu x_{t(x,\bar{y})}(\bar{y}) = t(\mu x_{t(x,\bar{y})}, \bar{y})$
2. If $t(s, \bar{y}) = s(\bar{y})$ then $\mu x_{t(x,\bar{y})} \leq s(\bar{y})$

## Theorem

*The system $\mu$Ł is standard complete.*

# The first order extension

The approach presented can be extended in principle to the first order Łukasiewicz logic Ł$\forall$.

In Ł$\forall$ the models are like in classical first order logic, but the predicates are [0,1]-valued relations, namely functions from the structure into [0,1].

Let $\mathfrak{A}$ be a model for Ł$\forall$. The truth values of the quantifiers is given by:

$$\|\exists x \varphi(x)\|_{\mathfrak{A}} := \bigvee_{a \in \mathfrak{A}} \|\varphi(a)\| \qquad \|\forall x \varphi(x)\|_{\mathfrak{A}} := \bigwedge_{a \in \mathfrak{A}} \|\varphi(a)\|$$

# Infinite dimensional space

In this case we have to use a different concept of subformula which give rise to infinite sets when quantifiers are considered.

## Definition

- $\mathrm{Sub}(\varphi) = \{\varphi\}$ if $\varphi$ is atomic,
- $\mathrm{Sub}(\varphi) = \{\varphi\} \cup \mathrm{Sub}(\psi) \cup \mathrm{Sub}(\xi)$ if $\varphi = \psi \oplus \xi$,
- $\mathrm{Sub}(\varphi) = \{\varphi\} \cup \mathrm{Sub}(\psi)$ if $\varphi = \neg\psi$,
- $\mathrm{Sub}(\varphi) = \{\varphi\} \cup \bigcup_{a \in \mathfrak{A}} \mathrm{Sub}(\psi(a/x))$ if $\varphi = \exists x \psi(x)$,
- $\mathrm{Sub}(\varphi) = \{\varphi\} \cup \bigcup_{a \in \mathfrak{A}} \mathrm{Sub}(\psi(a/x))$ if $\varphi = \forall x \psi(x)$.

# The system $\mu Ł\forall$

In this case the range of $\mu$ are second order variables $X_1, \ldots, X_n, \ldots$

So we consider formulas of the kind

$$\mu_{X_1}\varphi(y_1, \ldots, y_n, X_1).$$

## Definition

The axioms of $\mu Ł\forall$ are all the axioms of $Ł\forall$ plus

- $\mu_{X_1}\varphi(X_1) \leftrightarrow \varphi(\mu_{X_1}\varphi(X_1))$,
- If $\varphi(X/\psi) \leftrightarrow \psi$ then $\mu_{X_1}\varphi(X_1) \rightarrow \psi$.

# Interpretation

To interpret these new fixed points we can try to carry out the same construction of the propositional case.
Let $\varphi$ be a formula of $\mu Ł\forall$ and $\mathfrak{A}$ be a model, let us define the function

$$f_\varphi : \qquad (w_i)_{i \in \mathsf{Sub}(\varphi)} \in [0,1]^\omega \longmapsto (w'_i)_{i \in \mathsf{Sub}(\varphi)} \in [0,1]^\omega,$$

as follows

$$w'_{\psi \oplus \xi} := w_\psi \oplus w_\xi \qquad w'_{P(a_1,\dots,a_n)} = \|P(a_1,\dots,a_n)\|_{\mathfrak{A}}$$

$$w'_{\neg\psi} := \neg w_\psi \qquad w'_{\mu_{X_1}\psi} := w_\psi$$

$$w'_{\forall x \psi(x)} := \bigwedge_{a \in \mathfrak{A}} w_{\psi(a/x)} \qquad w'_{X_1} := w_\psi$$

# Fixed point theorems in infinite-dimesional spaces

In order to make the next step we need a generalisation of Brouwer Theorem.

**Theorem (Schauder 1930)**

*Let $A$ be Banach space and let $K$ be a compact convex subset then any continuous mapping $F$ of $K$ into itself has a fixed point.*

**Theorem (Cauty 2001)**

*Let $A$ be topological vector space and let $K$ be a compact convex subset then any continuous mapping $F$ of $K$ into itself has a fixed point.*

# Continuity

Since we need $[0,1]^\omega$ to be compact it is natural to endow it with the product topology.

Unfortunately this makes our functions not continuous.

# Continuity

$$
\begin{array}{ccccccccc}
(\frac{1}{2}, \frac{2}{3}) & \times & [0, \frac{2}{3}) & \times & [0, \frac{2}{3}) & \times & [0, \frac{2}{3}) & \times & \dots \\
(w_1, & & w_2, & & w_3, & & w_4, & & \dots)
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccccc}
(\bigvee_{n \in \mathcal{N}} \{w_n\}, & & w_2, & & w_3, & & w_4, & & \dots) \\
(\frac{1}{2}, \frac{2}{3}) & \times & [0, 1] & \times & [0, 1] & \times & [0, 1] & & \dots
\end{array}
$$

# Continuity

$$
\begin{array}{ccccccccc}
[0, \frac{2}{3}) & \times & \color{blue}{(\frac{1}{2}, \frac{2}{3})} & \times, & [0, \frac{2}{3}) & \times & [0, \frac{2}{3}) & \times & \dots \\
(w_1, & & w_2, & & w_3, & & w_4, & & \dots)
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccccc}
(\bigvee_{n \in \mathcal{N}} \{w_n\}, & & w_2, & & w_3, & & w_4, & & \dots) \\
(\frac{1}{2}, \frac{2}{3}) & \times & [0, 1] & \times & [0, 1] & \times & [0, 1] & & \dots
\end{array}
$$

# Continuity

$$
\begin{array}{ccccccccc}
[0,\frac{2}{3}) & \times & [0,\frac{2}{3}) & \times & (\frac{1}{2},\frac{2}{3}) & \times, & [0,\frac{2}{3}) & \times & \dots \\
(w_1, & & w_2, & & w_3, & & w_4, & & \dots)
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccccc}
(\bigvee_{n\in\mathcal{N}}\{w_n\}, & & w_2, & & w_3, & & w_4, & & \dots) \\
(\frac{1}{2},\frac{2}{3}) & \times & [0,1] & \times & [0,1] & \times & [0,1] & & \dots
\end{array}
$$

# Continuity

$$
\begin{array}{ccccccccc}
[0,\frac{2}{3}) & \times & [0,\frac{2}{3}) & \times & [0,\frac{2}{3}) & \times & (\frac{1}{2},\frac{2}{3}) & \times, & \dots \\
(w_1, & & w_2, & & w_3, & & w_4, & & \dots)
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccccc}
(\bigvee_{n\in\mathcal{N}}\{w_n\}, & & w_2, & & w_3, & & w_4, & & \dots) \\
(\frac{1}{2},\frac{2}{3}) & \times & [0,1] & \times & [0,1] & \times & [0,1] & & \dots
\end{array}
$$

# Maximum fixed points

Why only minimum fixed points are considered?

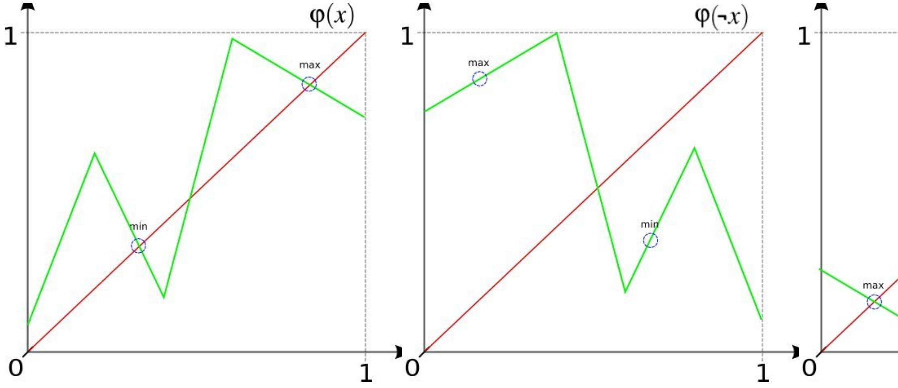
## Proposition

For every formula $\varphi(x, \bar{y})$, the term defined by $\nu_x \varphi(x, \bar{y}) := \neg\mu_x(\neg\varphi(\neg x))$ has the following properties:

- $\varphi(\nu_x \varphi(x)) = \nu_x \varphi(x)$
- If $\varphi(\psi) = \psi$ then $\psi \to \nu_x \varphi(x)$

Hence it interprets the maximum fixed point of $\varphi(x)$.

## Proof.

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# Expressive power

Do the nested fixed points actually increase the expressive power of the logic?

I don't know. This seems a non trivial question, as even the problem of collapsing of the $\mu$-hierarchy for $\mu$MV algebras is still unexplored.

In the view of the characterisation of the free $\mu$MV algebras, one strategy to prove it in the positive, could be to try to define irrational numbers with the operation $\oplus, \neg$ and the nested fixed points.

# Alternative semantics

Are there other more natural structures which give a semantics to $\mu$Ł?

I don't know.

A Kripke semantics would allow another natural interpretation of fixed points, in the style of the modal $\mu$-calculus. Unfortunately the Kripke semantics for Łukasiewicz logic is quite trivial as admits only frames in which the relation is a linear order.

On the other hand also the expansion of MV algebras with fixed points deserve more consideration as this class is linked to divisible groups through Mundici's functor.

Thank you for your attention.