Dualities

Joint thinking with O. Caramello (Institut des Hautes Études Scientifiques) *and* V. Marra (University of Milano)

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Online material

Here you can find the slides of this tutorial.
http://logica.dmi.unisa.it/lucaspada/wp-content/uploads/BuenosAires.pdf

Prerequisites

1. Some universal algebra,
2. Some category theory,
3. Some general topology.

Contents of the tutorial

1. Introduction
   1.1 Semantics.
   1.2 Stone duality.
   1.3 Priestley duality.
   1.4 Gelfand duality.
   1.5 Pontryagin duality.
   1.6 Piggy-back dualities.

Caveats

1. The general framework gives a uniform understanding of several dualities, however it does not dramatically simplify the mathematics behind.
2. This is still a work in progress, so proofs may be sub-optimal.
3. There is much space for further research.

An outlook on some prominent dualities

Primal	**Dual**	**Main Tool**	**Topology**
Boolean algebras	Stone spaces	Ultrafilters	Stone/ Zariski topology
Distributive lattices	Priestley Spaces	Prime filters	Patch topology
Commutative (unital) C*-algebras	Compact Hausdorff spaces	*-homomorphisms into $\mathbb{R}$ / Gelfand transform	weak-* topology
Abelian groups	Compact Abelian groups	Characters into $U(1)$ / Pontryagin transform	Compact-open topology
Fin'ly presented (unital) Riesz spaces	(compact) polyhedra	Maximal ideals	Hull-kernel topology
Fin'ly presented MV-alegrbas	Rational polyhedra	Maximal ideals	Hull-kernel topology
Modal algebras	Descriptive Kripke frames	Boolean ultrafilters	Stone + relations

Contents of the tutorial

1. Preliminaries
   1.1 Varieties, congruences, subdirect representations, and free algebras.
   1.2 Galois connections.
   1.3 Categories and functors.
   1.4 Adjoints and equivalencies.

Varieties, homomorphism, and congruences

1. By a **finitary** language I mean a language with possibly infinitely many operations, all with finite arity.
2. **Infinitary** languages may admit operation symbols with infinite arity.

From now on, fix a (possibly infinitary) functional signature $\mathcal{L}$.

Varieties, homomorphism, and congruences

Definition

An **(algebraic) variety** is the class of all structures in the signature $\mathcal{L}$ satisfying a given set of equations.

Definition

A **homomorphism** f between two structures A and B in the language $\mathcal{L}$ is a map that preserves alla operations in $\mathcal{L}$ i.e., for any n-ary operation $\star$ in $\mathcal{L}$ and any tuple $a_1, ..., a_n \in A$:

$$f(\star^A(a_1, ..., a_n)) = \star^B(f(a_1), ..., f(a_n)).$$

Varieties, homomorphism, and congruences

Definition

A **congruence** θ on an algebra A is an equivalence relation on A^2 which is compatible with the operations of A i.e., for any n-ary operation $\star$ in $\mathcal{L}$

$$\text{if } a_1 \theta a'_1, ..., a_n \theta a'_n \text{ then } \star^A(a_1, ..., a_n) \theta \star^A(a'_1, ..., a'_n)$$

Varieties, homomorphism, and congruences

Lemma (The homomorphism theorem)

In an algebraic variety $\mathcal{V}$, for every homomorphism $f\colon A \to B$ there exists a congruence θ on A such that $f = i \circ \pi_\theta$ where π_θ is surjective and i is injective.

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle \pi_\theta} \searrow \quad \nearrow {\scriptstyle i} & \\ & A/\theta & \end{array}$$

Free algebras

Definition

A **free algebra over μ generators** for a class of structures K (notation $\mathscr{F}_K(\mu)$) is an algebra in K with the following property:

> *for any algebra $A \in K$ and any assignment of the μ free generators into A, there exists a unique homomorphism form $\mathscr{F}_K(\mu)$ into A that extends the above assignment.*

Free algebras

1. Free algebra on a fixed number of generators are easily seen to be **unique (up to isomorphism)**.
2. The free algebra over μ generators in a variety $\mathcal{V}$ can be obtained in a syntactic way. One takes **the set of terms with variables in** $\{X_\alpha \mid \alpha < \mu\}$ in the language $\mathcal{L}$ with its obvious $\mathcal{L}$ structures, and **quotient** it over the congruence generated by the equalities holding in the variety:

$$\theta_{\mathcal{V}} := < \{ \big(s(X), t(X) \big) \mid \mathcal{V} \models s(X) = t(X) \} > .$$

Subdirect products

Definition

An algebra is said to be a **subdirect product** of the algebras A_i with $i \in I$, if

1. A embeds into the product $\prod_{i \in I} A_i$.
2. The projections $\pi_i : A \to A_i$ are surjective.

An algebra which cannot be decomposed into a subdirect product of simpler algebra is called **subdirectly irreducible**.

Subdirect products

Theorem (Birkhoff subdirect representation)

*Any algebra in a **finitary** variety is isomorphic to the subdirect product of subdirectly irreducible algebras.*

Notice that an equation holds in A if, and only if, it holds in all components of its subdirect representation. So, the study of equations holding in a variety can be safely reduced to the ones holding on the subdirectly irreducible.

Galois connections

Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets.

An **isotone Galois connection** between these posets consists of two isotone functions: $F \colon A \to B$ and $G \colon B \to A$, such that for all $a \in A$ and $b \in B$, we have

$$F(a) \leq b \text{ if and only if } a \leq G(b).$$

An **antitone Galois connection** between these posets consists of two antitone functions: $F \colon A \to B$ and $G \colon B \to A$, such that for all $a \in A$ and $b \in B$, we have

$$b \leq F(a) \text{ if and only if } a \leq G(b).$$

Galois connections, examples

Example

Let U be a set and pick a fixed subset L of U. Then the maps F and G, where $F(X) = L \cap X$, and $G(X) = (U \setminus L) \cup X$, form a **monotone** Galois connection from $\wp(U)$ into $\wp(U)$.

A similar Galois connection exists in any Heyting algebra. $F(x) = (a \wedge x)$ and $G(x) = (a \to x)$. In logical terms: "implication from a" is the upper adjoint of "conjunction with a".

Galois connections, examples

Example

Let $f\colon X \to Y$ be a function, $M \subseteq X$ and $N \subseteq Y$. Consider the image function $F(M) = f[M] = \{f(m) \mid m \in M\}$ and the preimage function $G(N) = f^{-1}[N] = \{x \in X \mid f(x) \in N\}$. Then F and G form a monotone Galois connection between the power sets of X and Y, both ordered by inclusion.

Galois connections, examples

Example

Let A and B be sets and $R \subseteq A \times B$. Consider the maps $F\colon A \to B$ and $G\colon B \to A$, given by

$$F(X) = \{b \in B \mid xRb \forall x \in X\} \text{ and } G(Y) = \{a \in A \mid aRy \forall y \in Y\}.$$

The maps F and G form an **antitone** Galois connection.

Actually all antitone Galois connections arise in this manner.

Galois connections, properties

If $F : A \rightleftarrows B : G$ is an antitone Galois connection, the pair F, G enjoys a number of immediate properties:

1. $b \leq F \circ G(b)$ **expansive**.
2. $F \circ G \circ F(a) = F(a)$ **fixed points**.
3. $F(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} F(a_i)$ **inverts arbitrary joins**.

Categories

A **category** is a collection of *objects* $A, B, C, \ldots$ together with a collection of *arrows* $f, g, h, \ldots$. such that

1. each arrow f is associated to a pair of objects called **domain** and **codomain** of f.
2. If f and g are arrows such that $dom(f) = A$, $cod(f) = B$, $dom(g) = B$, and $cod(g) = C$, then there exists an arrow $g \circ f$, called **composition** such that $dom(g \circ f) = A$ and $cod(g \circ f) = C$.
3. Composition is **associative**.
4. For any object A there exists an arrow id_A which is a **neutral element** w.r.t. compositions.

Functors

Let C and D be categories. A **functor** F from C to D is a mapping that

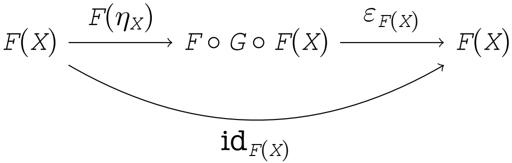
1. associates to each object $X \in C$ an object $F(X) \in D$,
2. associates to each morphism $f\colon X \to Y \in C$ a morphism $F(f)\colon F(X) \to F(Y) \in D$ **preserving the structure** of C i.e., such that
   2.1 $F(\mathbf{id}_X) = \mathbf{id}_{F(X)}$, for every object $X \in C$,
   2.2 $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$.

Equivalence of categories

Let C and D be categories. A pair of functors $F: C \to D$ $G: D \to C$ is called **categorical equivalence** if there exists a family of **isomorphisms** $\varepsilon_Y: FG(Y) \to Y$ and $\eta_X: X \to GF(X)$ natural in X and Y.

Adjoints

Let C and D be categories. A pair of functors $F\colon C \to D$ $G\colon D \to C$ is called **adjunction** if there exists a family of **maps** $\varepsilon_Y\colon FG(Y) \to Y$ and $\eta_X\colon X \to GF(X)$ natural in X and Y and such that



If F and G are adjoint, then there exists a natural bijection

$$\mathbf{hom}_{\mathcal{C}}(FY, X) \cong \mathbf{hom}_{\mathcal{D}}(Y, GX)$$

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   1.1 The Galois connection between ideals and affine varieties.
   1.2 An algebraic Galois connection.
   1.3 Definable functions.
   1.4 A general adjunction between varieties and topological spaces.

The Galois connection between ideals and affine varieties

Let k be an algebraically closed field. There is an antitone Galois connection between subsets of k^n and subsets of the ring of polynomials $k[x_1, \ldots, k_n]$. The connection carries

1. a subset $E \subseteq k[x_1, \ldots, k_n]$ to the set $\mathbb{V}(E) \subseteq k^n$ of common zeros of the polynomials in E;
2. a subset $S \subseteq k^n$ to the set $\mathbb{C}(S)$ of polynomials in $k[x_1, \ldots, x_n]$ vanishing over S.

- The affine sets of k^n that are fixed by this connection are called **affine varieties**.
- The ideals of $k[x_1, \ldots, x_n]$ that are fixed are precisely the **radical ideals**.

The latter statement is the content of Hilbert's *Nullstellensatz*.

An algebraic Galois connection

Let $\mathcal{V}$ be any variety of algebras and $\mathscr{F}(\mu)$ the free algebra over μ generators in $\mathcal{V}$. Let A be any algebra in $\mathcal{V}$.
For any $\theta \subseteq \mathscr{F}(\mu)^2$, put

$$\mathbb{V}(\theta) := \{s \in A^\mu \mid p(s) = q(s) \quad \forall (p,q) \in \mathscr{F}(\mu)\}.$$

Vice versa, given a subset S of A^μ one defines a congruence of $\mathscr{F}(\mu)$ as follows:

$$\mathbb{C}(S) := \{(p,q) \in \mathscr{F}(\mu)^2 \mid p(s) = q(s) \quad \forall s \in S\}.$$

An algebraic Galois connection

Lemma

The operators $\mathbb{C}$ and $\mathbb{V}$ form an antitone Galois connection between the powersets of k^{μ} and $\mathscr{F}^2(\mu)$.

Assume $R \subseteq \mathscr{I}(S)$ and let $p \in S$. If $(s,t) \in R$ then $s(p) = t(p)$ by the definition of $\mathscr{I}$, so that $p \in \mathscr{V}(R)$.

Conversely, assume $S \subseteq \mathscr{V}(R)$ and suppose $(s,t) \in R$. If $p \in S$ then $s(p) = t(p)$ by the definition of $\mathscr{V}$, so that $(s,t) \subseteq \mathscr{I}(S)$.

Definable functions

Definition

Given $S \subseteq A^{\mu}$ and $T \subseteq A^{\nu}$, a function $\lambda\colon S \to T$ is **definable** if there exists a ν-tuple of terms $(t_{\beta})_{\beta<\nu}$, with $t_{\beta} \in \mathscr{F}_{\mu}$, such that

$$\lambda(\,(p_{\alpha})_{\alpha<\mu}\,) = (\,t_{\beta}(\,(p_{\alpha})_{\alpha<\mu}\,)\,)_{\beta<\nu}$$

for every $(p_{\alpha})_{\alpha<\mu} \in S$. We call any such ν-tuple a family of **defining terms** for λ. In the special case that $\nu = 1$, the ν-tuple may be regarded as a single term $t \in \mathscr{F}_{\mu}$, called a *defining term* for λ.

The functor C: objects

For any $S \subseteq A^{\mu}$, it is easy to check that $\mathbb{C}(S)$ is a congruence on $\mathscr{F}_{\mu}$. In view of this, for any subset $S \subseteq A^{\mu}$ we define

$$\mathscr{C}(S) = \mathscr{F}_{\mu} / \mathbb{C}(S) \ .$$

The functor C: arrows

Given $S \subseteq A^\mu$ and $T \subseteq A^\nu$, let $\lambda\colon S \to T$ be a definable map, and let $(l_\beta)_{\beta<\nu}$ be a ν-tuple of defining terms for λ. Then there is an induced function

$$\mathscr{C}(\lambda)\colon \mathscr{C}(T) \to \mathscr{C}(S)$$

which acts on each $s \in \mathscr{F}_\nu$ by substitution as follows:

$$\frac{s\big((X_\alpha)_{\beta<\nu}\big)}{\mathbb{C}(T)} \in \mathscr{C}(T) \quad \stackrel{\mathscr{C}(\lambda)}{\longmapsto} \quad \frac{s\big([X_\beta \backslash l_\beta]_{\beta<\nu}\big)}{\mathbb{C}(S)} \in \mathscr{C}(S) \ .$$

The functor $\mathscr{C}$: arrows

There can be several distinct defining terms for a definable function $\lambda\colon S \to [0,1]$. However, $l \in \mathscr{F}_\mu$ is a defining term for λ and $(l, l') \in \mathbb{C}(S)$ if, and only if, l' is a defining term for λ.

Indeed, $(l, l') \in \mathbb{C}(S)$ if, and only if, $l(p) = l'(p)$ holds for each $p \in S$.

On the other hand, l' is a defining term for λ if, and only if, $\lambda(p) = l'(p)$ holds for each $p \in S$.

The stated equivalence then follows from the assumption that l defines λ, i.e. $\lambda(p) = l(p)$ for each $p \in S$.

The functor $\mathscr{C}$: arrows

The definition of $\mathscr{C}(\lambda)$ above does not depend on the choice of the representing term s, for if s' is another term such that $(s, s') \in \mathbb{C}(T)$, then $s\big([X_\beta \backslash l_\beta]_{\beta<\nu}\big)$ is congruent to $s'\big([X_\beta \backslash l_\beta]_{\beta<\nu}\big)$ modulo $\mathbb{C}(S)$, because substitutions commute with congruences.

Further, the definition of $\mathscr{C}(\lambda)$ does not depend on the choice of the family of defining terms $(l_\beta)_{\beta<\nu}$ either. Indeed, suppose $(l'_\beta)_{\beta<\nu}$ is another ν-tuple of defining terms for λ, and let $p \in S$. For each $\beta < \nu$ we have $(l_\beta, l'_\beta) \in \mathbb{C}(S)$ by *1* in this remark, so that $\big(s((l_\beta)_{\beta<\nu}),\ s((l'_\beta)_{\beta<\nu})\big) \in \mathbb{C}(S)$ because congruences are compatible with operations.

The functor $\mathscr{V}$: objects

Given $R = \{(s_i, t_i) \mid i \in I\} \subseteq \mathscr{F}_\mu \times \mathscr{F}_\mu$, for I an index set, we set

$$\mathscr{V}\left(\mathscr{F}_\mu / \theta\right) = \mathbb{V}\left(\theta\right).$$

The functor $\mathscr{V}$: arrows

Let $h\colon \mathscr{F}_\mu/\theta_1 \to \mathscr{F}_\nu/\theta_2$ be a homomorphism of MV-algebras. For each $\alpha < \mu$, let π_α be the projection term on the α^{th} coordinate, and let π_α/θ_1 denote the equivalence class of π_α modulo θ_1. Fix, for each α, an arbitrary $f_\alpha \in h(\pi_\alpha/\theta_1)$. For any $(p_\beta)_{\beta<\nu} \in \mathscr{V}(\theta_2)$, set

$$\mathscr{V}(h)((p_\beta)_{\beta<\nu}) = \big(\, f_\alpha(\,(p_\beta)_{\beta<\nu}\,)\,\big)_{\alpha<\mu}\,.$$

The functor $\mathscr{V}$: arrows

To see that $\mathscr{V}(h)$ is well-defined, fix $\alpha < \mu$. By definition, if p is a point of $\mathbb{V}(\theta_2)$, and if $g \in \mathscr{F}_\nu$ is such that $(f_\alpha, g) \in \theta_2$, then $f_\alpha(p) = g(p)$. Therefore, the definition of $\mathscr{V}(h)$ does not depend on the choices of the f_α's.

A general adjunction between varieties and topological spaces

Theorem

For any variety $\mathcal{V}$, for any $A \in \mathcal{V}$ the functors $\mathscr{V}$ and $\mathscr{C}$ are adjoint.

$$\begin{array}{ccc} \mathscr{V}\mathscr{C}(S) & \xrightarrow{\mathscr{V}\mathscr{C}(\lambda)} & \mathscr{V}\mathscr{C}(T) \\ \Big\downarrow \varepsilon_S & & \Big\downarrow \varepsilon_T \\ S & \xrightarrow{\quad\lambda\quad} & T \end{array}$$

$$\begin{array}{ccc} \mathscr{C}\mathscr{V}\left(\frac{\mathscr{F}_\mu}{\theta_1}\right) & \xrightarrow{\mathscr{C}\mathscr{V}(h)} & \mathscr{C}\mathscr{V}\left(\frac{\mathscr{F}_\nu}{\theta_2}\right) \\ \Big\uparrow \eta_{\frac{\mathscr{F}_\mu}{\theta_1}} & & \Big\uparrow \eta_{\frac{\mathscr{F}_\nu}{\theta_2}} \\ \frac{\mathscr{F}_\mu}{\theta_1} & \xrightarrow{\quad h\quad} & \frac{\mathscr{F}_\nu}{\theta_2} \end{array}$$

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   1.3 Gelfand transform.
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The Birkhoff transform.

Given a presented algebra $\mathscr{F}(\nu)/\theta$ one can decompose it as follows:

1. If a is any element of $\mathbb{V}(\theta)$, then $\theta \subseteq \mathbb{C}(a)$, so there exists a canonical epimorphism $q_a \colon \mathscr{F}(\nu)/\theta \to \mathscr{F}(\nu)/\mathbb{C}(a)$.
2. The above epimorphism sends any term of $\mathscr{F}(\nu)/\theta$, t/θ into the element $t(a)/\mathbb{C}(a)$.
3. So there is a map $\sigma_\theta \colon \mathscr{F}(\nu)/\theta \to \prod_{a \in \mathbb{V}(a)} \mathscr{F}(\nu)/\mathbb{C}(a)$.
4. Call this map the **Birkhoff transform**.

The algebraic Nullstellensatz

Theorem (Algebraic *Nullstellensatz*)

Fix a cardinal ν, and a congruence θ on $\mathscr{F}(\nu)$. The following are equivalent.

(i) $\mathbb{C}(\mathbb{V}(\theta)) = \theta$.

(ii) $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a)$.

(iii) *The Birkhoff transform* $\sigma_\theta \colon \mathscr{F}(\nu)/\theta \to \prod_{a \in \mathbb{V}(\theta)} \dfrac{\mathscr{F}(\nu)}{\mathbb{C}(a)}$ *is a subdirect embedding.*

Proof of the algebraic Nullstellensatz

Recall that

$$\theta \subseteq \mathbb{C}\left(\mathbb{V}\left(\theta\right)\right)$$

And notice that

$$\theta \subseteq \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a) \ .$$

(i)$\Rightarrow$(ii) By hypothesis $\theta = \mathbb{C}\left(\mathbb{V}\left(\theta\right)\right)$, so it is enough to prove

$$\bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a) \subseteq \mathbb{C}\left(\mathbb{V}\left(\theta\right)\right) \ .$$

If $(p, q) \in \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a)$, then for every $a \in \mathbb{V}(\theta)$, $p(a) = q(a)$, which means $(p, q) \in \mathbb{C}\left(\mathbb{V}\left(\theta\right)\right)$.

Proof of the algebraic Nullstellensatz

(ii)$\Rightarrow$(i) It is enough to prove

$$\mathbb{C}(\mathbb{V}(\theta)) \subseteq \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a).$$

Suppose that $(p, q) \in \mathbb{C}(\mathbb{V}(\theta))$, i.e. for all $a \in \mathbb{V}(\theta)$ $p(a) = q(a)$. The latter entails that, for all $a \in \mathbb{V}(\theta)$, $(p, q) \in \mathbb{C}(a)$, whence $(p, q) \in \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a)$.

Proof of the algebraic Nullstellensatz

$(ii) \Leftrightarrow (iii)$. The homomorphism

$$\sigma_\theta \colon \mathscr{F}(\nu)/\theta \to \prod_{a \in \mathbb{V}(\theta)} \frac{\mathscr{F}(\nu)/\theta}{\mathbb{C}(a)/\theta}$$

is an embedding if, and only if, $\bigcap_{a \in \mathbb{V}(\theta} \mathbb{C}(a)/\theta$ is the identity congruence on $\mathscr{F}(\nu)/\theta$.

It is clear by construction that if σ_θ is an embedding, then it is subdirect.

Since $a \in \mathbb{V}(\theta)$ iff $\theta \subseteq \mathbb{C}(a)$, by the Second Isomorphism Theorem

$$\forall a \in \mathbb{V}(a) : \quad \frac{\mathscr{F}(\nu)/\theta}{\mathbb{C}(a)/\theta} \cong \mathscr{F}(\nu)/\mathbb{C}(a).$$

Proof of the algebraic Nullstellensatz

Upon recalling that the mapping $\theta' \mapsto \theta'/\theta$ is an isomorphism of lattices between the lattice of congruences of $\mathscr{F}(\nu)$ extending θ and the lattice of congruences of $\mathscr{F}(\nu)/\theta$, we have

$$\bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a)/\theta = \Delta/\theta \quad \Longleftrightarrow \quad \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a) = \theta.$$

The Gelfand evaluation

For any $a \in A$ we have the following canonical surjection.

$$q_a\colon \mathscr{F}(\nu) \twoheadrightarrow \mathscr{F}(\nu)/\mathbb{C}(a).$$

By the universal property of the quotient homomorphism there exists a unique homomorphism

$$\gamma_a\colon \mathscr{F}(\nu)/\mathbb{C}(a) \longrightarrow A$$

So the following diagram commutes.

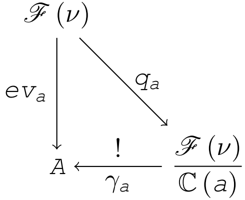


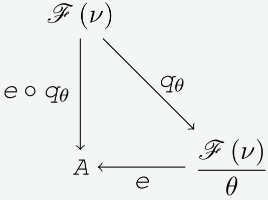
Figure: The Gelfand evaluation γ_a.

The GKS lemma

Lemma (GKS Lemma)

(i) *For each* $a \in A^{\nu}$*, the Gelfand evaluation* γ_a *is a monomorphism.*

(ii) *For each congruence* θ *on* $\mathscr{F}(\nu)$*, and each homomorphism* $e\colon \mathscr{F}(\nu)/\theta \to A$



If e is a monomorphism, then there exists $a \in A$ *such that* $\theta = \mathbb{C}(a)$*, and the commutative diagram above coincides with the one above.*

Proof of the GKS lemma

(i) Pick $p, q \in \mathscr{F}(\nu)$ such that $(p,q) \notin \mathbb{C}(a)$. Then, by definition, $p(a) \neq q(a)$, and therefore $ev_a(p) \neq ev_a(q)$. But then, by the definition of Gelfand evaluation, it follows that $\gamma_a(p) \neq \gamma_a(q)$.
(ii) Since e is monic, we have $\ker(e \circ q_\theta) = \ker q_\theta = \theta$. Explicitly,

$$\forall s, t \in \mathscr{F}(\nu): \quad (s,t) \in \theta \quad \Longleftrightarrow \quad e(q_\theta(s)) = e(q_\theta(t)).$$

Set $a := (e \circ q_\theta(X_\beta))_{\beta<\nu} \in A^\nu$, then the above yields

$$\forall s, t \in \mathscr{F}(\nu): \quad (s,t) \in \theta \quad \Longleftrightarrow \quad s(a) = t(a).$$

Therefore, we have $a \in \mathbb{V}(\theta)$, which is equivalent to

$$\theta \subseteq \mathbb{C}(a).$$

Proof of the GKS lemma

For the converse inclusion, if $(u, v) \in \mathbb{C}(a)$, then $u(a) = v(a)$, and therefore $(u, v) \in \theta$. This proves $\theta = \mathbb{C}(a)$, and therefore $q_\theta = q_a$. To show $ev_a = e \circ q_a$, note that, by the definition of ev_a and the universal property of $\mathscr{F}(\nu)$, they both are the (unique) extension of the function $X_\beta \mapsto ev(X_\beta, (e \circ q_\theta(X_\beta)))$, for $\beta < \nu$.

A picture

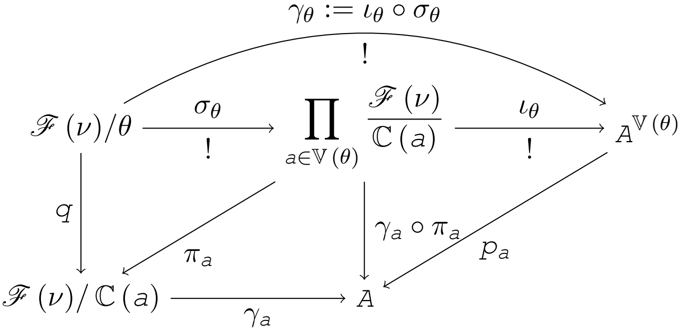


Figure: The Gelfand and Birkhoff transforms γ_θ and σ_θ.

The algebra of definable functions

Theorem

For any set $S \subseteq A^{\nu}$ *let* $\mathbf{D}(S)$ *be the algebra of definable functions into* A *restricted to* S. *The map* $\widehat{\gamma}_{\theta} \colon \mathscr{F}(\nu) \to \mathbf{D}(\mathbb{V}(\theta))$ *is an epimorphism. Furthermore,* $\widehat{\gamma}_{\theta}$ *is an isomorphism if, and only if,* $\theta = \mathbb{C}\,\mathbb{V}(\theta)$.

Proof of the representation

The map γ_θ sends an element $t(\overline{X})/\theta$ of $\mathscr{F}(\nu)/\theta$, into the definable function $\tilde{t}$ associated to $t(\overline{X})$ restricted to $\mathbb{V}(\theta)$.

- **Well defined**
  Suppose $(s,t) \in \theta$, then for all $a \in \mathbb{V}(\theta)$, $s(a) = t(a)$, so $\tilde{s} = \tilde{t}$.
- **homomorphism**
  Let $\star$ be an binary connective, then we need to prove that $\widetilde{s \star t}(a) = \tilde{s}(a) \star \tilde{t}(a)$ for all $a \in \mathbb{V}(\theta)$. Which is equivalent to say that $s \star t(a) = s(a) \star t(a)$ for all $a \in \mathbb{V}(\theta)$. The latter holds because the interpretation of connective is given pointwise.

Proof of the representation

- **surjective**
  For any definable map λ, there exists a defining term t, so $\lambda = \widehat{\gamma}_\theta(t)$.
- If $\theta = \mathbb{C}\,\mathbb{V}(\theta)$, then $\widehat{\gamma}_\theta$ is **injective**.
  It is enough to prove that $\ker(\widehat{\gamma}_\theta) = \dfrac{\mathbb{C}\,\mathbb{V}(\theta)}{\theta}$.
  The fact that $\widehat{\gamma}_\theta(s/\theta) = \widehat{\gamma}_\theta(t/\theta)$, is equivalent to the statement that for all $a \in \mathbb{V}(\theta)$ $s(a) = t(a)$, which in turn is equivalent to say that $(s, t) \in \mathbb{C}\,\mathbb{V}(\theta)$, so $(s/\theta, t/\theta) \in \mathbb{C}\,\mathbb{V}(\theta)/\theta$.

Choosing the right A

Proposition

Assume the variety $\mathcal{V}$ is finitary. The adjunction $\mathscr{V}, \mathscr{C}$ completely fixes the algebraic side if, and only if, A contains a copy of each subdirectly irreducible algebra in $\mathcal{V}$.

- Call a congruence **subdireclty irreducible** if it presents a subdirectly irreducible algebra.
- If A contains a copy of each subdirectly irreducible algebra, by the GKS-lemma, all subdirectly irreducible congruences are of the form $\mathbb{C}(a)$ for some point $a \in A^{\mu}$.

Choosing the right A

- If the variety is finitary, then every congruence is the intersection of subdirectly irreducible congruences, so by the algebraic Nullstellensatz every congruence is fixed.
- Vice versa, if every congruence is fixed by $\mathbb{C}\mathbb{V}$ on the algebraic side, then also all subdirectly irreducible congruences are.
- So, every subdirectly irreducible congruence θ can be written as $\bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(a)$.
- But if a subdirectly irreducible congruence can be written as an intersection of a family of congruences, then it must belong to that family, so there exists a such that $\theta = \mathbb{C}(a)$.

Choosing the right A

As an immediate corollary

Corollary

Assume the variety $\mathcal{V}$ is finitary. If the adjunction $\mathscr{V}, \mathscr{C}$ completely fixes the algebraic side then A generates the variety.

Indeed, by Birkhoff theorem the sudirectly irreducible algebras alone generates the full variety and they are all present in A as subalgebras.

We will see an example that shows that the implication cannot be inverted.

Contents of the tutorial

The dual of algebra is topology!

1. Co-Nullstellenstaz
   1.1 The Zariski topology.
   1.2 A-compact spaces.
   1.3 Two natural topologies.
   1.4 The definable functions.

The Zariski topology

Notice that $\mathscr{V} \circ \mathscr{C}(S) = \mathbb{V}\mathbb{C}(S)$.

Recall that by the fact that $\mathbb{C}$ and $\mathbb{V}$ are a Galois adjunction, for any $S \subseteq A^{\mu}$,

1. $S \subseteq \mathbb{V}\mathbb{C}(S)$,
2. If $S_1 \subseteq S_2$ then $\mathbb{V}\mathbb{C}(S_1) \subseteq \mathbb{V}\mathbb{C}(S_2)$, and
3. $\mathbb{V}\mathbb{C}\left(\mathbb{V}\mathbb{C}(S)\right) = \mathbb{V}\mathbb{C}(S)$.

So the composition $\mathbb{V}\mathbb{C}$ is a **closure operator**.

Although this does not guarantee that the sets of the form $\mathbb{V}\mathbb{C}(S)$ are the closed sets in some topology, in the cases we will meet, it always holds that $\mathbb{V}\mathbb{C}(S_1 \cup S_2) = \mathbb{V}\mathbb{C}(S_1) \cup \mathbb{V}\mathbb{C}(S_2)$, in which case the closure operator is **topological**

The Zariski topology

The fixed points are of the form

$$\mathbb{V}(\theta) = \{p \in A^{\mu} \mid s(p) = t(p) \text{ for all } (t, s) \in \theta\}.$$

Compare with the **Zariski topology** in algebraic geometry given by following closed sets:

$$V(S) = \{x \in \mathbb{A}^{n} \mid f(x) = 0 \text{ for all } f \in S\}$$

So we are interested in characterising the fixed points in this **generalised** Zariski topology.

Kelley's embedding lemma

Given a family F of continuous functions from a topological space X into topological spaces Y_f the *evaluation* map $e\colon X \to \prod_{f\in F} Y_f$ defined as

$$e(x)_f := f(x).$$

Theorem

The map e is

1. ***continuos*** *on X,*
2. ***injective*** *if, and only if, $p, q \in X$ there exists $f \in F$ such that $f(p) \neq f(q)$,*
3. ***open*** *if for any $p \in X$ and any closed subset C such that $p \notin C$ there exists $f \in F$ such that $f(p) \notin \overline{f[C]}$.*

A-compact spaces

Obviously, we are interested in using the above result with $Y_f = A$, to prove that any **abstract** topological space (satisfying certain properties) is homeomorphic to a closed set of a suitable power of A.

In the literature those spaces have been extensively studied under the name of **A-compact spaces**.

Two topologies

However there is a problem namely, there are two natural topologies on A^μ

1. the **Zariski topology** given by the μ-ary definable functions,
2. the **product topology** induced by the Zariski topology on A,

and in general the two are different!

Two topologies

Lemma

If A is Hausdorff and all definable functions are continuous with regard to the product topology then a set $S \subseteq A^{\mu}$ is closed in the product topology if, and only if, $\mathbb{V}(\mathbb{C}(S)) = S$.

Two topologies

- Let us start by noticing that **definable functions are continuous** w.r.t. the Zariski topology. Indeed, if S is closed in the Zariski topology, then there exits θ such that $S = \mathbb{V}(\theta)$. If f is a definable function, then

  $$f^{-1}[S] = \{p \in A^{\mu} \mid s(f(p)) = t(f(p)) \text{ for all } (s,t) \in \theta\},$$

  hence if $\theta' := \{(s \circ f, t \circ f) \mid (s,t) \in \theta\}$, $f^{-1}[S] = \mathbb{V}(\theta')$.
- Let us write $\overline{S}$ for the smallest closed set in the product topology that contains S. The product topology is coarser than the Zariski topology, because the projections are continuous in the Zariski topology. So, we have $\mathbb{V}\mathbb{C}(S) \subseteq \overline{S}$.

Two topologies

- To prove the other direction, recall that if X is any space, and Y is Hausdorff, then for any two continuous functions $f, g \colon X \to Y$ the solution set of the equation $f = g$ is a closed subset of X.
- Now, by assumption A is Hausdorff and definable functions are continuous, so for any pair of terms (s, t), the set $\mathbb{V}(s, t)$ is closed in product topology.
- On the other hand, $\mathbb{V}(R) = \mathbb{V}\big(\bigcup_{(s,t)\in R} \{(s, t)\}\big) = \bigcap_{(s,t)\in R} \mathbb{V}(s, t)$ so we conclude that $\mathbb{V}(R)$ is closed in the product topology for any subset R of $\mathscr{F}_\mu \times \mathscr{F}_\mu$. Thus we obtain the inclusion $\overline{S} \subseteq \mathbb{V}(\mathbb{C}(S))$.

Two topologies

Corollary

If the Zariski topology on A is discrete, then Zariski and product topology coincide.

- If the Zariski topology on A is discrete then it obviously is Hausdorff.
- In addition all finite products are also discrete, so all definable functions are continuous.
- Thus the assumptions of the above Lemma are met and the claim holds by its direct application.

Contents of the tutorial

1. Stone duality

Stone duality

Let $\mathcal{V}$ be the variety of Boolean algebras. Let A be the Boolean algebra whose support is $\{0,1\}$.

By the general framework we have an adjunction $\mathscr{V}$, $\mathscr{C}$ between $\mathcal{V}$ and the subsets of A^{μ} for μ ranging among all cardinals.

Recall that

Lemma

The only subdirectly irreducible Boolean algebra is A.

So, *a fortiori*, all subdirectly irreducible Boolean algebra embed into A, hence the adjunction fixes completely the algebraic side.

Stone duality

So there is a dual equivalence between the **variety of Boolean algebra** and the **closed subsets of the cubes** A^{μ} with the Zariski topology.

An easy checking shows that all such closed spaces are Stone spaces. (Notice that closed subspaces of Stone spaces are again Stone spaces.)

An intrinsic characterisation

Finally, we need to prove that any compact, Hausdorff, totally disconnected space X, there exists a cardinal μ and closed subset S of $\{0,1\}^{\mu}$ such that X is homeomorphic to S.

Given such a space X, consider the family F of all continuous functions from X to $\{0,1\}$. If C is a closed subset of X and $p \in X \setminus C$, then there exists a clopen K which extends C and does not contain p.

It is then easy to see that the following function is continuous.

$$f(x) := \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{otherwise.} \end{cases}$$

An intrinsic characterisation

So we can apply Kelley's embedding lemma to affirm that any Stone space can be realised as closed subspace of some Cantor cube $\{0,1\}^{\mu}$ with the Zariski topology.

Further, any continuous map between Stone spaces can be realised as a map between the respective closed supspaces of the Cantor cube, by composing with the homeomorphisms given by Kelley's embedding lemma.

Hence, the category of Stone spaces and continuous maps among them is equivalent to the category of closed subspaces of Cantors cubes with definable maps among them.

Stone duality

Corollary (Stone 1936)

The category of Boolean algebras with their homomorphisms is dually equivalent to the category of compact, Hausdorff, totally disconnected spaces with continuous maps among them.

Contents of the tutorial

1. Priestley duality

Priestley duality

Let $\mathcal{V}$ be the variety of distributive lattice. Let A be the distributive lattice whose support is $\{0,1\}$.

By the general framework we have an adjunction $\mathscr{V}$,$\mathscr{C}$ between $\mathcal{V}$ and the subsets of A^μ for μ ranging among all cardinals.

Recall that

Lemma

The only subdirectly irreducible distributive lattice is A.

So, *a fortiori*, all subdirectly irreducible distributive lattices embed into A, hence the adjunction fixes completely the algebraic side.

Priestley duality

So, there is a dual equivalence between the **variety of distributive lattices** and the **closed subsets of the cubes** A^{μ} with the Zariski topology.

As before all such closed spaces are Stone spaces, but now we will consider also their pointwise order. It is again easy to verify that they satisfy Priestley separation axiom:

> If $x \not\leq y$ then there exists a upward closed, clopen set containing x and omitting y.

Priestley duality

We also need to check that

1. definable functions are order preserving, and
2. every continuos, order preserving map among closed subspace of Cantor cubes is definable.

An intrinsic characterisation

Finally, we need to prove that any Priestley space X, there exists a cardinal μ and closed subset S of $\{0,1\}^\mu$ such that X is **order** homeomorphic to S.

Given such a space X, consider the family F of all continuous functions from X to $\{0,1\}$. If $p \neq q$ we may safely assume $p \not\leq q$, so there is a clopen K which contains p and does not contain q.

It is then easy to see that the following function is continuous.

$$f(x) := \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{otherwise.} \end{cases}$$

An intrinsic characterisation

So we can apply Kelley's embedding lemma to affirm that any Priestley space can be realised as closed subspace of some Cantor cube $\{0,1\}^{\mu}$ with the Zariski topology.

Since the evaluation map is also order preserving, any continuous and order preserving map between Priestley spaces can be realised as a map between the respective closed supspaces of the Cantor cube, by composing with the respective evaluation maps.

Hence, the category of Priestley spaces and continuous, order preserving maps among them is equivalent to the category of closed subspaces of Cantors cubes with definable maps among them.

Stone duality

Corollary (Priestley 1970)

The category of distributive lattices with their homomorphisms is dually equivalent to the category of Priestley spaces with continuous and order preserving maps among them.

Contents of the tutorial

1. The Duality for MV-algebras
   1.1 MV-algebras
   1.2 Semisimple MV-algebras
   1.3 Complete regularity
   1.4 Finitely presented MV-algebras.
   1.5 Rational polyhedra and $\mathbb{Z}$-maps.
   1.6 McNaughton theorem.

MV-algebras

Consider the structure $\langle [0,1], \oplus, \neg, 0$ where

$$x \oplus y = \min 1, x + y \text{ and } \neg x = 1 - x.$$

This is an MV-algebras. In general, MV-algebras are the algebras in the variety generated by the above structure.

MV-algebras provide the equivalent algebraic semantics for Łukasiewicz logic.

MV-algebras

Let $\mathcal{V}$ be the variety of MV-algebras and A be the above MV-algebra on $[0,1]$

The functors $\mathscr{V}$ and $\mathscr{C}$ provide a dual adjunction between the variety of MV-algebras and the closed subspaces of $[0,1]^{\mu}$.

Semisimple MV-algebras

Lemma

*An MV-algebra is **simple** if, and only if, it can be embedded into* $[0,1]$.

So all **simple congruences** are of the form $\mathbb{C}(a)$ for some point $a \in [0,1]^{\mu}$. The intersection of such congruences are exactly the **semisimple** congruences.

We then immediately have that the adjunction $\mathscr{V}, \mathscr{C}$ fixes exactly the semisimple MV-algebras.

Complete regularity

Lemma

Fro any open interval $(a,b) \subseteq [0,1]$ *and any* $p \in (a,b)$, *there exists a term* $l(x)$ *such that its definable function* $\tilde{l}\colon [0,1] \to [0,1]$ *satisfies* $\tilde{l}(p) > 0$ *and* $([0,1] \setminus (a,b)) \subseteq (\tilde{l})^{-1}(0)$.

So, if $C \subseteq [0,1]$ is a closed set in the Euclidean topology and p is an external point, by the above lemma there is a Zariski closed set $\mathbb{V}(t,0)$ which contains C but excludes p.

From this we immediately have that the Euclidean topology is coarser than the Zariski topology. So the two coincides.

The two topologies

Lemma

The product and Zariski topologies on A^{μ} coincide for any cardinal μ.

One implication is trivial, for the product topology is coarser than the Zariski.

Vice versa, every Zariski closed set is intersection of sets of the form $\mathbb{V}(s, t)$ where s and t are terms representing definable functions from $[0, 1]^{\mu}$ into $[0, 1]$. But $[0, 1]$ is Hausdorff and definable functions are continuous w.r.t. the product topology, so $\mathbb{V}(s, t)$ is closed in the product topology.

The intrinsic characterisation

Once more, as an application of Kelley's embedding lemma, one obtains that

Lemma

Every Tychonoff space is homeomorphic to a closed subset of $[0,1]^\mu$ *for a suitable cardinal* μ.

Finitely presented MV-algebras

Theorem (Wójcicki's Theorem)

Every finitely presented MV-algebra is semisimple.

So, in particular finitely presented algebras are fixed by the adjunction.

It is interesting to characterise the corresponding topological spaces.

Finitely presented MV-algebras

- A **convex combination** of a finite set of vectors $v_1, \ldots, v_u \in \mathbb{R}^m$ is any vector of the form $r_1 v_1 + \cdots + r_u v_u$, for non-negative real numbers $r_i \geq 0$ summing up to 1.
- If $S \subseteq \mathbb{R}^m$ is any subset, we let $\mathbf{conv}(S)$ denote the collection of all convex combinations of finite sets of vectors $v_1, \ldots, v_u \in S$.
- A **polytope** is any subset of $\mathbb{R}^m$ of the form $\mathbf{conv}(S)$, for some finite $S \subseteq \mathbb{R}^m$.
- A (**compact**) **polyhedron** is a union of finitely many polytopes in $\mathbb{R}^m$.
- A polyhedron is **rational** if it may be written as union of polytopes of the form $\mathbf{conv}(S)$ for some finite set $S \subseteq \mathbb{Q}^m$ of vectors with rational coordinates.

Finitely presented MV-algebras

Definition

Given a rational polyhedron $P \subseteq [0,1]^m$ and a continuous map $\zeta = (\zeta_1, ..., \zeta_n) : P \to [0,1]^n$, for $n \geq 0$ an integer, we say that ζ is a **$\mathbb{Z}$-map** if for each $i = 1, \ldots, n$, ζ_i is **piecewise linear with integer coefficients**.

McNaughton theorem

Theorem (McNaughton's Theorem for rational polyhedra)

Let $P \subseteq [0,1]^m$ be a rational polyhedron, and let $\lambda\colon P \to [0,1]$ be any function. Then λ is a $\mathbb{Z}$-map if, and only if, λ is a definable function.

The intrinsic characterisation

- S is a rational polyhedron if, and only if, there is a $\mathbb{Z}$-map $\zeta\colon [0,1]^m \to [0,1]$ vanishing precisely on S.
- If S is of the form $\mathbb{V}(R)$ for some finite $R \subseteq \mathscr{F}_m \times \mathscr{F}_m$, we may assume that R is a singleton $\{(s,0)\}$
- So S is the solution set over $[0,1]^m$ of the equation $\tilde{s} = 0$, where $\tilde{s}$ is the function defined by s.
- By McNaughton theorem, $\tilde{s}$ is a $\mathbb{Z}$-map, and therefore S is a rational polyhedron.

The intrinsic characterisation

- Conversely, if S is a rational polyhedron in $[0,1]^m$, there is a $\mathbb{Z}$-map $\zeta\colon [0,1]^m \to [0,1]$ such that $\zeta^{-1}(0) = S$.
- By McNaughton theorem there is a term $s \in \mathscr{F}_m$ such that ζ is the function defined by s.
- Therefore, since $S = \mathbb{V}(s,0)$, S correspond to a finitely presented MV-algebras.

The duality for finitely presented MV-algebras

Corollary (The duality theorem for finitely presented MV-algebras)

The adjunction $\mathscr{V}, \tilde{}$ restricts to an equivalence of categories between finitely presented MV-algebras and rational polyhedra with $\mathbb{Z}$-maps

Contents of the tutorial

1. A further generalisation
   1.1 Syntactic and semantic category.
   1.2 The category of relations and the category of subobjects.
   1.3 The adjunction $\mathscr{C}$, $\mathscr{V}$.
   1.4 The abstract Nullstellensatz.
   1.5 Example 1: Galois theory of field extensions.
   1.6 Example 2: Grothendieck theory of universal coverings.

Syntactic and semantic category

All categories in this paper are assumed to be locally small.

We consider the following.

- Two categories T and S.
- A functor $\mathscr{I}\colon \mathrm{T} \to \mathrm{S}$.
- An object $\triangle$ of T.

The category of subobjects

Objects are all pairs (t, s) where t is T-object and $s\colon \mathbf{dom}\, s \to \mathscr{I}(t)$ is an S-subobject.

Arrows $(t, s) \to (t', s')$ are the T-arrows $f\colon t \to t'$ such that $\mathscr{I}(f) \circ s$ factors through s'; that is, there exists an S-arrow $g\colon S \to S'$ such that the diagram

$$\begin{array}{ccc}
\mathscr{I}(t) & \xrightarrow{\mathscr{I}(f)} & \mathscr{I}(t') \\
\uparrow s & & \uparrow s' \\
\mathbf{dom}\, s & \xrightarrow[g]{} & \mathbf{dom}\, s'
\end{array}$$

commutes.

The category of subobjects

Objects are all pairs (t, R) where t is a T-object and R is a relation on $\mathrm{Hom}_{\mathrm{T}}(t, \triangle)$.

Arrows $(t, R) \to (t', R')$ are the T-arrows $f\colon t \to t'$ such that the function

$$- \circ f\colon \mathrm{Hom}_{\mathrm{T}}(t', \triangle) \to \mathrm{Hom}_{\mathrm{T}}(t, \triangle)$$

satisfies the property

$$(p', q') \in R' \quad \Longrightarrow \quad (p' \circ f, q' \circ f) \in R.$$

We say in this case that *f preserves R'* (*with respect to R*).

The Galois connection $\mathbb{C}$, $\mathbb{V}$

Definition

For any $(t, s) \in \mathrm{D}$, we define the following equivalence relation on $\mathrm{Hom}_{\mathrm{T}}(t, \triangle)$:

$$\mathbb{C}(s) := \left\{(p, q) \in \mathrm{Hom}^2_{\mathrm{T}}(t, \triangle) \mid \mathscr{I}(p) \circ s = \mathscr{I}(q) \circ s\right\}.$$

The Galois connection $\mathbb{C}$, $\mathbb{V}$

In order to define $\mathbb{V}$ it is necessary to assume that S has enough limits. It is sufficient to make the following

Assumption: Henceforth, we assume that S has equalisers of pairs of parallel arrows, and intersections of arbitrary families of subobjects.

Definition

For any (t, R) in R, we set

$$\mathbb{V}(R) := \bigwedge_{(p,q)\in R} Eq(\mathscr{I}(p), \mathscr{I}(q)),$$

The Galois connection $\mathbb{C}$, $\mathbb{V}$

Lemma (Galois connection)

For any T-*object* t, *any relation* R *on* $\mathrm{Hom}_{\mathrm{T}}(t, \triangle)$, *and any* S-*subobject* $s\colon \mathbf{dom}\, s \to \mathscr{I}(t)$, *we have*

$$R \subseteq \mathbb{C}(s) \quad \textit{if, and only if,} \quad s \leq \mathbb{V}(R). \tag{1}$$

The Galois connection $\mathbb{C}$, $\mathbb{V}$

Theorem (Weak *Nullstellensatz*)

Fix an R-*object* (t, R). *For any family* $\Sigma = \{\sigma_i\}_{i \in I}$ *of subobjects of* $\mathscr{I}(t)$ *such that for each* σ_i *there exists* m_i *with* $\sigma_i = \mathbb{V}(R) \circ m_i$ *(i.e.* $\sigma_i \leq \mathbb{V}(R)$*) and the family of* S-*arrows* $\{m_i\}_{i \in I}$ *is jointly epic in* S, *the following are equivalent.*

(i) $R = \mathbb{C}(\mathbb{V}(R))$, *i.e.* R *is fixed by the Galois connection* (1).

(ii) $R = \bigcap_{i \in I} \mathbb{C}(\sigma_i)$.

The end.

Thank you for your (extended) attention!