

LUKASIEWICZ LOGIC AND MV-ALGEBRAS

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Course Contents (4 hours):

- From classical logic to many-valued logics
- Łukasiewicz logic and other truth-functional logics
- MV-algebras
- Advances in MV-algebras

Section 1: classical propositional logic

Classical propositional logic axiomatizes the connectives OR, AND, NOT, IMPLIES, ETC through **truth tables**

A	B	$A \wedge B$	A	B	$A \vee B$	A	$\neg A$
0	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0
1	0	0	1	0	1		
1	1	1	1	1	1		

Truth tables offer a **complete** description of the connective behaviour in all possible cases (which, in classical logic, are assumed to be two).

LAW OF EXCLUDED MIDDLE

So, the truth tables give an **interpretation**
 (or meaning, or semantics) to the symbols

$\wedge, \vee, \rightarrow, \neg$

But there are several reasons to relax the requirement that **absolute true** and **absolute false** are the only possibilities... our world is not like that!

This possibility has been studied for ages... but once we have a **formal system** to frame cope with the problem, things become clearer, if not easier.

A	B	$A \wedge B$	$A \vee B$	$\neg B$
0	0	0	0	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	1	0	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
$\frac{1}{2}$	1	$\frac{1}{2}$	1	
1	0	0	1	
1	$\frac{1}{2}$	$\frac{1}{2}$	1	
1	1	1	1	

KLEENE
LOGIC

UNKNOWN

$$\begin{aligned} \text{N}(A \wedge B) &= \min\{\text{N}(A), \text{N}(B)\} \\ \text{N}(A \vee B) &= \max\{\text{N}(A), \text{N}(B)\} \\ \text{N}(\neg B) &= 1 - \text{N}(B) \end{aligned}$$

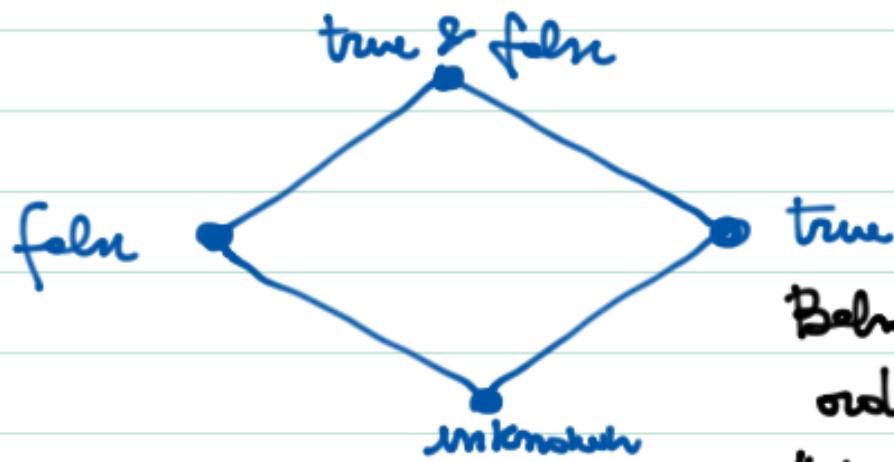
Having more than one truth value, one has to decide what it means be a **tautology**

If we take only formulas that always get value equal one (where the most natural interpretation is $1/2 = \text{undetermined}$, K_3) then we get no tautologies at all!! If we allow both 1 and $1/2$ (where $1/2$ can be interpreted as being overvalued, both true and false, P_3) then we get exactly classical tautologies.

K_3 suffers of being **too restrictive** (there are good reasons to ask that a logic makes true at least $A \rightarrow A$)

On the other hand, P_3 is **too relaxed**, as the formula $A \wedge \neg A$ is not a contradiction but $A \wedge \neg A$ can be read as $A \wedge A \rightarrow \perp$ so, here **modus ponens** is failing!

Another popular many-valued logic is **Belnap four-valued logic** which combines the two above by using 4 truth values, 0, 1, a value for unknown and one for both true and false.



Belnap 4 values
ordered by their
"degree of informati"

truth functionality:

Notice that here we are not deviating from the "**extensionality principle**", namely the truth value of compound formulas only depends from the truth values of their constituting parts.

Of course this principle can be dropped, but the price to pay is to leave many important paradoxes of classical logic too.

Lukasiewicz logic (1920) was one of the earliest many valued logic. The first version was three valued:

A	B	$A \odot B$	$A \oplus B$	$\neg B$
0	0	0	0	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	1	0	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	
$\frac{1}{2}$	$\frac{1}{2}$	0	1	
$\frac{1}{2}$	1	$\frac{1}{2}$	1	
1	0	0	1	
1	$\frac{1}{2}$	$\frac{1}{2}$	1	
1	1	1	1	

It was soon generalized, by Lukasiewicz and Tarski to an infinite-valued logic

$$\begin{aligned} n(A \odot B) &= \max \{0, n(A) + n(B) - 1\} & n(\neg A) &= 1 - n(A) \\ n(A \oplus B) &= \min \{1, n(A) + n(B)\} & n(A \rightarrow B) &= \min \{1, 1 - n(A) + n(B)\} \end{aligned}$$

Notice that the functions just defined serve, just as truth tables, to give a complete description of the connectives in all possible cases; the cases are just infinite now!

In Łukasiewicz logic **tautologies** are defined as formulas that always get evaluated into 1. The tautologies of Łukasiewicz logic are **strictly fewer** than the classical ones (for instance, $A \rightarrow A \circ A$ is not a tautology) but **modus ponens** holds. Furthermore one can verify that the connectives \wedge, \vee are definable, for instance

$$A \vee B = \neg (\neg A \oplus B) \oplus B$$

Now that truth tables are infinite, checking whether something is a tautology or not is not just a matter of time anymore. One has to prove it abstractly. So having a system of axioms is even more important.

The following axioms were found to be
complete by Rose & Rosser (1958, Trans.AMS)

T AXIOMS

- 1) $A \rightarrow (B \rightarrow A)$
- 2) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- 3) $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$
- 4) $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Warning: the article is 52 pages
written in Polish notation

Although the proof is mostly syntactical, the main idea of associating functions to formulas comes from a former, and extremely important, result in Lukasiewicz logic, due to McNaughton (1951, JSL). We will speak about this later.

Lukasiewicz logic is not the only infinite-valued logic extending the classical one.

Consider

$$N(A \times B) = N(A) \cdot N(B) \quad (\text{usual product})$$

$$N(A \rightarrow B) = \begin{cases} 1 & \text{if } N(A) \leq N(B) \\ \frac{N(B)}{N(A)} & \text{otherwise} \end{cases}$$

The logical system modelling these connectives is called Product logic. Again it has infinitely many truth values, its set of tautologies is strictly smaller than classical logic, and satisfies modus ponens.

Also in this case is possible to give a complete axiomatisation of the tautologies.

Finally, the connectives of K_3 can also be extended to the full $[0,1]$ real interval:

$$N(A \wedge B) = \min(N(A), N(B))$$

$$N(A \rightarrow B) = \begin{cases} 1 & \text{if } N(A) \leq N(B) \\ N(B) & \text{otherwise} \end{cases}$$

This logic is called Gödel-Dummett Logic and its finite versions were first studied by Gödel (1932) to show that Intuitionistic logic cannot be seen as a (finite) many valued logic.

Product, minimum, and Łukasiewicz conjunction are not the only possibility to enlarge the scope of logic to the infinite truth values set $[0,1]$.

Taking the matter more abstractly, one may ask what are the natural properties of conjunction, that we want to retain

- truth functionality
- Commutativity
- Associativity
- Extending classical conjunction
- Monotonicity
- Continuity

Algebraically we get:

- 1) $n * y = y * n$
- 2) $n * (y * z) = (n * y) * z$
- 3) $n * 1 = n \quad n * 0 = 0$
- 4) $n \leq y \Rightarrow n * z \leq n * y$

The above properties axiomatize the concept of a t-norm.

In P. Hájek "Metamathematics of fuzzy logic"
1998 KLUWER, it was for the first
time presented the logic **BL** (**BASIC LOGIC**)
inspired by the above considerations.

The system BL is axiomatised as follows

- 1) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- 2) $(A \bullet (A \rightarrow B)) \rightarrow (B \bullet (B \rightarrow A))$
- 3) $(A \rightarrow (B \rightarrow C)) \leftrightarrow (A \bullet B \rightarrow C)$
- 4) $((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)$
- 5) $\perp \rightarrow A$

The only deduction rule is *modus ponens*

It turns out that Łukasiewicz, product and Gödel logic are axiomatic extensions of BL.
E.g. Łukasiewicz logic is $BL + (A \rightarrow \perp) \rightarrow \perp \leftrightarrow A$

But the importance of the system BL does not stop there ...

Theorem (Cignoli, Esteva, Godo, Torrens 2000)

BL is the logic of all continuous t-norms -

So, loosely speaking, theorems of BL correspond to the formulas which are always evaluated to one, no matter what continuous t-norm we use to interpret the conjunction -

To understand more formally the content of the theorem, we need to make more explicit the definition of interpretation

Going back to classical logic, we may remind of a pivotal tool in the study of propositional calculi:

Boolean algebras.

The connection between classical logic and Boolean algebras is just one instance of a more general phenomenon called now

equivalent algebraic semantics (Blok & Pigozzi, J. Amer. Math. Soc., 1989)

The idea, in its primitive form, is that it is possible to associate formulas of a logic with terms in some class of algebras, in such a way that the provability of formulas becomes a property expressible by some term.

When such a correspondence can be translated back and forth without changing the set of tautologies we say that the class of algebras is the algebraic semantics of the logic, and the properties of such a class get an immediate translation into properties of the logic.

Definition A **BL-algebra** is a structure $\langle A, \wedge, \vee, \odot, \Rightarrow, 0, 1 \rangle$ such that :

- 1) $\langle A, \odot, 1 \rangle$ is a commutative monoid.
- 2) $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice.
- 3) \odot, \Rightarrow are a residuated pair, i.e.
$$z \leq (x \Rightarrow y) \iff x \odot z \leq y$$
- 4) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (pre-linearity)
- 5) $x \wedge y = x \odot (x \Rightarrow y)$ (combinability)

Example Let $*$ be a t-norm and \Rightarrow its residuum, then the algebra $\langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle$ is a BL-algebra.

One may recognise that the class of BL-algebras is a sub-class of the (commutative, integral) residuated lattices

As these latter structures are closely related to substructural logics (in a sense that will be made precise in a moment), one obtains that the many-valued logics extending BL can be considered substructural too.

Góbel, Jipsen, Kowalski, On "Residuated lattices: an algebraic glimpse at substructural logics" 2007

We can now define what is an algebraic interpretation of a BL-formula:

it is a map from propositional variables into a BL-algebra A that extends to the full set of formulas according to the following rules:

$$\begin{aligned}e(A \bullet B) &= e(A) \odot e(B) \\e(A \rightarrow B) &= e(A) \Rightarrow e(B) \\e(\perp) &= 0\end{aligned}$$

Theorem The logic BL is complete w.r.t. the class of all BL-algebras.

In other words, a formula φ is true in BL if, and only if, its interpretation $e(\varphi)$ equals 1 in every BL-algebra.

Theorem BL is complete w.r.t. the class of all standard BL-algebras, i.e. of the form

$$\langle [0..1], *, \Rightarrow, \min, \max, 0, 1 \rangle$$

for $*$, \Rightarrow cont. t-norm and its zero-dimum-

It makes sense to ask for the continuity of the t-norm, both because it is desirable for applications and because it guarantees the existence of a residuum -

However one can do with less, just asking for left-continuity. The logical system can be obtained from BL with simple modifications (namely, drop divisibility) and is called MTL.

Also for MTL there is a

standard completeness theorem

Theorem (Jenei-Montagna 20)

MTL is the logic of all left-continuous t-norm and their residua -

For an extensive account of recent accomplishments in the theory of BL, MTL, and more, see Cintula, Hájek, Noguera Ed's "Handbook of Mathematical Fuzzy Logic Vol I and II, 2011

Let's go back to Łukasiewicz logic.
 As this system can be obtained as BL + double negation law, its algebraic semantics corresponds to BL-algebras + $\neg\neg n = n$

This class of structures were known much earlier than BL and BL-algebras.
 They were introduced by Chang (1958, Trans AMS) under the name of MV-algebras (MV stands for Many-Valued) -

Gigow, D'Ottoniano, Mundici - "Algebraic foundations of Many-valued reasoning", Springer 2000

Definition: A structure $\langle A, \oplus, \neg, 0 \rangle$ is called MV-algebra if the following hold:

- 1) $n \oplus (y \oplus z) = (n \oplus y) \oplus z$
- 2) $n \oplus y = y \oplus n$
- 3) $n \oplus 0 = n$
- 4) $\neg \neg n = n$
- 5) $\neg(\neg n \oplus y) \oplus y = \neg(\neg y \oplus n) \oplus z$

Note that $n \neg y = \neg n \oplus y$ and $n \neg y = \neg(\neg n \oplus y)$

As before, also in this case it is possible to define a lattice order using the basic operations

Lemma Let $u \vee y = \neg(\neg u \oplus y) \odot y$ and $u \wedge y = \neg(\neg u \vee \neg y)$, then if A is an MV-obj. $\langle A, \vee, \wedge \rangle$ is a lattice.

As usual in algebra, in the study of MV-algebras it is crucial to consider homomorphisms which in turn correspond to congruences which in turn correspond to ideals

$$f \text{ is a } B_f = \{(u, y) \mid f(u) = f(y)\}$$

$$f \text{ is a } K_f = [u \mid f(u) = 0]$$

Definition Let A be an MV-algebra and $I \subseteq A$ we say that I is an ideal if

- 1) $a, b \in I \Rightarrow a \oplus b \in I$
- 2) $a \in A, b \in I, a \leq b \Rightarrow a \in I$

An MV-algebra A is said linearly ordered if the order induced by the lattice operations ($n \leq y \Leftrightarrow n \wedge y = n$) is linear.

An ideal P of an MV-algebra A is said prime if A/P is linearly ordered.

An MV-algebra is said simple if it can be embedded into the standard MV-algebra $[0,1]$.

An MV-algebra is said semisimple if it is the subdirect product of simple MV-obj.

An ideal H of an MV-algebra A is called maximal if A/H is simple.

Example : The algebra $([0,1], \oplus, \neg, 0)$ where $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1-x$ is an MV-algebra and it is called the Standard MV-algebra.

Theorem (Chay⁵⁸) Every MV-algebra is a **subdirect product** of linearly ordered MV-algebras.

Proof The proof hinges on the following Lemma

Lemma If $a \neq 0$, there exists a prime ideal containing a .

To prove the theorem, we consider the product of the algebras A_p for P prime ideals in A . The above lemma gives injectivity at once.

Theorem (Standard completeness) A formula is true in Łukasiewicz logic if its interpretation is true in the standard MV-algebra $[0,1]$.

The proof of this theorem hinges on the following fact:

Theorem (Chang) The Lindenbaum algebra of Łukasiewicz propositional logic is complete.

Granted this result standard completeness becomes easy to prove. Indeed, suppose that some formula φ does not hold in Łukasiewicz logic. Then it is easy to see that its interpretation $e(\varphi)$ fails in the Lindenbaum algebra of the calculus. But we can consider a decomposition of the algebra into a subdirect product of MV algebras which are embeddable in $[0,1]$. Furthermore $e(\varphi)$ fails in some of them, hence it fails in $[0,1]$.

Although it does not transpire from the above sketch of the standard completeness in its original proof it was crucial to establish a connection between MV-algebras and ℓ -groups.

Definition A structure $\langle L, +, -, \circ, \wedge, \vee \rangle$ is called **(Abelian) ℓ -group** if $\langle L, +, - \rangle$ is an (Abelian) group, $\langle L, \vee, \wedge \rangle$ is a lattice and

$$x \leq y \Rightarrow x+z \leq y+z$$

Proposition Given an ℓ -group L and a $m \in L$ with $m > 0$, the algebra

$$A = \{ n \in L \mid 0 \leq n \leq m \}, \quad n \oplus y = (n+y) \wedge m \\ \neg n = m - n$$

is an MV-algebra.

Theorem (Mundici, 1986, Adv. Math.)

The example above gives the **most general** MV-algebra, the construction is **functorial**, and the functor is **faithful**, **full** and **essentially surjective** on the class of **unital l-groups**.

(A unital l-group is a group possessing an element u such that $\forall n > 0 \exists m \in \mathbb{N} \underbrace{u + \dots + u}_{n \text{ times}} \geq n_-$)

Mundici equivalence has been used in many cases to transfer information from the more classical class of **ul-groups** into the (at that time) less known variety of **MV-algebras**.

Another pivotal result in the theory of MV-algebras is the following:

Theorem (McNaughton, 1953, JSL)

The free k -generated MV-algebra is isomorphic to the algebra of continuous piece-wise linear functions with integer coefficients.

Beside offering a concrete representation of important object in the class of MV-algebras, McNaughton theorem unlock a crucial correspondence between the class of finitely presented MV-algebras and the class of rational polyhedra.

Corollary The category of finitely presented MV-algebra is dually equivalent to the category of rational polyhedra and \mathbb{Z} -maps among them.

This correspondence is built exactly in the same way it is done for (geometric) varieties and ideals of a ring of polynomials.

$$I \subseteq \text{Free}_k \mapsto V(I) = \{u \in [0,1]^{\omega} \mid f(u) = 0 \ \forall f \in I\}$$

$$S \subseteq [0,1]^{\omega} \mapsto \mathcal{D}(S) = \{f \in \text{Free}_k \mid f(u) = 0 \ \forall u \in S\}$$

$$f: A \rightarrow B \mapsto f^*: V(B) \rightarrow V(A)$$

$$u \mapsto h(u) \text{ with } h_i \in f(\pi_i)$$

$$f: S \rightarrow T \mapsto f^*: \mathcal{D}(T) \rightarrow \mathcal{D}(S)$$

$$h \mapsto h \circ f$$

Proposition N and \mathfrak{J} are a Galois correspondence, i.e.

- 1) $\mathfrak{J}N(I) \ni I$ and $\mathfrak{J}(S) \ni S$
- 2) $N\mathfrak{J}N(I) = N(I)$ and $\mathfrak{J}N\mathfrak{J}(S) = \mathfrak{J}(S)$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ N\mathfrak{J}(A) & \longrightarrow & N\mathfrak{J}(B) \end{array}$$

$$\begin{array}{ccc} S & \longrightarrow & T \\ \uparrow & & \uparrow \\ NJ(S) & \longrightarrow & NJ(T) \end{array}$$

$$\mathfrak{J}(S) \rightarrow \mathfrak{J}N\mathfrak{J}(S) \rightarrow \mathfrak{J}(S)$$

$\underbrace{\hspace{1cm}}_{id}$

$$NJN(A) \rightarrow NJ(A) \rightarrow NJ(A)$$

$\underbrace{\hspace{1cm}}_{id}$

Extensions of this duality are possible in several directions and constitute at the moment an active field of research.

Needless to say, the tools that come in play, thanks to this duality constitute an important weaponry in the study of MV-algebras.

As Łukasiewicz logic generalizes classical logic, MV-algebras generalizes Boolean algebras

As Boolean algebras form the algebraic structure of the space of the events in classical probability, one may think to extend this idea to many-valued events

Definition A state s on a MV-algebra A , is a function $s: A \rightarrow [0,1]$ such that

$$s(1) = 1$$

$$s(n \otimes y) = s(n) + s(y) \quad \text{whenever } n \otimes y = 0$$

Theorem (Kroupa-Panti 2006)

For every semimodel there is a regular Borel probability measure μ , such that

$$s(n) = \int n \, d\mu$$

Theorem (Mundici, 2009, APAL)

A probability measure on a MV algebra is coherent if, and only if, there exists no Dutch book for it, if and only if, there exists a regular Borel probability measure extending it.

States were internalized in algebraic language, leading to a logical system that may take into account at once both vagueness and uncertainty
(Flaminio - Montagna, J Log Comp., 2011)

For a lot more details on probability measures on MV-algebras, as well as the connection with polyhedral geometry see

Mundici "Advanced Łukasiewicz calculus and MV-algebras", 2011

Finally, just a few words about first order Łukasiewicz logic.

With minor variation on the chemical approach, one may set up a formal system to study first order formulas in Łukasiewicz logic. The axioms to be added are:

$$\forall n \Psi(n) \rightarrow \Psi(1) \quad [\text{t substitutable in } \Psi]$$

$$\forall n (\Psi \rightarrow \Psi) \rightarrow \Psi \rightarrow \forall n \Psi \quad [n \text{ not free in } \Psi]$$

then \exists can be recovered as $\neg \forall \neg$.

The interpretation of \forall is

$$\| \forall n \Psi \|_M = \bigwedge_{a \in M} \| \Psi(a)_n \|_M$$

Notice that also in this case the notions of tautology, satisfiable formulas may be taken either w.r.t 1 or to ≥ 0

Theorem (Scarpellini 1962 JSL)

The set of standard (1-) tautologies is **not** recursively enumerable (actually is Π_2)