Some consequences of compactness in Łukasiewicz Predicate Logic*

Luca Spada

Department of Mathematics and Computer Science, University of Salerno
via Ponte don Melillo, 84084 Fisciano (SA), Italy
e-mail: lspada@unisa.it, web site: http://logica.dmi.unisa.it/lucaspada/

Abstract. The Łoś-Tarski Theorem and the Chang-Łoś-Susko Theorem, two classical results in Model Theory, are extended to the infinite-valued Łukasiewicz logic. The latter is used to settle a characterisation of the class of generic structures introduced in the framework of model theoretic forcing for Łukasiewicz logic [1].

Key-words: First Order Many-Valued Logic, Łukasiewicz logic, Model Theory.

1 Introduction

Łukasiewicz logic is just one possibility in the myriad of infinite-valued generalisations of classical logic. An infinite-valued generalisation of classical logic is obtained by simply picking up some functions on an infinite superset of $\{0, 1\}$ which behave on 0 and 1 accordingly to the classical connectives which they generalise; in other words, by extending the truth tables of the classical connectives.

Among those generalisations some are meaningless, for they have very little in common with a *logic*. Yet, when one requires a few natural properties to be fulfilled, the systems arising allow deep mathematical investigations; this is the case of continuous t-norm based logics. In these systems the conjunction is interpreted in an associative, commutative and weakly-increasing continuous function from $[0, 1]^2$ to $[0, 1]$, which behaves accordingly to classical conjunction in the limit cases 0 and 1, namely a continuous t-norm.

As a matter of fact the most important many-valued logics studied in mathematics are based on continuous t-norms; this is the case, for instance, of Łukasiewicz logic or Gödel logic. The logical system BL, introduced in [2], encompasses all logics based on continuous t-norms.

The setting based on continuous t-norm[1], or equivalently BL, has been quite successful, for it provides a general mathematical framework for investigations on many-valued logics and offers an utter bridge towards fuzzy set theory and fuzzy logic, as t-norms are a pivotal tool in fuzzy logic.

* Partially supported by FWF project P 19872-N18.

[1] The requirement on continuity is sufficient for the existence of a residual operation, which plays the role of implication. Such a requirement can be relaxed to only left-continuity leading to a logical system called MTL [3].

Yet Łukasiewicz logic stands out among those logics because of some of its properties. Indeed, Łukasiewicz logic is the *only* one, among continuous t-norm based logics, with a continuous implication and therefore the only logic whose whole set of formulae can be interpreted as continuous functions. Furthermore the Łukasiewicz negation is *involutive*, namely it is such that $\neg\neg\varphi \leftrightarrow \varphi$. Those two features, inherited from classical logic, makes Łukasiewicz logic a promising setting to test how far the methods of model theory can reach in the realm of many-valued logics.

A model theoretic study of many-valued logic is especially important in the light of the negative results already obtained in the first order theory of these logics: the predicate version BL has a (standard) tautology problem whose complexity is not arithmetical, the same problem is Π_2-complete for Łukasiewicz logic. Thus the favourable duality between syntax and semantics vanishes when switching to t-norm based logics and new tools must be developed.

The results so far are encouraging: in [1] the Robinson finite and infinite forcing were generalised to Łukasiewicz logic; here some basic results for a model theory of Łukasiewicz logic are presented and used to settle an open problem left therein.

2 Preliminaries

The language of the infinite-valued Łukasiewicz propositional logic, Ł, is built from a countable set of propositional variables, $Var = \{p_1, p_2, \dots, p_n, \dots\}$, and two connectives $\to$ and $\neg$. The axioms of Ł are the following:

$$\varphi \to (\psi \to \varphi); \qquad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi));$$
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi); \qquad (\neg\varphi \to \neg\psi) \to (\psi \to \varphi),$$

Modus ponens is the only rule of inference. The notions of proof and tautology are defined as usual.

The equivalent algebraic semantics for Ł (in the sense of [4]) is given by the variety of MV-algebras [5]. An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid, * is an involution and the following equations hold: $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ and $x \oplus 0^* = 0^*$.

An Ł∀ language $\mathcal{L}$ is defined similarly to a language for classical first order logic, without functional symbols[2], taking as primitive the connectives: $\to, \neg, \exists$. This allows the syntactical concepts of term, (atomic) formula, free or bounded variable, substitutable variable for a term, formal proof, formal theorems, etc. to be defined just as usual. The set $V = \{x, y, z, ...\}$ is a fixed set of variables and **Form** will be used to indicate the set of formulae of $\mathcal{L}$.

The axioms of Ł∀ are:

(i) All the axioms of the infinite-valued propositional Łukasiewicz calculus;

[2] Functional symbols can be added, in principle, to the language, however this requires a discussion on how equality has to be treated which goes beyond the scope of this article.

(ii) $\forall x\varphi \to \varphi(t)$, where the term t is substitutable for x in φ;
(iii) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is not free in φ;

The inference rules are *Modus ponens*: from φ and $\varphi \to \psi$, derive ψ; *Generalisation*: from φ, derive $\forall x\varphi$. The fact that φ is a formal theorem of Ł$\forall$ is indicated by $\vdash \varphi$. A **theory** is a set of formulae. If T is a theory and φ is a formula then the formal inference $T \vdash \varphi$ is defined in the usual way.

Let $\mathcal{L}$ be a Ł$\forall$ language with n predicate symbols and m constant symbols. Let A be an MV-algebra. An A-structure has the form

$$\mathcal{M} = \langle M, P_1^{\mathcal{M}}, ..., P_n^{\mathcal{M}}, c_1^{\mathcal{M}}, ..., c_m^{\mathcal{M}} \rangle$$

where M is a non-empty set (called the universe of the structure); if P_i is a predicate symbol in $\mathcal{L}$ of arity k then $P_i^{\mathcal{M}}$ is a k-ary A-valued relation on A, namely $P_i^{\mathcal{M}} : M^k \to A$; if c_j is a constant symbol in $\mathcal{L}$ then $c_j^{\mathcal{M}}$ is an element of M.

Let $\mathcal{M}$ be an A-structure. An **evaluation** of $\mathcal{L}$ in $\mathcal{M}$ is a function $e : V \to M$. Given any two evaluations e, e' of $\mathcal{L}$ and for $x \in V$ let $e \equiv_x e'$ iff $e\ |_{V\setminus\{x\}} = e'\ |_{V\setminus\{x\}}$. For any term t of $\mathcal{L}$ and any evaluation in $\mathcal{M}$ let

$$t^{\mathcal{M}}(e) = \begin{cases} e(x) & \text{if } t \text{ is a variable } x \\ c^{\mathcal{M}} & \text{if } t \text{ is a constant } c \end{cases}$$

Given any evaluation in $\mathcal{M}$, e and any formula φ of $\mathcal{L}$, the element $\|\varphi(e)\|_{\mathcal{M}}$ of A is defined by induction, and it is called the **truth value** of φ:

- if φ is of the form $P(t_1, ..., t_n)$ then $\|\varphi(e)\| = P^{\mathcal{M}}(t_1^{\mathcal{M}}(e), ..., t_n^{\mathcal{M}}(e))$;
- if $\varphi = \neg\psi$ then $\|\varphi(e)\| = \|\psi(e)\|^*$;
- if $\varphi = \psi \to \chi$ then $\|\varphi(e)\| = \|\psi(e)\| \Rightarrow \|\chi(e)\|$;
- if $\varphi = \exists x\psi$ then $\|\varphi(e)\| = \bigvee\{\|\psi(e')\| \mid e' \equiv_x e\}$.

An A-structure $\mathcal{M}$ is called **safe** if for any evaluation $e : V \to M$ and for any formula ψ of $\mathcal{L}$, the supremum $\bigvee\{\|\psi(e')\| \mid e' \equiv_x e\}$ exists in A (in this case the infimum $\bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}$ also exists).

If $\|\varphi\|_{\mathcal{M}} = 1$ then φ is said to be true in $\mathcal{M}$, this can be alternatively written as $\mathcal{M} \models \varphi$. A safe A-structure $\mathcal{M}$ is a **model** of a theory T if $\mathcal{M} \models \varphi$ for all $\varphi \in T$. A **standard structure** is a $[0, 1]$-structure, which is always safe. A **standard model** of a theory T is a $[0, 1]$-structure which is a model of T. If φ every A-model of a theory T is also an A-model of a formula φ then φ is said an **A-logical consequence** of T, in symbols $T \models_A \varphi$, in particular, when this is true for standard models then I write $T \models_{[0,1]} \varphi$.

Definition 1. A formula φ is **generally satisfiable** if there exists a model $\mathcal{M}$ such that $\|\varphi\|_{\mathcal{M}} = 1$. If the model can be taken standard then φ is called just **satisfiable**. The previous definitions naturally generalise to theories. A theory T is **consistent** if $T \nvdash \bot$.

All the results in the next section hinge on the following theorems.

Theorem 1 (Weak Completeness [6]). *Any consistent theory T of Ł∀ has a standard model.*

As can be easily guessed from the considerations at the end of Section 1, the notion of compactness in Łukasiewicz logic splits in two, furthermore, if one considers also the two classes of general and standard models, then four different re-statements of compactness emerge (see e.g. [2] for more details). The situation in Ł∀ is fully described below.

Theorem 2 (Compactness). *Let T be a theory in Ł∀:*

(i) If T is finitely generally satisfiable then T is generally satisfiable.
(ii) If T is finitely satisfiable then T is satisfiable.
(iii) If $T \models \varphi$ then there exists a finite $T_0 \subseteq T$ such that $T_0 \models \varphi$
*(iv) If $T \models_{[0,1]} \varphi$ then **in general it is false** that there exists a finite $T_0 \subseteq T$ such that $T_0 \models_{[0,1]} \varphi$.*

3 Main results

Henceforth $\mathcal{L}$ is assumed to be a fixed language of Ł∀ and all structures are standard.

In Łukasiewicz logic, all the connectives are continuous, thus each formula is equivalent to one in *prenex form.* This allows to define a total hierarchy on the lines of the *arithmetical hierarchy* in classical logic.

Definition 2. A formula of $\mathcal{L}$ belongs to the set Σ_{n} (Π_{n}, respectively) if it is equivalent to a formula with n blocks of quantifier, where each block is either empty or constituted of an uninterrupted sequence of the same quantifier, $\exists$ or $\forall$, and the first block is made of $\exists$'s ($\forall$'s respectively).

As in the classical case one has $\Sigma_{\mathrm{n}} \cup \Pi_{\mathrm{n}} \subseteq \Sigma_{\mathrm{n+1}} \cap \Pi_{\mathrm{n+1}}$.

Let $\mathcal{M}$ be an A-structure, $\mathcal{L}(\mathcal{M})$ is the expansion of the language $\mathcal{L}$ with a new constant symbol for each element of M. The **diagram** of $\mathcal{M}$, i.e. the set of atomic formulae φ in $\mathcal{L}(\mathcal{M})$ such that $\|\varphi\|_{\mathcal{M}} = 1$, is indicated by $D(\mathcal{M})$; $\mathrm{Th}(\mathcal{M})$ is the set of formulae φ such that $\|\varphi\|_{\mathcal{M}} = 1$.

Definition 3. If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are two A-structures and for any $\varphi \in D(\mathcal{M}_1)$, $\mathcal{M}_1 \models_A \varphi$ iff $\mathcal{M}_2 \models_A \varphi$ then $\mathcal{M}_1$ is a **substructure of** $\mathcal{M}_2$, in symbols $\mathcal{M}_1 \leq \mathcal{M}_2$. If the same is true for *any* sentence of $\mathcal{L}(\mathcal{M}_1)$ than $\mathcal{M}_1$ is an **elementary substructure of** $\mathcal{M}_2$, written $\mathcal{M}_1 \preceq \mathcal{M}_2$

Proposition 1. *Let T be a theory, let $T_\forall$ be the set of logical consequences of T which are in Π_1 and let $\mathbf{K}$ be the class of all substructures of models of T. Then $\mathbf{K}$ is the class of model of $T_\forall$.*

Proof. It follows directly from the definition of the interpretation of universal quantifiers that if $\mathcal{M} \in \mathbf{K}$ then $\mathcal{M} \models_{[0,1]} T_\forall$. Vice-versa, let $\mathcal{M}$ be a model of $T_\forall$, then it follows that $D(\mathcal{M}) \cup T$ is finitely satisfiable. Indeed if it is not then there exist finite subsets $\Psi \subseteq D(\mathcal{M})$ and $\Phi \subseteq T$ such that $\bigwedge \Psi \models_{[0,1]} \neg \bigwedge \Phi$, but $\neg \bigwedge \Phi$ is in Π_1 and this contradicts the fact that $\bigwedge \Phi \in D(\mathcal{M})$. By compactness, $D(\mathcal{M}) \cup T$ has a model, say $\mathcal{N}$. Obviously $\mathcal{M}$ embeds in $\mathcal{N}$ which is a model of T, hence $\mathcal{M} \in \mathbf{K}$.

Corollary 1 (Łoś-Tarski Theorem for Łukasiewicz logic).
A theory is preserved under substructure if, and only if, it is equivalent to a universal (i.e. Π_1) theory.

Let α be an ordinal and $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ a family of $\mathcal{L}$-structure. The structure $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ is a **chain** if for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \leq \mathcal{M}_{\lambda_2}$. If for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \preceq \mathcal{M}_{\lambda_2}$ then $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ is called **elementary chain**.

Lemma 1 ([1, Lemma 4.2]). *Let $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ be an elementary chain. Then for every $\lambda \in \alpha$, $\mathcal{M}_\lambda \preceq \bigcup_{\lambda\in\alpha} \mathcal{M}_\lambda$*

T is an **inductive** theory if it is closed under unions of chains.

Theorem 3 (Chang-Łoś-Suszko Theorem for Łukasiewicz logic).
A theory is inductive if, and only if, it is equivalent to a Π_2 theory.

Proof. For the non-easy direction suppose that a theory T is inductive. Let $\mathcal{M} \models T_{\forall_2}$ then $T \cup \mathrm{Th}_\exists(\mathcal{M})$ is satisfiable, for if it is not then there exist two finite sets $\Psi \subseteq \mathrm{Th}_\exists(\mathcal{M})$ and $\Phi \subseteq T$ such that $\bigwedge \Phi \models_{[0,1]} \neg \bigwedge \Psi$, but then $\neg \wedge \Psi \in T_\forall$, whence it holds in $\mathcal{M}$, which is a contradiction. So $T \cup \mathrm{Th}_\exists(\mathcal{M})$ has a model $\mathcal{N}$, and $\mathcal{M} \subseteq \mathcal{N}$. Every existential sentence of $L(\mathcal{M})$ which is true in $\mathcal{N}$ holds in $\mathcal{M}$, hence $D(\mathcal{N}) \cup \mathrm{Th}(\mathcal{M})$ is satisfiable, so it has a model $\mathfrak{M}_1$ which is an extension of $\mathcal{N}$ and an elementary extension of $\mathcal{M}$. Repeating such a construction countably many times an infinite chain is produced:

$$\mathcal{M} \leq \mathcal{N} \leq \mathcal{M}_1 \leq \mathcal{N}_1 \leq \ldots$$

Let $\mathcal{O}$ be the limit of this chain. $\mathcal{O}$ is a model of T, for T is inductive; furthermore $\mathcal{O}$ is an elementary extension of $\mathcal{M}$, because the chain $\{\mathcal{M}_i\}_{i\in\omega}$ is elementary. Therefore $\mathcal{M}$ is a model of T.

The above characterisation is extremely useful, when dealing with model complete theories.

Corollary 2. *When the model companion of a theory is axiomatisable, it is equivalent to a $\forall\exists$ theory.*

Proof. In a model companion every chain is elementary.

From this it is also easy to see that

Corollary 3. *There exists at most one model companion of a theory.*

In [1] both the notions of finite and infinite model theoretic forcing [7,8] were extended to Łukasiewicz logic. Following the lines of Robinson, the family of *generic models*, $\mathfrak{G}_{\mathbf{K}}$, of a given class $\mathbf{K}$ was studied and proved to contain the class of models existentially closed in $\mathbf{K}$. Theorem 3 enables to complete this result.

Proposition 2. *Let T be an inductive theory, then if $\mathfrak{G}_{\mathrm{Mod}(T)}$ is axiomatisable then it is the class of existentially closed models of T.*

Proof. One direction is given in [1, Proposition 5.7]. For the other let $\mathcal{M}$ be a existentially closed model of T, then, by [1, Theorem 5.10], it embeds in a model $\mathcal{N} \in \mathfrak{G}_{\mathrm{Mod}(T)}$. By [1, Theorem 5.9] $\mathfrak{G}_{\mathrm{Mod}(T)}$ is inductive, so if it is axiomatisable then by Theorem 3 it is equivalent to a Π_2 theory. Since $\mathcal{M}$ is existentially closed, it is easy to see that it satisfies the same Π_2 formulae of $\mathcal{N}$, whence $\mathcal{M} \in \mathfrak{G}_{\mathrm{Mod}(T)}$.

References

1. Di Nola, A., Georgescu, G., Spada, L.: Forcing in Łukasiewicz predicate logic. Studia Logica **89**(1) (May 2008) 111–145
2. Hájek, P.: Metamathematics of Fuzzy Logic. Volume 4 of Trends in Logic, Studia Logica Library. Kluwer Academic, Berlin (1998)
3. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Systems **124** (2001) 271–288
4. Blok, W.J., Pigozzi, D.: Algebraizable logics. Memoirs of American Mathematics Society **77**(396) (1989) 78pp
5. Cignoli, R., D'Ottaviano, I., Mundici, D.: Algebraic Foundations of Many-valued Reasoning. Volume 7 of Trends in Logic, Studia Logica Library. Kluwer Academic (2000)
6. Belluce, L.P., Chang, C.C.: A weak completeness theorem for infinite valued first-order logic. Journal of Symbolic Logic **28**(1) (1963) 43–50
7. Robinson, A.: Forcing in model theory. In: Actes du Congrès International des Mathématiciens, Nice. Volume 1., Paris, Gauthier-Villars (1970) 145–150
8. Robinson, A.: Infinite forcing in model theory. In: Proceedings of the Second Scandinavian Logic Symposium (Oslo, 1970). Volume 63 of Studies in Logic and the Foundations of Mathematics., Amsterdam, North-Holland (1971) 317–340